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ON THE CONNECTEDNESS OF THE SET OF FIXED POINTS OF A COMPACT OPERATOR IN THE FRÉCHET SPACE $C^m((b,\infty),\mathbb{R}^n)$

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INTRODUCTION

Several authors (e.g. N. Aronszajn in [2], M. Hukuhara in [7], M. A. Krasnosel’skij and A. I. Perov in [8], G. Stampacchia in [14], F. E. Browder and G. P. Gupta in [4], G. Vidossich in [19], S. Szufia in [15]–[18], R. R. Achmerov, M. I. Kamenskij, A. S. Potapov in [1], M. A. Krasnosel’skij, P. P. Zabrejko in [9] and B. N. Sadovskij in [13]) have investigated the compactness as well as the connectedness of the set of all fixed points of a compact operator or an operator of a more general type mostly in a Banach space. Only few of them have been interested in this problem in a more general space (P. Morales in [12], Š. Belohorec in [3], Z. Kubáček in [10] and K. Czarnowski, T. Pruszko in [5]). Here the results from a Banach space will be extended to a Fréchet space. Our considerations will be based on the following results which are given as Lemmas.

Lemma 1 ([10], p. 422). Let $X$ be a Hasdorff topological vector space, $M$ a non-empty closed subset of $X$, $F : M \to X$ a compact mapping, and let $B$ denote the neighborhood base of the point 0 consisting of balanced sets. Let the following conditions be satisfied:

(i) for each set $U \in B$ there exists a compact mapping $F_U : M \to X$ such that

$$F(x) - F_U(x) \in U \text{ for each } x \in M;$$

(ii) for each $U \in B$ and for each $x \in U$ the equation

$$y - F_U(y) = x$$

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has a unique solution \( y \in M \).

Then the set \( S \) of fixed points of the mapping \( F \) is nonempty, compact and connected.

**Lemma 2** ([6], pp. 89–90, [20], pp. 55–56). Let \((X, \| \cdot \|)\) be a real Banach space, \( \Omega \) a non-empty open and bounded subset of \( X \), \( F : \bar{\Omega} \rightarrow X \) a compact mapping which satisfies the strengthened Leray – Schauder condition:

there exists an \( x_0 \in \Omega \) such that

\[
F(x) - x_0 \neq \lambda (x - x_0) \quad \text{for each} \ x \in \partial \Omega \ \text{and each} \ \lambda \geq 1.
\]

Further, let there exist a sequence of compact mappings \( F_p : \bar{\Omega} \rightarrow X, \ p = 1, 2, \ldots \) with the properties

a) \( \delta_p = \sup \{ \| F_p(x) - F(x) \| : x \in \bar{\Omega} \} \rightarrow 0 \) for \( p \rightarrow \infty \);

b) the equation (in \( y \))

\[
y - F_p(y) = F(x) - F_p(x)
\]

has at most one solution in \( \bar{\Omega} \) for each \( x \in \Omega \).

Then the set \( S \) of fixed points of the mapping \( F \) is non-empty, compact and connected.

The next Lemma is a consequence of the theorem in [11], p. 111.

**Lemma 3.** Let \((X, d)\) be a metric space and \( \{ S_m : m = 1, 2, \ldots \} \) a sequence of non-empty compact and connected sets such that

\[
S_{m+1} \subset S_m \quad \text{for} \ m = 1, 2, \ldots.
\]

Then \( \bigcap_{m=1}^{\infty} S_m \) is a non-empty compact and connected set.

We shall use the following notation.

Let \( -\infty < b < \infty \) and let \( n > 0, \ k \geq 0 \) be integers, \( I_b = (b, \infty) \), \( | \cdot | \) a norm in \( \mathbb{R}^n \).

Let

\[
X = C^k(I_b, \mathbb{R}^n), p_m(x) = \max \{|x(t)| + \ldots + |x^{(k)}(t)| : b \leq t \leq b + m\}
\]

for each \( x \in X \) and each \( m = 1, 2, \ldots \). The space \((X, \{p_m\})\) is a real Fréchet space and the convergence in this space means the uniform convergence of the functions and their first \( k \) derivatives on each interval \((b, b + m), m = 1, 2, \ldots\).

Further, let

\[
X_m = C^k((b, b + m), \mathbb{R}^n) \quad \text{for each} \ m = 1, 2, \ldots
\]
Then $p_m$ is a norm in $X_m$ and $(X_m, p_m)$ is a real Banach space.

