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_Czechoslovak Mathematical Journal_, Vol. 42 (1992), No. 4, 577–588

Persistent URL: [http://dml.cz/dmlcz/128365](http://dml.cz/dmlcz/128365)

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ON THE CONNECTEDNESS OF THE SET OF FIXED POINTS OF A COMPACT OPERATOR IN THE FRÉCHET SPACE $C^m((b, \infty), \mathbb{R}^n)$

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(Received April 4, 1990)

INTRODUCTION

Several authors (e.g. N. Aronszajn in [2], M. Hukuhara in [7], M. A. Krasnosel’skij and A. I. Perov in [8], G. Stampacchia in [14], F. E. Browder and G. P. Gupta in [4], G. Vidossich in [19], S. Szufia in [15]–[18], R. R. Achmerov, M. I. Kamenskij, A. S. Potapov in [1], M. A. Krasnosel’skij, P. P. Zabrejko in [9] and B. N. Sadovskij in [13]) have investigated the compactness as well as the connectedness of the set of all fixed points of a compact operator or an operator of a more general type mostly in a Banach space. Only few of them have been interested in this problem in a more general space (P. Morales in [12], Š. Belohorec in [3], Z. Kubáček in [10] and K. Czarnowski, T. Pruszko in [5]). Here the results from a Banach space will be extended to a Fréchet space. Our considerations will be based on the following results which are given as Lemmas.

Lemma 1 ([10], p. 422). Let $X$ be a Hausdorff topological vector space, $M$ a non-empty closed subset of $X$, $F : M \to X$ a compact mapping, and let $B$ denote the neighborhood base of the point 0 consisting of balanced sets. Let the following conditions be satisfied:

(i) for each set $U \in B$ there exists a compact mapping $F_U : M \to X$ such that

$$F(x) - F_U(x) \in U \text{ for each } x \in M;$$

(ii) for each $U \in B$ and for each $z \in U$ the equation

$$y - F_U(y) = z$$
has a unique solution \( y \in M \).

Then the set \( S \) of fixed points of the mapping \( F \) is nonempty, compact and connected.

**Lemma 2** ([6], pp. 89–90, [20], pp. 55–56). Let \((X, ||\cdot||)\) be a real Banach space, \( \Omega \) a non-empty open and bounded subset of \( X \), \( F: \overline{\Omega} \to X \) a compact mapping which satisfies the strengthened Leray – Schauder condition:

there exists an \( x_0 \in \Omega \) such that

\[
F(x) - x_0 \neq \lambda (x - x_0) \quad \text{for each } x \in \partial \Omega \text{ and each } \lambda \geq 1.
\]

Further, let there exist a sequence of compact mappings \( F_p: \overline{\Omega} \to X, p = 1, 2, \ldots \) with the properties

a) \( \delta_p = \sup\{||F_p(x) - F(x)||: x \in \overline{\Omega}\} \to 0 \) for \( p \to \infty \);

b) the equation (in \( y \))

\[
y - F_p(y) = F(x) - F_p(x)
\]

has at most one solution in \( \overline{\Omega} \) for each \( x \in \Omega \).

Then the set \( S \) of fixed points of the mapping \( F \) is non-empty, compact and connected.

The next Lemma is a consequence of the theorem in [11], p. 111.

**Lemma 3.** Let \((X, d)\) be a metric space and \( \{S_m: m = 1, 2, \ldots\} \) a sequence of non-empty compact and connected sets such that

\[
S_{m+1} \subset S_m \quad \text{for } m = 1, 2, \ldots
\]

Then \( \bigcap_{m=1}^{\infty} S_m \) is a non-empty compact and connected set.

We shall use the following notation.

Let \(-\infty < b < \infty \) and let \( n > 0, k \geq 0 \) be integers, \( I_b = (b, \infty) \), \( \cdot \) a norm in \( \mathbb{R}^n \). Let

\[
X = C^k(I_b, \mathbb{R}^n), p_m(x) = \max\{|x(t)| + \ldots + |x^{(k)}(t)|: b \leq t \leq b + m\}
\]

for each \( x \in X \) and each \( m = 1, 2, \ldots \). The space \((X, \{p_m\})\) is a real Fréchet space and the convergence in this space means the uniform convergence of the functions and their first \( k \) derivatives on each interval \((b, b + m), m = 1, 2, \ldots\).

