ON COMPLEX RADON MEASURES I

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1. INTRODUCTION

If \( \Psi \) is a right continuous complex function of finite variation in \( \mathbb{R}^n \), then \( \Psi \) induces a complex Lebesgue-Stieltjes measure \( m_\Psi \) on \( \mathbb{R}^n \), whose domain is a \( \delta \)-ring \( M_\Psi \) containing all the compact subsets of \( \mathbb{R}^n \). If \( A = M_\Psi \cap \mathcal{A}(\mathbb{R}^n) \), it is well-known (e.g. vide [6], [8]) that \( M_\Psi \) (resp., \( m_\Psi \)) is the Lebesgue completion of \( A \) (resp., of \( m_\Psi | A \)) with respect to \( v(m_\Psi | A, A) \) (resp., to \( A \)). Conversely, if \( \mu \) is a complex measure on a \( \delta \)-ring \( \mathcal{D} \) containing the compact subsets of \( \mathbb{R}^n \) and \( \mathcal{D} \) is the Lebesgue completion of the \( \delta \)-ring \( A = \mathcal{D} \cap \mathcal{A}(\mathbb{R}^n) \) with respect to \( |\mu| \) and \( \{ E \in \mathcal{A}(\mathbb{R}^n) : |\mu|^*(E) < \infty \} = A \), then there is a right continuous function \( \Psi \) of finite variation on \( \mathbb{R}^n \) such that \( M_\Psi = \mathcal{D} \) and \( \mu = m_\Psi \). This result is essentially the same as Theorem 54.2 of [6], where McShane considers \( \mu \) to be real.

The object of the present paper and the succeeding one is to generalize the above mentioned results to complex Radon measures on a locally compact Hausdorff space \( X \). Since \( \mathcal{K}_0 \neq \mathcal{K} \) in general, we are led to the study of various types of regular extensions of positive and complex measures defined on \( \mathcal{D}(\mathcal{K}_0) \).

With each \( \theta \in \mathcal{K}(X)^* \) we associate canonically a unique complex measure \( \mu_\theta \) defined on a \( \delta \)-ring \( M_\theta \) containing \( \mathcal{D}(\mathcal{K}) \) and in the present paper we study the regularity properties of \( \mu_\theta \). Among other results, we show that \( M_\theta = M_{|\theta|} \) and \( \mu_{|\theta|} = v(\mu_\theta, M_\theta) \). In the succeeding paper [9] we obtain the generalization of the said theorem of [6] and give some characterizations of bounded and unbounded complex Radon measures.

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2. Preliminaries

In this section we fix the notation and terminology. Also we give some definitions and results from the literature.

$X$ denotes a locally compact Hausdorff space and $C_c(X)$ (resp., $C_c^*(X)$) is the vector space of all continuous complex (resp., real) valued functions with compact support in $X$. $\mathcal{K}_0$ (resp., $\mathcal{K}$) is the family of all compact $G_\delta$s (resp., compact subsets) of $X$, while $\mathcal{U}$ is the class of all open subsets of $X$. For a class $\mathcal{C}$ of subsets of $X$, $\mathcal{I}(\mathcal{C})$ (resp., $\mathcal{I}(\mathcal{C})$) is the $\delta$-ring (resp., $\sigma$-ring) generated by $\mathcal{C}$. We denote $\mathcal{J}(\mathcal{C})$ by $\mathcal{I}(\mathcal{C})$ and $\mathcal{I}(\mathcal{X}_0)$ by $\mathcal{I}_0(X)$. The members of $\mathcal{I}(X)$ are called the Borel subsets of $X$; those of $\mathcal{I}_0(X)$ the Baire subsets of $X$ and finally, those of $\mathcal{I}_c(X)$ are called $\sigma$-Borel (since $E \in \mathcal{I}_c(X)$ belongs to $\mathcal{I}_c(X)$ if and only if $E$ is $\sigma$-bounded).

The locally convex spaces $\mathcal{K}(X)$ and $\mathcal{K}(X, \mathbb{R})$ are as in Bourbaki [1] and $\mathcal{K}(X, \mathbb{R})^*$ (resp., $\mathcal{K}(X, \mathbb{R})^*$) is the topological dual of $\mathcal{K}(X)$ (resp., of $\mathcal{K}(X, \mathbb{R})$). For a functional $\theta \in \mathcal{K}(X)^*$, we refer to [1] for the concepts of $\text{Re}\theta$, $\text{Im}\theta$, $|\theta|$ and for those of $\theta^+$ and $\theta^-$ when $\theta$ is real in the sense of [1].

In the following proposition is given the concept of Lebesgue completion of a complex measure on a $\delta$-ring. As its proof is of routine nature (vide [3]) we omit it.