Let $h > 0$ and $\psi \in C^k((-h, 0), \mathbb{R}^n)$. Let $\varphi, \varphi_p \in C(I_b, (0, \infty))$, $p = 1, 2, \ldots$ where the sequence $\{\varphi_p\}$ is nonincreasing in $I_b$ and $\lim_{p \to \infty} \varphi_p(t) = 0$ for each $t \in I_b$.

Denote

$$M = \{x \in X: |x(t) - \psi(0)| + \ldots + |x^{(k)}(t) - \psi^{(k)}(0)| \leq \psi(t)$$
for each $t \in I_b$ and $x^{(j)}(b) = \psi^{(j)}(0), j = 0, 1, \ldots, k\},$

$$M_m = \{x \in X_m: |x(t) - \psi(0)| + \ldots + |x^{(k)}(t) - \psi^{(k)}(0)| \leq \varphi(t),$$
t $\in (b, b + m)$ and $x^{(j)}(b) = \psi^{(j)}(0), j = 0, 1, \ldots, k\}, \quad m = 1, 2, \ldots.$

$M(M_m)$ is a closed, convex and bounded set in $X$ (in $X_m, m = 1, 2, \ldots$). Clearly, if $x \in M$ or $x \in M_{m+p}$, then $x\big|_{(b, b + m)} \in M_m$ for each $m = 1, 2, \ldots, p = 1, 2, \ldots$.

Here and in the sequel $f\big|_{(a, b)}$ denotes the restriction of the function $f$ to the interval $(a, b)$.

**Main results**

The approximation Lemma which follows represents the main tool in obtaining the new results.

**Lemma 4.** Let the spaces $X, X_m, m = 1, 2, \ldots$, the functions $\psi, \varphi$ and the sets $M, M_m, m = 1, 2, \ldots$ be as above. Let there exist mappings $T: M \to X, T_m: M_m \to X_m, m = 1, 2, \ldots$ with the properties

1. $x\big|_{(b, b + m)} = y\big|_{(b, b + m)} \Rightarrow T(x)\big|_{(b, b + m)} = T(y)\big|_{(b, b + m)}$ for each $x, y \in M, m = 1, 2, \ldots$;
2. $T_m(x\big|_{(b, b + m)}) = T(x)\big|_{(b, b + m)}$ for each $x \in M, M = 1, 2, \ldots$;
3. $x\big|_{(b, b + m)} = y\big|_{(b, b + m)} \Rightarrow T_{m+p}(x)\big|_{(b, b + m)} = T_{m+p}(y)\big|_{(b, b + m)}$ for each $x, y \in M_{m+p}, m = 1, 2, \ldots, p = 1, 2, \ldots$;
4. $T_m(x\big|_{(b, b + m)}) = T_{m+p}(x)\big|_{(b, b + m)}$ for each $x \in M_{m+p}, m = 1, 2, \ldots, p = 1, 2, \ldots$.

Further, let the set $S_m^*\alpha$ of all fixed points of the operator $T_m$ be nonempty, compact and connected in the space $X_m$. Then the set $S$ of all fixed points of the operator $T$ is nonempty, compact and connected in the space $X$.

**Proof.** Let $m_0 \geq 1$ be a fixed integer. Let

$$S_m = \{x\big|_{(b, b + m_0)} : x \in S_m^*\} \text{ for all } m \geq m_0.$$
Fix an arbitrary $m \geq m_0$. Clearly $S_m \neq \emptyset$. Since the mapping from $X_m$ to $X_{m_0}$ which to each function $x \in X_m$ assigns the restriction $x|_{(b, b + m_0)}$ is continuous, $S_m$ is compact and connected. Since $m \geq m_0$ is arbitrary, by Lemma 3 we get that

$$P_{m_0} = \bigcap_{m=m_0}^{\infty} S_m \neq \emptyset,$$

and it is a compact and connected set.

Denote by $S$ the set of all fixed points of the operator $T$. If $x \in S$, then in view of (6)

$$y_m = x|_{(b, b + m)} \in S_m^* \text{ for each } m \geq m_0$$

and hence

$$y = y_m|_{(b, b + m_0)} = x|_{(b, b + m_0)} \in P_{m_0}.$$  

Conversely, let $y \in P_{m_0}$. Then for each $m \geq m_0$ there is a $y_m \in S_m^*$ such that $y_m|_{(b, b + m_0)} = y$. We shall show that there is an $x \in S$ such that $y = x|_{(b, b + m_0)}$.