Further, let

\[
X_m = C^k((b, b + m), \mathbb{R}^n) \quad \text{for each } m = 1, 2, \ldots
\]
Then $p_m$ is a norm in $X_m$ and $(X_m, p_m)$ is a real Banach space.

Let $h > 0$ and $\psi \in C^k((-h, 0), \mathbb{R}^n)$. Let $\varphi, \varphi_p \in C(I_b, (0, \infty))$, $p = 1, 2, \ldots$ where the sequence $\{\varphi_p\}$ is nonincreasing in $I_b$ and $\lim_{p \to \infty} \varphi_p(t) = 0$ for each $t \in I_b$.

Denote

$$M = \{x \in X : |x(t) - \psi(0)| + \ldots + |x^{(k)}(t) - \psi^{(k)}(0)| \leq \psi(t)$$

for each $t \in I_b$ and $x^{(j)}(b) = \psi^{(j)}(0)$, $j = 0, 1, \ldots, k \}$,

$$M_m = \{x \in X_m : |x(t) - \psi(0)| + \ldots + |x^{(k)}(t) - \psi^{(k)}(0)| \leq \varphi(t),$$

$t \in (b, b + m)$ and $x^{(j)}(b) = \psi^{(j)}(0)$, $j = 0, 1, \ldots, k \}$, $m = 1, 2, \ldots$.

$M(M_m)$ is a closed, convex and bounded set in $X$ (in $X_m$, $m = 1, 2, \ldots$). Clearly, if $x \in M$ or $x \in M_{m+p}$, then $x|_{(b, b+m)} \in M_m$ for each $m = 1, 2, \ldots, p = 1, 2, \ldots$.

Here and in the sequel $f|_{(a, b)}$ denotes the restriction of the function $f$ to the interval $(a, b)$.

**Main results**

The approximation Lemma which follows represents the main tool in obtaining the new results.

**Lemma 4.** Let the spaces $X$, $X_m$, $m = 1, 2, \ldots$, the functions $\psi$, $\varphi$ and the sets $M$, $M_m$, $m = 1, 2, \ldots$ be as above. Let there exist mappings $T: M \to X$, $T_m: M_m \to X_m$, $m = 1, 2, \ldots$ with the properties

1. $x|_{(b, b + m)} = y|_{(b, b + m)} \Rightarrow T(x)|_{(b, b + m)} = T(y)|_{(b, b + m)}$ for each $x, y \in M$, $m = 1, 2, \ldots$;

2. $T_m(x|_{(b, b + m)}) = T(x)|_{(b, b + m)}$ for each $x \in M$, $M = 1, 2, \ldots$;

3. $x|_{(b, b + m)} = y|_{(b, b + m)} \Rightarrow T_{m+p}(x)|_{(b, b + m)} = T_{m+p}(y)|_{(b, b + m)}$ for each $x, y \in M_{m+p}$, $m = 1, 2, \ldots, p = 1, 2, \ldots$;

4. $T_{m+p}(x|_{(b, b + m)}) = T_{m+p}(x)|_{(b, b + m)}$ for each $x \in M_{m+p}$, $m = 1, 2, \ldots, p = 1, 2, \ldots$.

Further, let the set $S_m$ of all fixed points of the operator $T_m$ be nonempty, compact and connected in the space $X_m$. Then the set $S$ of all fixed points of the operator $T$ is nonempty, compact and connected in the space $X$.

**Proof.** Let $m_0 \geq 1$ be a fixed integer. Let

$$S_m = \{x|_{(b, b + m_0)} : x \in S_m^* \} \text{ for all } m \geq m_0.$$
Fix an arbitrary \( m \geq m_0 \). Clearly \( S_m \neq \emptyset \). Since the mapping from \( X_m \) to \( X_{m_0} \) which to each function \( x \in X_m \) assigns the restriction \( x|_{(b, b + m_0)} \) is continuous, \( S_m \) is compact and connected. Since \( m \geq m_0 \) is arbitrary, by Lemma 3 we get that

\[
P_{m_0} = \bigcap_{m=m_0}^{\infty} S_m \neq \emptyset,
\]

and it is a compact and connected set.

Denote by \( S \) the set of all fixed points of the operator \( T \). If \( x \in S \), then in view of (6)

\[
y_m = x|_{(b, b + m)} \in S_m^* \quad \text{for each } m \geq m_0
\]

and hence

\[
y = y_m|_{(b, b + m_0)} = x|_{(b, b + m_0)} \in P_{m_0}.
\]

Conversely, let \( y \in P_{m_0} \). Then for each \( m \geq m_0 \) there is a \( y_m \in S_m^* \) such that \( y_m|_{(b, b + m_0)} = y \). We shall show that there is an \( x \in S \) such that \( y = x|_{(b, b + m_0)} \).