**Proposition 2.1.** Let $\nu$ be a complex measure on a $\delta$-ring $\mathcal{A}$. Let $\mathcal{A}^\ast = \{E \cup N : E \in \mathcal{A}, N \subseteq M \in \mathcal{A} \text{ with } |\nu|(M) = 0\}$, where $|\nu| = \nu(\nu, \mathcal{A})$. For such $E \cup N \in \mathcal{A}^\ast$, let $\tilde{\nu}(E \cup N) = \nu(E)$. Then $\mathcal{A}^\ast$ is a $\delta$-ring, $\mathcal{A}^\ast \supset \mathcal{A}$, $\tilde{\nu}$ is well defined, $\tilde{\nu}|\mathcal{A} = \nu$ and $\tilde{\nu}$ is a complex measure on $\mathcal{A}^\ast$. We say that $\tilde{\nu}$ (resp., $\mathcal{A}^\ast$) is the Lebesgue completion of $\nu$ (resp., of $\mathcal{A}$) relative to $\mathcal{R}$ (resp., to $\nu$).

By measure (on a ring) we mean a positive measure.

From Chapter IV of Bourbaki [1], we have the following theorem. The reader may also refer to [5], [7], [8] and [10].

**Theorem 2.2.** Let $\theta$ be a positive linear functional on $C_c(X)$. Let $M^+(X) = \{f : X \rightarrow [0, \infty], \ f \text{ lower semi-continuous}\}$ and $\mathcal{P}(X) = \{f : X \rightarrow [0, \infty]\}$. Let

$$\mu^\theta_\phi(E) = \inf_{\chi E \subseteq \phi \in M^+(X)} \sup \{\theta(\psi) : \psi \leq g, \psi \in C_c^+(X)\}.$$

for $E \subset X$. Then:

(i) $\mu^\theta_\phi$ is an outer measure on $\mathcal{P}(X)$.

(ii) Let $M^\ast_\phi = \{E \subseteq X : E \text{ is } \mu^\phi_\mu\text{-measurable}\}$. Then $M^\ast_\phi$ is a $\sigma$-algebra and contains $\mathcal{P}(X)$. We denote $\mu^\phi_\phi|_{M^\ast_\phi}$ by $\tilde{\mu}\phi$.

(iii) $\mu^\theta_\phi(K) < \infty$, $K \in \mathcal{K}$.

(iv) $\mu^\theta_\phi(E) = \inf\{\mu^\theta_\phi(U) : E \subset U \in \mathcal{U}\}$, $E \subset X$.

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(v) \( \mu_\theta^*(E) = \sup\{\mu_\theta^*(K) : K \subset E, K \in \mathcal{K}\} \), if \( \mu_\theta^*(E) \) is \( \sigma \)-finite and \( E \in M_{\mu_\theta^*} \) or if \( E \in \mathcal{U} \).

(vi) Given \( E \in M_{\mu_\theta^*} \) with \( \mu_\theta^*(E) \) \( \sigma \)-finite, there exist \( A, B \in \mathcal{B}(X) \) such that \( A \subset E \subset B, A \sigma \)-compact and \( \mu_\theta^*(B \setminus A) = 0 \). Consequently, \( \mu_\theta^*(E) = \mu_\theta^*(A) \).

(vii) Let \( \mu_\theta = \mu_\theta|\mathcal{B}(X) \). Then

\[
\theta(f) = \int_X f \, d\mu_\theta, \quad f \in C_c(X).
\]

Besides, if \( \nu \) is a measure on \( \mathcal{B}(X) \) satisfying (iii)-(v) above and if \( \theta(f) = \int_X f \, d\nu, \) \( f \in C_c(X) \), then \( \nu = \mu_\theta \).

3. Regular extensions of positive measures

We introduce several notions of regularity for positive and complex measures defined on certain \( \delta \)-rings or \( \sigma \)-rings of subsets of \( X \) and study the existence of regular extensions. These results play a key role in the next section.

**Definition 3.1.** A measure \( \mu \) on \( \mathcal{B}_0(X) \) (resp., on \( \mathcal{B}_\sigma(X) \), on \( \mathcal{B}(X) \)) is a Baire (resp., a \( \sigma \)-Borel, a Borel) measure if \( \mu(K) < \infty \) for \( k \in \mathcal{K}_0 \) (resp., for \( K \in \mathcal{K} \)).

**Definition 3.2.** Let \( \mathcal{R} \) be a ring of sets in \( X \) with \( \mathcal{D}_{\mathcal{K}_0} \subset \mathcal{R} \) or \( \mathcal{D}(\mathcal{K}) \subset \mathcal{R} \). A measure \( \mu \) defined on \( \mathcal{R} \) is said to be \( \mathcal{R} \)-regular if

(i) \( \mu(K) < \infty, K \in \mathcal{K} \cap \mathcal{R} \);

(ii) \( \mu(E) = \inf\{\mu(U) : E \subset U \in \mathcal{U} \cap \mathcal{R}\} \) for \( E \in \mathcal{R} \); and

(iii) \( \mu(E) = \sup\{\mu(C) : C \subset E, C \in \mathcal{K} \cap \mathcal{R}\} \) for \( E \in \mathcal{R} \).