Consider the sequence $\{y_m\}_{m=m_0+1}^{\infty}$. As by (4) the sequence $\{y_m|_{(b, b + m_0 + 1)}\} \subseteq S_{m_0+1}^*$ and the last set is compact, there exists a subsequence $\{y_{m_r}\}$ of the sequence $\{y_m\}$ and a point $y_1 \in S_{m_0+1}^*$ such that the sequence $\{y_{m_j}|_{(b, b + m_0 + 1)}\}$ converges uniformly to $y_1^{(j)}$ on $(b, b + m_0 + 1)$, $j = 0, \ldots, k$. By mathematical induction we get a sequence of sequences

$$\{y_{m_1}\}, \{y_{m_2}\}, \ldots, \{y_{m_r}\}, \ldots$$

such that

(i) the sequence $\{y_{m_1}\}$ is a subsequence of the sequence $\{y_m\}$;

(ii) $\{y_{m_{r+1}}\}$ is a subsequence of the sequence $\{y_{m_r}\}$ for $r = 1, 2, \ldots$;

(iii) the sequence $\{y_{m_{j}}|_{(b + m_0 + r)}\}$ converges uniformly on $(b, b + m_0 + r)$ for $j = 0, \ldots, k$ and $\{y_{m_r}|_{(b, b + m_0 + r)}\} \subseteq S_{m_0+r}^*$.

Then the diagonal sequence $\{y_{m_m}\}$ possesses the property that $\{y_{m_m}^{(j)}\}$ converges uniformly on each interval $(b, b + m_0 + r)$ to $x^{(j)}$ for $j = 0, \ldots, k$ where $x \in X$ is a certain function. As $y_{m_m}|_{(b, b + m_0 + m)} \in S_{m_0+m}^*$, also $x|_{(b + m_0 + m)} \in S_{m_0+m}^*$ and by (2), $x \in S$.

Hence $S \neq \emptyset$ and $P_{m_0}$ is the set of restrictions to $(b, b + m_0)$ of all functions belonging to $S$, for each $m_0 = 1, 2, \ldots$. Now we prove that $S$ is a compact set in $X$.

Let $\{x_p\} \subseteq S$ be a sequence of points. Then by the compactness of the sets $P_1$, $P_2$, $\ldots$ in the spaces $X_1$, $X_2$, $\ldots$ respectively we get that there exist sequences

$$\{x_{p,1}\}, \{x_{p,2}\}, \ldots$$
such that

(i) \( \{x_{p,1}\} \) is a subsequence of the sequence \( \{x_p\} \);
(ii) \( \{x_{p,r+1}\} \) is a subsequence of the sequence \( \{x_{p,r}\} \) for \( r = 1, 2, \ldots \);
(iii) the sequence \( \{x_{p,r}\} \) together with its first \( k \) derivatives converges uniformly on \( (b, b + r) \).

Then the diagonal sequence \( \{x_{p,p}\} \) converges in the space \( X \) to a point \( x \in X \)
with the property that \( x \mid (b, b + m) \in S_m^* \) and by (2), \( x \in S \).

Finally, we prove that \( S \) is connected. If not, the set \( S \) can be decomposed into the union

\[
S = \hat{K}_1 \cup \hat{K}_2
\]

where \( \hat{K}_1, \hat{K}_2 \) are two non-empty, disjoint and compact sets. Let \( m \geq 1 \) be a natural number. Denote by \( \hat{K}_{1m} \) and \( \hat{K}_{2m} \) the sets of restrictions to \( (b, b + m) \) of the functions from \( \hat{K}_1 \) and \( \hat{K}_2 \), respectively. Hence we have

\[
P_m = \hat{K}_{1m} \cup \hat{K}_{2m}.
\]

The compactness of \( \hat{K}_1, \hat{K}_2 \) implies that \( \hat{K}_{1m}, \hat{K}_{2m} \) are nonempty, compact sets in \( X_m \). If they were disjoint, then \( P_m \) would not be connected in \( X_m \). Hence there exist two elements \( x_m \in \hat{K}_1, y_m \in \hat{K}_2, x_m \neq y_m \) such that their restrictions to \( (b, b + m) \) coincide. Thus

\[
x_m \mid (b, b + m) = y_m \mid (b, b + m).
\]