Consider the sequence \( \{y_m\}_{m=m_0+1}^{\infty} \). As by (4) the sequence \( \{y_m|_{(b, b + m_0 + 1)}\} \subset S_{m_0+1}^* \) and the last set is compact, there exists a subsequence \( \{y_{m_r}\} \) of the sequence \( \{y_m\} \) and a point \( \tilde{y}_1 \in S_{m_0+1}^* \) such that the sequence \( \{y_{m_j}|_{(b, b + m_0 + 1)}\} \) converges uniformly to \( \tilde{y}_1^{(j)} \) on \( (b, b + m_0 + 1) \), \( j = 0, \ldots, k \). By mathematical induction we get a sequence of sequences

\[
\{y_{m_1}\}, \{y_{m_2}\}, \ldots, \{y_{m_r}\}, \ldots
\]

such that

(i) the sequence \( \{y_{m_1}\} \) is a subsequence of the sequence \( \{y_m\} \);

(ii) \( \{y_{m_{r+1}}\} \) is a subsequence of the sequence \( \{y_{m_r}\} \) for \( r = 1, 2, \ldots \);

(iii) the sequence \( \{y_{m_j}|_{(b + m_0 + r)}\} \) converges uniformly on \( (b, b + m_0 + r) \) for \( j = 0, \ldots, k \) and \( \{y_{m_r}|_{(b, b + m_0 + r)}\} \subset S_{m_0+r}^* \).

Then the diagonal sequence \( \{y_{m_u}\} \) possesses the property that \( \{y_{m_u}^{(j)}\} \) converges uniformly on each interval \( (b, b + m_0 + r) \) to \( x^{(j)} \) for \( j = 0, \ldots, k \) where \( x \in X \) is a certain function. As \( y_{m_u}|_{(b, b + m_0 + m)} \in S_{m_0+m}^* \), also \( x|_{(b + m_0 + m)} \in S_{m_0+m}^* \) and by (2), \( x \in S \).

Hence \( S \neq \emptyset \) and \( P_{m_0} \) is the set of restrictions to \( (b, b + m_0) \) of all functions belonging to \( S \), for each \( m_0 = 1, 2, \ldots \). Now we prove that \( S \) is a compact set in \( X \).

Let \( \{x_p\} \subset S \) be a sequence of points. Then by the compactness of the sets \( P_1, P_2, \ldots \) in the spaces \( X_1, X_2, \ldots \) respectively we get that there exist sequences

\[
\{x_{p,1}\}, \{x_{p,2}\}, \ldots
\]
such that

(i) \( \{x_{p,1}\} \) is a subsequence of the sequence \( \{x_p\} \);

(ii) \( \{x_{p,r+1}\} \) is a subsequence of the sequence \( \{x_{p,r}\} \) for \( r = 1, 2, \ldots \);

(iii) the sequence \( \{x_{p,r}\} \) together with its first \( k \) derivatives converges uniformly on \( (b, b + r) \).

Then the diagonal sequence \( \{x_{p,p}\} \) converges in the space \( X \) to a point \( x \in X \) with the property that \( x|_{(b, b + m)} \in S_m^* \) and by (2), \( x \in S \).

Finally, we prove that \( S \) is connected. If not, the set \( S \) can be decomposed into the union

\[ S = \hat{K}_1 \cup \hat{K}_2 \]

where \( \hat{K}_1, \hat{K}_2 \) are two non-empty, disjoint and compact sets. Let \( m \geq 1 \) be a natural number. Denote by \( \hat{K}_{1m} \) and \( \hat{K}_{2m} \) the sets of restrictions to \( (b, b + m) \) of the functions from \( \hat{K}_1 \) and \( \hat{K}_2 \), respectively. Hence we have

\[ P_m = \hat{K}_{1m} \cup \hat{K}_{2m}. \]

