Ján Jakubík

Partially ordered sets with nondistributive lattices of maximal antichains

Czechoslovak Mathematical Journal, Vol. 42 (1992), No. 4, 685–691

Persistent URL: `http://dml.cz/dmlcz/128363`

Czechoslovak Mathematical Journal, 42 (117) 1992, Praha

PARTIALLY ORDERED SETS WITH NONDISTRIBUTIVE LATTICES OF MAXIMAL ANTICHAINS

JÁN JAKUBÍK, Košice

(Received June 12, 1991)

All partially ordered sets which are dealt with in the present paper are assumed to be finite.

For a partially ordered set X we denote by $MA(X)$ the system of all maximal antichains in X; this system is considered to be partially ordered (cf. Section 1 below). Then $MA(X)$ is a lattice (cf. [1]).

A convex subset of X which is isomorphic to the partially ordered set on Fig. 1 or Fig. 2 will be called a serpentine set or a serpentine cycle in X, respectively.

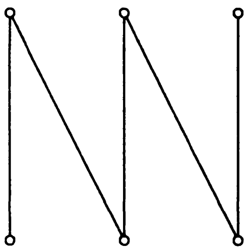

Fig. 1

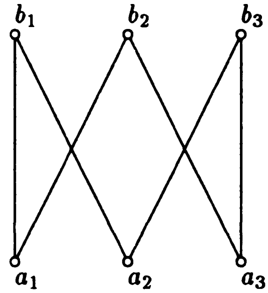

Fig. 2

In [1] the question was proposed to find an internal characterization of those partially ordered sets for which the lattice $MA(X)$ is distributive or modular, respectively.

In [3] it was shown that if $MA(X)$ is nonmodular, then X possesses a serpentine subset. Next, the notion of a regular serpentine subset was introduced and it was shown that $MA(X)$ is nonmodular iff X has a regular serpentine subset.

In this paper the notion of a regular serpentine cycle will be defined. The following result will be established:

(α) Let the lattice $MA(X)$ be modular. Then $MA(X)$ is nondistributive iff X possesses a regular serpentine cycle.

1. Preliminaries

Let X be a partially ordered set. We denote by $A(X)$ the system of all antichains in X. For $B_1, B_2 \in A(X)$ we put $B_1 \leqslant B_2$ if for each $b_1 \in B_1$ there exists $b_2 \in B_2$ with $b_1 \leqslant b_2$. Then $\leqslant$ is a partial order on $A(X)$.

Next, we denote by $MA(X)$ the set of all $B \in A(X)$ having the property that for each $C \in A(X)$ with $B \subseteq C$ the relation $B = C$ is valid. The elements of $MA(X)$ are called maximal antichains in X.

The system $MA(X)$ is partially ordered by the relation $\leqslant$ inherited from $A(X)$. In [1] it was proved that $MA(X)$ is a lattice.

Let $a, b \in X$. If a is covered by b, then we write $a \prec b$ or $b \succ a$. The same symbols are applied for denoting the covering relation in the lattice $MA(X)$. The notation $a \mid b$ means that the elements a and b are incomparable. For A, B in $A(X)$ we write $A <_1 B$ if $A < B$ and if $a \in A$, $b \in B$, $a < b$ implies $a \prec b$.

A convex subset $X_1 \neq \emptyset$ of X will be called to be a short subset of X if there exist A and B in $MA(X)$ with $A <_1 B$ having the property that $X_1 = \{x_1 \in X :$ there are $a \in A$ and $b \in B$ such that $a \leqslant x_1 \leqslant b\}$. Hence whenever x and x' are elements of X_1 with $x < x'$, then $x \prec x'$.

The following result will be proved:

(β) The lattice $MA(X)$ is distributive iff for each short subset X_1 of X the lattice $MA(X_1)$ is distributive.

For an analogous result concerning modularity cf. [3]. From [3] we also recall the following notion.

We denote by $N(X)$ the set of all triples (P_1, P_2, P_3) of mutually disjoint subsets of X such that

(i) $P_2 \neq \emptyset \neq P_3$ and each element of P_2 is covered by each element of P_3;

(ii) both sets $P_1 \cup P_2$ and $P_1 \cup P_3$ belong to $MA(X)$.

A serpentine cycle S of X will be said to be regular if there exist (B_1, B_2, A_2), (B_1', B_2', A_2') and (B_1'', B_2'', A_2'') in $N(X)$ such that (under the notation as in Fig. 2) we have

(i) $A_2 \cup A_2' \cup A_2'' \in A(X)$;

(ii) $B_1 \cup B_2 = B_1' \cup B_2' = B_1'' \cup B_2''$;

(iii) $a_1 \in A_2''$, $a_2 \in A_2'$, $a_3 \in A_2'$, $b_1 \in B_1, b_2 \in B_1'$, $b_3 \in B_1''$.