Consider the sequences \( \{x_m\}, \{y_m\} \). As \( \{x_m\} \subset \hat{K}_1, \{y_m\} \subset \hat{K}_2 \) and \( \hat{K}_1, \hat{K}_2 \) are compact in \( X \), there exist two subsequences \( \{x_{m_i}\}, \{y_{m_i}\} \) of the sequences \( \{x_m\}, \{y_m\} \), respectively, and there exist two elements \( x \in \hat{K}_1, y \in \hat{K}_2 \) such that \( \lim_{i \to \infty} x_{m_i} = x, \lim_{i \to \infty} y_{m_i} = y \) in \( X \). Then with respect to (7) we have \( x = y \). This contradicts the fact that \( \hat{K}_1 \cap \hat{K}_2 = \emptyset \). Hence \( S \) is connected. \( \square \)

Now by means of Lemmas 1 and 2 a sufficient condition for the sets \( S_m^* \) in Lemma 4 to be non-empty, compact and connected can be given. This is the content of the next theorem.

**Theorem 1.** Suppose that all assumptions of Lemma 4 are satisfied except the assumption on the sets \( S_m^* \), \( m = 1, 2, \ldots \) Suppose, further, that for each \( m = 1, 2, \ldots \)

\[
T_m: M_m \subset X_m \to X_m \text{ is a compact mapping},
\]
and there exists a sequence \( \{T_{mp}\}_{p=1}^{\infty} \) of mappings

\[ T_{mp} : M_m \rightarrow X_m \]

with the following properties: For each \( p = 1, 2, \ldots \)

9. \( T_{mp} : M_m \subset X_m \rightarrow X_m \) is a compact mapping;

10. \( |T_m(x)(t) - T_{mp}(x)(t)| + \ldots + |(T_m(x))^{(k)}(t) - (T_{mp}(x))^{(k)}(t)| \leq \varphi_p(t) \) for each \( x \in M_m \) and each \( t \in (b, b + m) \),

and either

11. there exists a function \( \varphi^*_p \in C(I_\delta, (0, \infty)) \) such that

\[ \varphi^*_p + \varphi_p \leq \varphi \text{ in } I_\delta \]

and

\[ |T_{mp}(x)(t) - \psi(0)| + \ldots + |(T_{mp}(x))^{(k)}(t) - \psi^{(k)}(0)| \leq \varphi^*_p(t) \]

for all \( x \in M_m \) and all \( t \in (b, b + m) \);

12. the operator \( H_{mp} : M_m \rightarrow X_m \) which is defined by the relation

\[ H_{mp}(x) = x - T_{mp}(x) \]

for all \( x \in M_m \)

is injective on \( M_m \),

or

13. there exists an \( x_m \in \mathcal{M}_m \) (the interior of \( M_m \)) such that

\[ T_m(x) - x_m \neq \lambda(x - x_m) \]

for each \( x \in \partial M \) and each \( \lambda \geq 1 \);

14. the equation

\[ H_{mp}(y) = x \]

has at most one solution in \( M_m \) for each \( x \in X_m \) such that

\[ |x(t)| + \ldots + |x^{(k)}(t)| \leq \varphi_p(t), \quad b \leq t \leq b + m. \]

(Here \( H_{mp} \) has the same meaning as in (12)).

Then the set \( S \) of all fixed points of the operator \( T \) is non-empty, compact and connected in the space \( X \).

Proof. With respect to Lemma 4 it suffices to show that the set \( S_m^* \) of all fixed points of the operator \( T_m \) is non-empty, compact and connected for each \( m = 1, 2, \ldots \). Hence, let \( m \geq 1 \) be an arbitrary but fixed number. Consider the case when
the assumptions (11), (12) are satisfied. Then we apply Lemma 1 to the operator $T_m$ in the space $X_m$. In this space we have two systems of balanced neighborhoods of 0:

$$U(0, \frac{1}{j}) = \{ x \in X_m : \rho_m(x) < \frac{1}{j} \}, \quad j = 1, 2, \ldots ;$$

$$U_p = \{ x \in X_m : \|x(t)\| + \ldots + \|x^{(k)}(t)\| \leq \varphi_p(t), b \leq t \leq b + m \}, \quad p = 1, 2, \ldots .$$

By the Dini theorem the sequence $\{\varphi_p\}$ converges uniformly to 0 on $(b, b + m)$ and both systems of neighborhoods determine the same topology in $X_m$. For each $U_p$ there exists a compact mapping $T_{mp} : M_m \subset X_m \to X_m$ such that, in view of (10), $T_m(x) - T_{mp}(x) \in U_p$ for each $x \in M_m$.