The compactness of \( \hat{K}_1, \hat{K}_2 \) implies that \( \hat{K}_{1m}, \hat{K}_{2m} \) are nonempty, compact sets in \( X_m \). If they were disjoint, then \( P_m \) would not be connected in \( X_m \). Hence there exist two elements \( x_m \in \hat{K}_1, y_m \in \hat{K}_2, x_m \neq y_m \) such that their restrictions to \( (b, b + m) \) coincide. Thus

\[ (7) \quad x_m|_{(b, b + m)} = y_m|_{(b, b + m)}. \]

Consider the sequences \( \{x_m\}, \{y_m\} \). As \( \{x_m\} \subset \hat{K}_1, \{y_m\} \subset \hat{K}_2 \) and \( \hat{K}_1, \hat{K}_2 \) are compact in \( X \), there exist two subsequences \( \{x_{m_1}\}, \{y_{m_1}\} \) of the sequences \( \{x_m\}, \{y_m\} \), respectively, and there exist two elements \( x \in \hat{K}_1, y \in \hat{K}_2 \) such that \( \lim_{n \to \infty} x_{m_i} = x, \lim_{l\to\infty} y_{m_i} = y \) in \( X \). Then with respect to (7) we have \( x = y \). This contradicts the fact that \( \hat{K}_1 \cap \hat{K}_2 = \emptyset \). Hence \( S \) is connected. \( \square \)

Now by means of Lemmas 1 and 2 a sufficient condition for the sets \( S_m^* \) in Lemma 4 to be non-empty, compact and connected can be given. This is the content of the next theorem.

**Theorem 1.** Suppose that all assumptions of Lemma 4 are satisfied except the assumption on the sets \( S_m^* \), \( m = 1, 2, \ldots \). Suppose, further, that for each \( m = 1, 2, \ldots \)

\[ (8) \quad T_m : M_m \subset X_m \rightarrow X_m \text{ is a compact mapping}, \]
and there exists a sequence \( \{T_{mp}\}_{p=1}^{\infty} \) of mappings

\[
T_{mp} : M_m \rightarrow X_m
\]

with the following properties: For each \( p = 1, 2, \ldots \)

(9) \( T_{mp} : M_m \subset X_m \rightarrow X_m \) is a compact mapping;

(10) \(|T_m(x)(t) - T_{mp}(x)(t)| + \ldots + |(T_m(x))^{(k)}(t) - (T_{mp}(x))^{(k)}(t)| \leq \varphi_p(t) \) for each \( x \in M_m \) and each \( t \in (b, b + m) \),

and either

(11) there exists a function \( \varphi_{*p} \in C(I_b, (0, \infty)) \) such that

\[
\varphi_{*p} + \varphi_p \leq \varphi \quad \text{in } I_b
\]

and

\[
|T_{mp}(x)(t) - \psi(0)| + \ldots + |(T_{mp}(x))^{(k)}(t) - \psi^{(k)}(0)| \leq \varphi_{*p}(t)
\]

for all \( x \in M_m \) and all \( t \in (b, b + m) \);

(12) the operator \( H_{mp} : M_m \rightarrow X_m \) which is defined by the relation

\[
H_{mp}(x) = x - T_{mp}(x) \quad \text{for all } x \in M_m
\]

is injective on \( M_m \),

or

(13) there exists an \( x_m \in M_m \) (the interior of \( M_m \)) such that

\[
T_m(x) - x_m \neq \lambda(x - x_m)
\]

for each \( x \in \partial M \) and each \( \lambda \geq 1 \);

(14) the equation

\[
H_{mp}(y) = x
\]

has at most one solution in \( M_m \) for each \( x \in X_m \) such that

\[
|x(t)| + \ldots + |x^{(k)}(t)| \leq \varphi_p(t), \quad b \leq t \leq b + m.
\]

(Here \( H_{mp} \) has the same meaning as in (12)).

Then the set \( S \) of all fixed points of the operator \( T \) is non-empty, compact and connected in the space \( X \).

Proof. With respect to Lemma 4 it suffices to show that the set \( S_m^* \) of all fixed points of the operator \( T_m \) is non-empty, compact and connected for each \( m = 1, 2, \ldots \). Hence, let \( m \geq 1 \) be an arbitrary but fixed number. Consider the case when
the assumptions (11), (12) are satisfied. Then we apply Lemma 1 to the operator $T_m$ in the space $X_m$. In this space we have two systems of balanced neighborhoods of 0:

$$U(0, \frac{1}{j}) = \{x \in X_m : p_m(x) < \frac{1}{j}\}, \quad j = 1, 2, \ldots;$$

$$U_p = \{x \in X_m : |x(t)| + \ldots + |x^{(k)}(t)| \leq \varphi_p(t), b \leq t \leq b + m\}, \quad p = 1, 2, \ldots.$$

By the Dini theorem the sequence $\{\varphi_p\}$ converges uniformly to 0 on $(b, b + m)$ and both systems of neighborhoods determine the same topology in $X_m$. For each $U_p$ there exists a compact mapping $T_{mp} : M_m \subset X_m \rightarrow X_m$ such that, in view of (10), $T_m(x) - T_{mp}(x) \in U_p$ for each $x \in M_m$.