1.1. Lemma. *Let $B_1, B_2 \in MA(X)$. The following conditions are equivalent:*

(i) $B_1 \leqslant B_2$.

(ii) *For each* $b_2 \in B_2$ *there exists* $b_1 \in B_1$ *such that* $b_1 \leqslant b_2$.

The proof is easy; it is omitted.

1.2. Lemma. *Let* $A, B \in MA(X)$, $A < B$, *and let* $X_1 = A \cup B$ *be a short subset of* X. *Then the set* $MA(X_1)$ *coincides with the interval* $[A, B]$ *of the lattice* $MA(X)$.

Proof. Let $C \in MA(X_1)$. First we shall verify that C belongs to $MA(X)$. By way of contradiction, suppose that C does not belong to $MA(X)$. Hence there is $C' \in MA(X)$ such that $C \subset C'$. Thus there is $c' \in C' \setminus C$. Then clearly $a' \notin A \cup B$.

Since $c' \notin A$ there exists $a \in A$ such that a and c' are comparable. Hence a cannot belong to C; thus a is comparable with an element c of C. Suppose that $c' < a$. If $a < c$, then $c' < c$, which is impossible. Thus $c < a$. Hence $c \notin A$ and then $c \in B$. By virtue of $A < B$ there is $b_1 \in B$ with $a < b_1$; we obtain that $c < b_1$. This cannot hold since both b_1 and c belong to B. Therefore $a < c'$.

An analogous consideration (applying 1.1) leads to the existence of $b \in B$ such that $c' < b$. From $a < c' < b$ and from the convexity of X_1 we infer that $c' \in X_1$, which is a contradiction. Thus $C \in MA(X)$.

Let $c \in C$. Then either $c \in B_2$ or $c \in A$. In the latter case there is $b' \in B$ with $c \leqslant b'$. Hence $C \leqslant B$. Analogously we obtain that $A \leqslant C$. Hence C belongs to the interval $[A, B]$ of $MA(X)$.

Conversely, let C belong to the interval $[A, B]$ of $MA(X)$. Let $c \in C$. There are $a \in A$ and $b \in B$ such that $a \leqslant c \leqslant b$. The relation $a < c < b$ is impossible, since $A \cup B$ is a short subset of X. Therefore $c \in A \cup B$. Now it is clear that $C \in MA(X_1)$. □

2. Short subsets

We denote by M the modular nondistributive lattice with five elements. A sublattice L_1 of a lattice L is said to be saturated if, whenever x and y are elements of L_1 such that x is covered by y in L_1, then x is covered by y in L. The following result is well-known (cf. [2], p. 151).

2.1. Proposition. *Let* L *be a finite modular lattice. Then the following conditions are equivalent:*

(i) L *is nondistributive.*

(ii) *There exists a saturated sublattice* M_1 *of* L *such that* M_1 *is isomorphic to* M.

Let X be a partially ordered set.

2.2. Lemma. *Let A, A' and B be elements of $MA(X)$ such that $A \prec B$, $A' \prec B$ and $A \neq A'$ Then there exists a short subset X_1 of X such that $B \in X_1$ and $A \wedge A' \in X_1$.*

Proof. This is a consequence of [3], Lemma 3.6. □

Proof of (β). Let X_1 be a short subset of X and let A, B be as in Section 1 (with respect to the given X_1). Then $MA(X_1)$ is a convex sublattice of $MA(X)$ with the least element A and the greatest element B. Hence if $MA(X)$ is distributive, then $MA(X_1)$ is distributive as well.

Conversely, suppose that $MA(X)$ fails to be distributive. First assume that $MA(X)$ is nonmodular. Thus in view of [3], Theorem 3.11, there exists a short subset X_1 of X having the property that $MA(X_1)$ is nonmodular, and so $MA(X_1)$ is nondistributive. Next, assume that $MA(X)$ is modular. Then according to 2.1, there exists a five-element saturated sublattice $M_1 = \{B, A, A', A'', C\}$ of $MA(X)$ such that M_1 is isomorphic to M, B is the greatest element of M_1 and C is the least element of M_1. Lemma 2.2 yields that there exists a short subset X_1 of X such that $M_1 \subseteq X_1$. Hence according to 1.2, $MA(X_1)$ is nondistributive. □