As to the assumption (ii) in Lemma 1, by the assumption (12) it suffices to show that the equation

$$H_{mp}(y) = x$$

has at least one solution in $M_m$ for each $x \in U_p$. So let us fix an arbitrary $x \in U_p$. Since $M_m$ is a closed and convex set in $X_m$, the operator $T_{mp} + x : M_m \subset X_m \to X_m$ is compact and moreover

$$|T_{mp}(y)(t) - \psi(0)| + |x(t)| + \ldots + |(T_{mp}(y))^{(k)}(t) - \psi^{(k)}(0)| + |x^{(k)}(t)|$$

$$\leq \varphi_p(t) + \varphi_p(t) \leq \varphi(t) \quad \text{for each} \quad t \in (b, b + m),$$

which means that $T_{mp} + x : M_m \to M_m$, by the Schauder fixed point theorem the equation (15) has a solution in $M_m$ and the statement of the theorem follows.

When the assumptions (13) and (14) are fulfilled, then we use Lemma 2. We take $(X_m, \rho_m)$ for the real Banach space, $\hat{M}_m$ for $\Omega$ and $T_m : M_m \subset X_m \to X_m$ for the compact mapping $F$. By (13) $T_m$ satisfies the strengthened Leray-Schauder condition. When $\{T_{mp}\}_{p=1}^{\infty}$ is a sequence of compact mappings which approximates the mapping $T_m$, then by (10)

$$\delta_p = \sup \{\rho_m(T_{mp}(x) - T_m(x)) : x \in M_m\}$$

$$= \max \{\varphi_p(t) : b \leq t \leq b + m\} \to 0 \quad \text{for} \quad p \to \infty .$$

Let $x \in \hat{M}_m$. Then again by (10) $T_m(x) - T_{mp}(x) \in U_p$ and then (14) implies that the assumption b) of Lemma 2 is satisfied, too. By this Lemma the theorem is true. □
Theorem 1 will be applied to the initial value problem for a functional differential equation. First we consider a similar problem for an ordinary differential equation.

Let \( \omega \in C(I_b, (0, \infty)) \), let \( F \in C((0, \infty), (0, \infty)) \) be a non-decreasing function and let \( c \geq 0 \). Then one can find that a necessary and sufficient condition for the problem

\[
y'(t) = \omega(t)F(y + c), \quad y(b) = 0
\]

to have a unique solution on \((b, \infty)\) is that

\[
\int_b^\infty \omega(s) \, ds \leq \int_0^\infty \frac{dv}{F(v + c)}.
\]

Further, denote \( H = C((-h, 0), \mathbb{R}^n) \), \( ||x|| = \max\{|x(s)| : -h \leq s \leq 0\} \) for each \( x \in H \). Then \((H, ||\cdot||)\) is a Banach space. If \( x : (b - h, \infty) \to \mathbb{R}^n \) is a continuous function, then \( x \in H \) is defined by \( x_t(s) = x(t + s), -h \leq s \leq 0 \), for each \( t \in I_b \). In the space \( X^* = C((b - h, \infty), \mathbb{R}^n) \) let the topology be defined by the seminorms \( q_m(x) = \max\{|x(t)| : b - h \leq t \leq b + m\}, m = 1, 2, \ldots, x \in X^* \). Clearly \((X^*, \{q_m\}_{m=1}^{\infty})\) is a Fréchet space.