As to the assumption (ii) in Lemma 1, by the assumption (12) it suffices to show that the equation

$$(15) \quad H_{mp}(y) = x$$

has at least one solution in $M_m$ for each $x \in U_p$. So let us fix an arbitrary $x \in U_p$. Since $M_m$ is a closed and convex set in $X_m$, the operator $T_{mp} + x : M_m \subset X_m \rightarrow X_m$ is compact and moreover

$$|T_{mp}(y)(t) - \psi(0)| + |x(t)| + \ldots + |(T_{mp}(y))^{(k)}(t) - \psi^{(k)}(0)| + |x^{(k)}(t)|$$

$$\leq \varphi_* + \varphi_p(t) \leq \varphi(t) \quad \text{for each } t \in (b, b + m),$$

which means that $T_{mp} + x : M_m \rightarrow M_m$, by the Schauder fixed point theorem the equation (15) has a solution in $M_m$ and the statement of the theorem follows.

When the assumptions (13) and (14) are fulfilled, then we use Lemma 2. We take $(X_m, p_m)$ for the real Banach space, $\hat{M}_m$ for $\Omega$ and $T_m : M_m \subset X_m \rightarrow X_m$ for the compact mapping $F$. By (13) $T_m$ satisfies the strengthened Leray-Schauder condition. When $\{T_{mp}\}_{p=1}^{\infty}$ is a sequence of compact mappings which approximates the mapping $T_m$, then by (10)

$$\delta_p = \sup\{p_m(T_{mp}(x) - T_m(x)) : x \in M_m\}$$

$$= \max\{\varphi_p(t) : b \leq t \leq b + m\} \rightarrow 0 \quad \text{for } p \rightarrow \infty.$$

Let $x \in \hat{M}_m$. Then again by (10) $T_m(x) - T_{mp}(x) \in U_p$ and then (14) implies that the assumption b) of Lemma 2 is satisfied, too. By this Lemma the theorem is true. 

\[\square\]
Theorem 1 will be applied to the initial value problem for a functional differential equation. First we consider a similar problem for an ordinary differential equation.

Let \( \omega \in C(I_b, (0, \infty)) \), let \( F \in C((0, \infty), (0, \infty)) \) be a non-decreasing function and let \( c \geq 0 \). Then one can find that a necessary and sufficient condition for the problem

\[
y'(t) = \omega(t)F(y + c), \quad y(b) = 0
\]

to have a unique solution on \((b, \infty)\) is that

\[
\int_b^\infty \omega(s) \, ds \leq \int_0^\infty \frac{dv}{F(v + c)}.
\]

Further, denote \( H = C((-h, 0), \mathbb{R}^n) \), \( \|x\| = \max\{|x(s)|: -h \leq s \leq 0\} \) for each \( x \in H \). Then \((H, \|\cdot\|)\) is a Banach space. If \( x: (b - h, \infty) \to \mathbb{R}^n \) is a continuous function, then \( x_t \in H \) is defined by \( x_t(s) = x(t + s), -h \leq s \leq 0 \), for each \( t \in I_b \). In the space \( X^* = C((-h, \infty), \mathbb{R}^n) \) let the topology be defined by the seminorms \( q_m(x) = \max\{|x(t)|: b - h \leq t \leq b + m\} \), \( m = 1, 2, \ldots, x \in X^* \). Clearly \((X^*, \{q_m\}_{m=1}^\infty)\) is a Fréchet space.