From (β) and from 3.11 in [3] we obtain as a corollary:

2.3. Proposition. *The following conditions are equivalent:*

(i) *$MA(X)$ is modular and non-distributive.*

(ii) *There exists a short subset X_1 of X such that $MA(X_1)$ is nondistributive, and $MA(X_2)$ is modular for each short subset X_2 of X.*

3. Nondistributivity

In this section we suppose that X is a partially ordered set such that the lattice $MA(X)$ is modular and non-distributive. Thus there exists a saturated sublattice M_1 of $MA(X)$ with the properties as in the proof of (β) in Section 2. Denote

$$B_2 = B \setminus A, \quad B_1 = B \setminus B_2, \quad A_2 = A \setminus B_1,$$

and let B_2', B_1', A_2', B_2'', B_1'' and A_2'' be defined analogously.

3.1. Lemma. $B \cap C = B_1 \cap B_1'$.

Proof. This is a consequence of 3.6 in [3]. □

3.2. Corollary. $B_1 \cap B_1' = B_1 \cap B_1'' = B_1' \cap B_1''$.

Denote $X_2 = \{x_2 \in X_1 : c \leqslant x_2 \leqslant b$ for some $c \in C \setminus B$ and some $b \in B \setminus C\}$. For each $P \in MA(X_1)$ (where X_1 is the interval $[C, B]$ of $MA(X)$) we have $B \cap C \subseteq P$ and the mapping $P \to P \setminus (B \cap C)$ is an isomorphism of the lattice $MA(X_1)$ onto the lattice $MA(X_2)$.

The above consideration shows that $MA(X_2)$ is modular and nondistributive as well; hence without loss of generality we can suppose that $B \cap C = \emptyset$. Thus in view of 3.2 we assume that

$$B_1 \cap B_1' = B_1 \cap B_1'' = B_1' \cap B_1'' = \emptyset.$$

(For an analogous procedure cf. [3], Section 4.)

Denote $Y(A, A') = C \setminus (A_2 \cup A_2')$.

3.3. Lemma. $A_2 \cap A_2' = A_2 \cap A_2'' = A_2' \cap A_2'' = \emptyset$.

From 3.3 and [3], Lemma 3.6 we infer:

3.4. Lemma. $A_2'' \subseteq Y(A, A')$.

3.5. Lemma. *Each of the sets A_2, A_2', A_2'', B_1, B_1' and B_1'' is nonempty.*

Proof. This follows from [3], Lemmas 4.2 and 4.4. □

3.6. Lemma. *Let $y \in Y(A, A')$, $b_1 \in B_1$ and $b_1' \in B_1'$. Then $y \prec b_1$ and $y \prec b_1'$.*

Proof. We have $y \in C$ and $b_1 \in A$. Next, $C \prec A$ is valid. In view of Lemma 3.6.1 in [3] there exists b_1^* in B_1 such that $y \prec b_1^*$. Also, $b_1 \in B_2'$ and according to 3.3 there is $a_2' \in A_2'$; hence $a_2' \prec b_1$. Therefore from 2.7 in [3] we infer that $y \prec b_1$ is valid. Similarly we obtain that the relation $y \prec b_1'$ holds. □

3.7. Lemma. *Let $a'' \in A_2''$, $b_1 \in B_1$ and $b_1' \in B_1'$. Then $a'' \prec b_1$ and $a'' \prec b_1'$.*

Proof. This follows immediately from 3.4 and 3.5. □

Similarly we have

3.7.1. Lemma. *Let $a \in A_2$, $a' \in A_2'$, $b_1'' \in B_1''$. Next, let b_1 and b_1' be as in 3.7. Then $a \prec b_1'$, $a \prec b_1''$, $a' \prec b_1$ and $a' \prec b_1''$.*

3.8. Proposition. *Assume that $MA(X)$ is modular and nondistributive. Then X possesses a regular serpentine cycle.*

Proof. In view of 2.1 there exists a saturated sublattice $\{C, A, A', A'', B\}$ of $MA(X)$ which is isomorphic to the lattice M. Let us apply the notation as above. According to 3.5 there exist elements a, a', a'', b_1, b_1' and b_1'' with the properties as in 3.7.1. Then a, a' and a'' are distinct elements belonging to C, hence they are mutually incomparable. Next, b_1, b_1' and b_1'' are distinct elements belonging to B, hence they are mutually incomparable as well. It is easy to verify that the elements a, a', a'', b_1, b_1', b_1'' are distinct. Therefore in view of 3.6.1 the set consisting of these elements is a regular serpentine cycle in X. □