**Theorem 2.** Let \( \psi \in H, f \in C(I_b \times H, \mathbb{R}^n) \). Let \( \omega \in C(I_b, (0, \infty)) \), let \( F \in C((0, \infty), (0, \infty)) \) be a non-decreasing function and

\[
\int_b^\infty \omega(s) \, ds \leq \int_0^\infty \frac{dv}{F(v + |\psi(0)|)}.
\]

Let

\[
|f(t, x)| \leq \omega(t)F(||x||) \quad \text{for each } (t, x) \in I_b \times M^{**},
\]

where

\[
M^{**} = \{x_t \in H : x \in C((b - h, \infty), \mathbb{R}^n), |x(t) - \psi(0)| \leq \varphi(t) \text{ for each } t \in I_b, x_b = \psi\}
\]

and \( \varphi \) is the solution of (16) on \( I_b \) with \( c = |\psi(0)| \).

Then the problem

\[
x'(t) = f(t, x_t), \quad b \leq t < \infty
\]

\[
x_b = \psi
\]
has a solution satisfying the inequality

\[(21)\quad |x(t) - \psi(0)| \leq \varphi(t) \quad \text{for each } t \in I_b.\]

The set of all such solutions is compact and connected in the space \(X^*\).

Proof. Consider the Fréchet space \((X, \{p_m\}_{m=1}^\infty)\) where \(X = C(I_b, \mathbb{R}^n)\), and the seminorms \(p_m(x) = \max\{|x(t)|: b \leq t \leq b + m\}, m = 1, 2, \ldots, x \in X\). This space corresponds to the case \(k = 0\) mentioned above. By virtue of (21) the problem (19), (20) is equivalent to the fixed point (f.p. for short) problem for the operator \(T^*: M^* \to X^*\) which is defined by

\[T^*(x)(t) = \begin{cases} \psi(0) + \int_b^t f(s, x_s) \, ds, & b \leq t < \infty, \\ \psi(t - b), & b - h \leq t \leq b \end{cases}\]

on the set \(M^* = \{x \in X^*: x_b = \psi \text{ and } |x(t) - \psi(0)| \leq \varphi(t) \text{ for each } t \in I_b\}\).

Let

\[V = \{x \in X: x(b) = \psi(0)\}, \]
\[V^* = \{x \in X^*: x_b = \psi\}.

Define the mapping \(P: V \to V^*\) by

\[P(x)(t) = \begin{cases} x(t), & b \leq t < \infty, \\ \psi(t - b), & b - h \leq t \leq b, \end{cases}\quad \text{for each } x \in V.

Then \(P\) is a bijection of \(V\) onto \(V^*\) and since \(x_p \to x\) in \(V \subset X\) for \(p \to \infty\) is equivalent to \(P(x_p) \to P(x)\) in \(V^* \subset X^*\) for \(p \to \infty\), \(P\) is a homeomorphism of \(V\) onto \(V^*\). Clearly the inverse mapping \(P^{-1}\) of \(P\) is defined by

\[P^{-1}(x) = x|_{(b, \infty)} \quad \text{for each } x \in V^*.

Let \(M = \{x \in X: |x(t) - \psi(0)| \leq \varphi(t) \text{ for each } t \in I_b \text{ and } x(b) = \psi(0)\}\). Consider now the mapping \(T = P^{-1} \circ T^* \circ P|_M\). Then \(T: M \to X\) and

\[(22)\quad T(x)(t) = \psi(0) + \int_b^t f(s, x_s) \, ds, \quad b \leq t < \infty, \quad x \in M, \quad x_b = \psi.

(In fact, the operator \(T\) should be defined by

\[T(x)(t) = \psi(0) + \int_b^t f(s, (P(x))_s) \, ds, \quad b \leq t < \infty, \quad x \in M,

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but it is clear what (22) means. The same notation will be used for the operators \( T_p, T_m \) and \( T_{mp} \), which will be defined on \( M \) in a similar way.)

Clearly \( u \in M \) is a f.p. of \( T \) iff \( P(u) \in M^* \) is a f.p. of \( T^* \), and in view of the property of \( P \), the set of all f.p. of \( T^* \) in \( M^* \) is non-empty, compact and connected in \( M^* \) iff the set of all f.p. of \( T \) in \( M \) has the same property. Thus we can apply Theorem 1 to the operator \( T \).