**Theorem 2.** Let \( \psi \in H \), \( f \in C(I_b \times H, \mathbb{R}^n) \). Let \( \omega \in C(I_b, (0, \infty)) \), let \( F \in C((0, \infty), (0, \infty)) \) be a nondecreasing function and

\[
\int_b^\infty \omega(s) \, ds \leq \int_0^\infty \frac{dv}{F(v + |\psi(0)|)}.
\]

Let

\[
|f(t, \chi)| \leq \omega(t)F(\|\chi\|) \quad \text{for each} \quad (t, \chi) \in I_b \times M^{**},
\]

where

\[
M^{**} = \{x_t \in H: x \in C((-h, \infty), \mathbb{R}^n), |x(t) - \psi(0)| \leq \varphi(t) \text{ for each } t \in I_b, x_b = \psi\}
\]

and \( \varphi \) is the solution of (16) on \( I_b \) with \( c = |\psi(0)| \).

Then the problem

\[
x'(t) = f(t, x_t), \quad b \leq t < \infty
\]

(19)

\[
x_b = \psi
\]

(20)
has a solution satisfying the inequality

\[ |x(t) - \psi(0)| \leq \varphi(t) \quad \text{for each } t \in I_b. \]

The set of all such solutions is compact and connected in the space \( X^* \).

**Proof.** Consider the Fréchet space \( (X, \{p_m\}_{m=1}^{\infty}) \) where \( X = C(I_b, \mathbb{R}^n) \), and the seminorms \( p_m(x) = \max\{|x(t)| : b \leq t \leq b + m\} \), \( m = 1, 2, \ldots, x \in X \). This space corresponds to the case \( k = 0 \) mentioned above. By virtue of (21) the problem (19), (20) is equivalent to the fixed point (f.p. for short) problem for the operator \( T^*: M^* \to X^* \) which is defined by

\[
T^*(x)(t) = \begin{cases} 
\psi(0) + \int_b^t f(s, x_s) \, ds, & b \leq t < \infty, \\
\psi(t-b), & b - h \leq t \leq b 
\end{cases}
\]

on the set \( M^* = \{x \in X^*: x_b = \psi \text{ and } |x(t) - \psi(0)| \leq \varphi(t) \text{ for each } t \in I_b\} \).

Let

\[
V = \{x \in X: x(b) = \psi(0)\}, \\
V^* = \{x \in X^*: x_b = \psi\}.
\]

Define the mapping \( P: V \to V^* \) by

\[
P(x)(t) = \begin{cases} 
x(t), & b \leq t < \infty, \\
\psi(t-b), & b - h \leq t \leq b 
\end{cases}
\]

for each \( x \in V \).

Then \( P \) is a bijection of \( V \) onto \( V^* \) and since \( x_p \to x \) in \( V \subseteq X \) for \( p \to \infty \) is equivalent to \( P(x_p) \to P(x) \) in \( V^* \subseteq X^* \) for \( p \to \infty \), \( P \) is a homeomorphism of \( V \) onto \( V^* \). Clearly the inverse mapping \( P^{-1} \) of \( P \) is defined by

\[
P^{-1}(x) = x|_{(b, \infty)} \quad \text{for each } x \in V^*.
\]

Let \( M = \{x \in X: |x(t) - \psi(0)| \leq \varphi(t) \text{ for each } t \in I_b \text{ and } x(b) = \psi(0)\} \). Consider now the mapping \( T = P^{-1} \circ T^* \circ P|_M \). Then \( T: M \to X \) and

\[
T(x)(t) = \psi(0) + \int_b^t f(s, x_s) \, ds, \quad b \leq t < \infty, \ x \in M, \ x_b = \psi.
\]

(In fact, the operator \( T \) should be defined by

\[
T(x)(t) = \psi(0) + \int_b^t f(s, (P(x))_s) \, ds, \quad b \leq t < \infty, \ x \in M,
\]
but it is clear what (22) means. The same notation will be used for the operators $T_p, T_m$ and $T_{mp}$, which will be defined on $M$ in a similar way.)

Clearly $u \in M$ is a f.p. of $T$ iff $P(u) \in M^*$ is a f.p. of $T^*$, and in view of the property of $P$, the set of all f.p. of $T^*$ in $M^*$ is non-empty, compact and connected in $M^*$ iff the set of all f.p. of $T$ in $M$ has the same property. Thus we can apply Theorem 1 to the operator $T$.