Let $C_0 \in MA(X)$ and $A_0 \in A(X)$. Assume that $A_0 < C_0$ is valid in $A(X)$ and that, whenever $a_0 \in A_0$, $c_0 \in C_0$ and $a_0 \leqslant c_0$, then $a_0 \prec c_0$. Put $Q = \{c_0 \in C_0 \colon c_0 \mid a_0$ for each $a_0 \in A_0\}$. Next, let Q_1 be the set of all $x \in X$ such that

(i) $x \mid y$ for each $y \in A_0 \cup Q$;

(ii) there exists $c_0 \in C_0$ with $x \prec c_0$;

(iii) if $c \in C_0$ and $x \leqslant c$, then $x \prec c$.

We set $C^* = A_0 \cup Q \cup Q_1$. It is obvious that $C^* \in A(X)$ and that, whenever $t \in X \setminus C^*$, $t \leqslant c$ for some $c \in C_0$, then t is comparable with an element of C^*. Hence we obtain from Lemma 2.1 in [3]:

3.9. Lemma. *Under the above notation, C^* belongs to $MA(X)$.*

Also, from the construction of C^* we immediately conclude:

3.10. Lemma. *Let A_0, C_0 and C^* be as above. Let $D \in MA(X)$ be such that $A_0 \subseteq D$ and $D \subseteq C_0$. Then $D \leqslant C^*$.*

3.11. Proposition. *Assume that $MA(X)$ is modular and that X possesses a regular serpentine cycle. Then $MA(X)$ is nondistributive.*

Proof. Let us assume that X possesses a regular serpentine cycle S. Next, let (i), (ii) and (iii) be as in Section 1.

Denote $B = B_1 \cup B_2$, $A = B_1 \cup A_2$, $A' = B_1' \cup A_2'$, $A'' = B_1'' \cup A_2''$. Then B, A, A' and A'' belong to $MA(X)$. In view of (ii) and [3], Lemma 2.7 we have

$$A \prec B, \quad A' \prec B, \quad A'' \prec B. \tag{1}$$

Since $b_1 \in B_1$, $a' \in A_2'$ and $a' \prec b_1$ we infer that a' does not belong to A_2 and clearly $a' \notin B$. Therefore $A \neq A'$. Similarly we can verify that $A \neq A''$ and $A' \neq A''$.

Put $A_0 = A_2 \cup A_2' \cup A_2''$ and $C_0 = B$. Let C^* be as in Lemma 3.9.

We have $a \in A$ and $a \in C^*$, hence $a \in A \vee C^*$. Clearly $a \notin B$, thus $A \vee C^* \neq B$. Since $A \leqslant A \vee C^* \leqslant B$, according to (1) we obtain that $A \vee C^* = A$ and therefore $C^* \leqslant A$. Similarly we obtain that $C^* \leqslant A'$ and $C^* \leqslant A''$. Hence

$$C^* \leqslant A \wedge A' \wedge A''. \tag{2}$$

The fact that $A_2 \cup A_2' \cup A_2''$ is an antichain in X and that $A_2 \subseteq A$, $A_2' \subseteq A$ and $A_2'' \subseteq A''$ implies that

$$A_2 \cup A_2' \cup A_2'' \subseteq A \wedge A' \wedge A''$$

is valid. Thus (2) and 3.10 yield

$$C^* = A \wedge A' \wedge A''.$$

Now from Lemma 3.7 in [3] and by applying the relation $A_2 \cup A_2' \cup A_2'' \in A(X)$ again we infer that $A_2'' \subseteq A \wedge A'$. Thus

$$A_2 \cup A_2' \cup A_2'' \subseteq A \wedge A'$$

and hence $C^* \leqslant A \wedge A'$. Therefore $A \wedge A' \wedge A'' = A \wedge A'$. Similarly we infer that

$$A \wedge A'' = A \wedge A' \wedge A'' = A' \wedge A''.$$

Thus the sublattice of $MA(X)$ consisting of the elements A, A', A'', $A \wedge A'$ and B is nondistributive. □

From 3.8 and 3.11 we obtain that (α) holds.

References

[1] *G. Behrendt*: Maximal antichains in partially ordered sets, Ars conbinatoria *25C* (1988), 149–157.

[2] *G. Grätzer*: Lectures on lattice theory, Vol I, San Francisco, 1971.

[3] *J. Jakubík*: Maximal antichains in a partially ordered set., Czechoslov. Math. J. *41* (1991), 75–84.

Author's address: Matematický ústav SAV, dislokované pracovisko, Grešákova 6, 040 01 Košice, Czechoslovakia.