The set \( M \) is closed in the Fréchet space \( X \). Define operators \( T_p : M \to X \) by

\[
T_p(x)(t) = \begin{cases} 
\psi(0), & b \leq t \leq b + \frac{1}{p}, \\
\psi(0) + \int_b^{t-1/p} f(s, x_s) \, ds, & b + \frac{1}{p} \leq t < \infty, x \in M, \ x_b = \psi.
\end{cases}
\]

Then (18) yields

\[
|T(x)(t) - T_p(x)(t)| \leq \begin{cases} 
\int_b^t \omega(s) F(\varphi(s) + |\psi(0)|) \, ds, & b \leq t \leq b + \frac{1}{p}, \\
\int_{t-1/p}^t \omega(s) F(\varphi(s) + |\psi(0)|) \, ds, & b + \frac{1}{p} \leq t < \infty, x \in M, \ x_b = \psi.
\end{cases}
\]

Denote by \( \varphi_p(t) \) the right-hand side of the last inequality. Hence

\[
\varphi_p(t) = \begin{cases} 
\int_b^t \omega(s) F(\varphi(s) + |\psi(0)|) \, ds, & b \leq t \leq b + \frac{1}{p}, \\
\int_{t-1/p}^t \omega(s) F(\varphi(s) + |\psi(0)|) \, ds, & b + \frac{1}{p} \leq t < \infty, p = 1, 2, \ldots
\end{cases}
\]

Clearly \( \{ \varphi_p \} \) is a nonincreasing sequence on \( I_b \) and \( \lim_{p \to \infty} \varphi_p(t) = 0 \) for each \( t \in (b, \infty) \).

Further, when we define

\[
\varphi_{*p}(t) = \begin{cases} 
0, & b \leq t \leq b + \frac{1}{p}, \\
\int_b^{t-1/p} \omega(s) F(\varphi(s) + |\psi(0)|) \, ds, & b + \frac{1}{p} \leq t < \infty, p = 1, 2, \ldots
\end{cases}
\]

then

\[
|T_p(x)(t) - \psi(0)| \leq \varphi_{*p}(t), t \in I_b, \ p = 1, 2, \ldots, x \in M, \ x_b = \psi,
\]

and by (16)

\[
\varphi_{*p}(t) + \varphi_p(t) = \int_b^t \omega(s) F(\varphi(s) + |\psi(0)|) \, ds = \varphi(t),
\]

for each \( t \in I_b \).

Further, the operators \( T_m, T_{mp} : M_m \subset X_m \to X_m \) defined by

\[
T_m(x)(t) = \psi(0) + \int_b^t f(s, x_s) \, ds, b \leq t \leq b + m, x_b = \psi,
\]

\[
T_{mp}(x)(t) = \begin{cases} 
\psi(0), & b \leq t \leq b + \frac{1}{p}, \\
\psi(0) + \int_b^{t-1/p} f(s, x_s) \, ds, & b + \frac{1}{p} \leq t \leq b + m \\
\text{for } m = 1, 2, \ldots, p = 1, 2, \ldots
\end{cases}
\]

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are compact. This can be shown in the usual way.

The last step in checking the assumptions of Theorem 1 consists of proving (12). Let the mapping $H_{mp}: M_m \rightarrow X_m$ be defined by

$$H_{mp}(x) = x - T_{mp}(x) \text{ for all } x \in M_m, \ x_b = \psi, \ m = 1, 2, \ldots, \ p = 1, 2, \ldots.$$ 

Consider two elements $x, y \in M_m, x \neq y$. Then there exists a $t_0: b < t_0 \leq b + m$ such that $x(t_0) \neq y(t_0)$. Two cases may occur:

a) If $t_0 \in (b, b + \frac{1}{\rho})$, then $H_{mp}(x)(t_0) = x(t_0) - \psi(0) \neq y(t_0) - \psi(0) = H_{mp}(y)(t_0)$;

b) there is a $t_1 \geq b + \frac{1}{\rho}$ such that $T_1 = \sup\{\tau > b: x(t) = y(t) \text{ for } t \in (b, \tau)\}$. 

Then there exists a $t_0 \in (t_1, t_1 + \frac{1}{\rho})$ such that $x(t_0) \neq y(t_0)$. This implies that

$$T_{mp}(x)(t_0) = \psi(0) + \int_{b}^{t_0} f(s, x_s) ds = \psi(0) + \int_{b}^{t_0} f(s, y_s) ds = T_{mp}(y)(t_0)$$

and hence $H_{mp}(x)(t_0) \neq H_{mp}(y)(t_0)$.

In both cases the operator $H_{mp}$ is injective on $M_m$ and all assumptions of Theorem 1 are satisfied. By this theorem the statement of Theorem 2 follows. 

References


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