The set $M$ is closed in the Fréchet space $X$. Define operators $T_p : M \to X$ by

$$T_p(x)(t) = \begin{cases} 
\psi(0), & b \leq t \leq b + \frac{1}{p}, \\
\psi(0) + \int_b^{t-1/p} f(s, x_s) \, ds, & b + \frac{1}{p} \leq t < \infty, \quad x \in M, \quad x_b = \psi.
\end{cases}$$

Then (18) yields

$$|T(x)(t) - T_p(x)(t)| \leq \begin{cases} 
\int_b^t \omega(s) F(\varphi(s) + |\psi(0)|) \, ds, & b \leq t \leq b + \frac{1}{p}, \\
\int_{t-1/p}^t \omega(s) F(\varphi(s) + |\psi(0)|) \, ds, & b + \frac{1}{p} \leq t < \infty, \\
x \in M, \quad x_b = \psi.
\end{cases}$$

Denote by $\varphi_p(t)$ the right-hand side of the last inequality. Hence

$$\varphi_p(t) = \begin{cases} 
\int_b^t \omega(s) F(\varphi(s) + |\psi(0)|) \, ds, & b \leq t \leq b + \frac{1}{p}, \\
\int_{t-1/p}^t \omega(s) F(\varphi(s) + |\psi(0)|) \, ds, & b + \frac{1}{p} \leq t < \infty, \quad p = 1, 2, \ldots.
\end{cases}$$

Clearly $\{\varphi_p\}$ is a nonincreasing sequence on $I_b$ and $\lim_{p \to \infty} \varphi_p(t) = 0$ for each $t \in (b, \infty)$.

Further, when we define

$$\varphi^{*p}(t) = \begin{cases} 
0, & b \leq t \leq b + \frac{1}{p}, \\
\int_b^{t-1/p} \omega(s) F(\varphi(s) + |\psi(0)|) \, ds, & b + \frac{1}{p} \leq t < \infty, \quad p = 1, 2, \ldots
\end{cases}$$

then

$$|T_p(x)(t) - \psi(0)| \leq \varphi^{*p}(t), \quad t \in I_b, \quad p = 1, 2, \ldots, \quad x \in M, \quad x_b = \psi$$

and by (16)

$$\varphi^{*p}(t) + \varphi_p(t) = \int_b^t \omega(s) F(\varphi(s) + |\psi(0)|) \, ds = \varphi(t),$$

for each $t \in I_b$.

Further, the operators $T_m, T_{mp} : M_m \subset X_m \to X_m$ defined by

$$T_m(x)(t) = \psi(0) + \int_b^t f(s, x_s) \, ds, \quad b \leq t \leq b + m, \quad x_b = \psi,$$

$$T_{mp}(x)(t) = \begin{cases} 
\psi(0), & b \leq t \leq b + \frac{1}{p}, \\
\psi(0) + \int_b^{t-1/p} f(s, x_s) \, ds, & b + \frac{1}{p} \leq t \leq b + m
\end{cases}$$

for $m = 1, 2, \ldots, \quad p = 1, 2, \ldots$
are compact. This can be shown in the usual way.

The last step in checking the assumptions of Theorem 1 consists of proving (12). Let the mapping \( H_{mp} : M_m \to X_m \) be defined by

\[
H_{mp}(x) = x - T_{mp}(x) \quad \text{for all} \quad x \in M_m, \quad x_b = \psi, \quad m = 1, 2, \ldots, \quad p = 1, 2, \ldots
\]

Consider two elements \( x, y \in M_m, x \neq y \). Then there exists a \( t_0 : b < t_0 \leq b + m \) such that \( x(t_0) \neq y(t_0) \). Two cases may occur:

a) If \( t_0 \in (b, b + \frac{1}{p}) \), then \( H_{mp}(x)(t_0) = x(t_0) - \psi(0) \neq y(t_0) - \psi(0) = H_{mp}(y)(t_0) \);

b) there is a \( t_1 \geq b + \frac{1}{p} \) such that \( T_1 = \sup \{ \tau > b : x(t) = y(t) \text{ for } t \in (b, \tau) \} \). Then there exists a \( t_0 \in (t_1, t_1 + \frac{1}{p}) \) such that \( x(t_0) \neq y(t_0) \). This implies that

\[
T_{mp}(x)(t_0) = \psi(0) + \int_b^{t_0-1/p} f(s, x_s) \, ds = \psi(0) + \int_b^{t_0-1/p} f(s, y_s) \, ds = T_{mp}(y)(t_0)
\]

and hence \( H_{mp}(x)(t_0) \neq H_{mp}(y)(t_0) \).

In both cases the operator \( H_{mp} \) is injective on \( M_m \) and all assumptions of Theorem 1 are satisfied. By this theorem the statement of Theorem 2 follows. \( \square \)

References


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