A complex measure \( \nu \) on \( \mathcal{R} \) is said to be \( \mathcal{R} \)-regular if, given \( E \in \mathcal{R} \) and \( \epsilon > 0 \), there exist \( C \in \mathcal{K} \cap \mathcal{R} \) and \( U \in \mathcal{U} \cap \mathcal{R} \) such that \( C \subset E \subset U \) and \( |\nu(G)| < \epsilon \) for every \( G \in \mathcal{R} \) with \( G \subset U \setminus C \).

**Definition 3.3.** A Borel measure \( \mu \) on \( \mathcal{B}(X) \) is said to be Radon-regular if

(i) \( \mu(E) = \inf\{\mu(U) : E \subset U \in \mathcal{U}\} \) for \( E \in \mathcal{B}(X) \)

and

(ii) \( \mu(U) = \sup\{\mu(C) : C \subset U, C \in \mathcal{K}\} \), for \( U \in \mathcal{U} \).

**Proposition 3.4.**

(i) A Baire measure is \( \mathcal{B}_0(X) \)-regular.

(ii) A measure \( \mu_0 \) on \( \mathcal{D}(\mathcal{K}_0) \) with \( \mu_0(K) < \infty \) for \( K \in \mathcal{K}_0 \) is \( \mathcal{D}(\mathcal{K}_0) \)-regular.

(iii) A complex measure \( \nu_0 \) on \( \mathcal{D}(\mathcal{K}_0) \) is \( \mathcal{D}(\mathcal{K}_0) \)-regular.

(iv) If \( \mu \) is a Radon-regular measure on \( \mathcal{B}(X) \), then (ii) of Definition 3.3 holds for \( E \in \mathcal{B}(X) \) if \( \mu(E) \) is \( \sigma \)-finite.
Proof. By Proposition 13, §14 of [2], $\mathcal{X} \cap \mathcal{A}_0(X) = \mathcal{X} \cap \mathcal{D}(\mathcal{X}_0) = \mathcal{X}_0$ and hence (i) holds by the first part of Theorem 52.G of [4]. Corollary 1 on p. 347 of [2] implies (ii) and (iii) while (iv) is immediate from Theorem 10.30 of [5].

Lemma 3.5. Let $\mu$ be a Radon-regular measure on $\mathcal{A}(X)$. Then:

(i) If $\nu = \mu\mid\mathcal{D}(\mathcal{X})$, then $\nu$ is $\mathcal{D}(\mathcal{X})$-regular.

(ii) If $w = \mu\mid\mathcal{A}_c(X)$, then $w$ is $\mathcal{A}_c(X)$-regular.

Proof. Because of Proposition 3.4 (iv) it suffices to verify (ii) of Definition 3.2 for $\nu$ and $w$. If $E \in \mathcal{D}(\mathcal{X})$, then by Proposition 11, §14 of [2] and by the Radon-regularity of $\mu$, the result holds for $\nu$. If $E \in \mathcal{A}_c(X)$, then $E \subset \bigcup_{n=1}^{\infty} C_n$, $C_n \in \mathcal{X}$ and applying (i) to $E \cap C_n$ for each $n$, it can be shown that $w(E)$ satisfies (ii) of Definition 3.2.

Lemma 3.6. (i) Let $\mu_1$ and $\mu_2$ be $\mathcal{A}_c(X)$-regular measures. If

$$\int_X f \, d\mu_1 = \int_X f \, d\mu_2, \quad f \in C_c(X),$$

then $\mu_1 = \mu_2$.

(ii) If (1) holds for two Baire measures $\mu_1$ and $\mu_2$, then $\mu_1 = \mu_2$.

Proof. (i) This is immediate by Theorem 56.E of [4].

(ii) For $K \in \mathcal{X}_0$, by Theorem 55.A of [4] there exists $f_n \downarrow \chi_K$, $f_n \in C^+_c(X)$ so that by the Lebesgue dominated convergence theorem and by the hypothesis, $\mu_1(K) = \mu_2(K)$. Consequently, $\mu_1 = \mu_2$ by Proposition 3.4 (i).

Theorem 3.7. Let $\mu_0$ be a Baire measure on $X$. Then:

(i) $\mu_0$ has a unique extension $\mu$ to $\mathcal{A}_c(X)$ such that $\mu$ is a $\mathcal{A}_c(X)$-regular measure.

(ii) $\mu_0$ has a unique extension $\nu$ to $\mathcal{A}(X)$ such that $\nu$ is a Radon-regular measure.

(iii) If $\mu$ and $\nu$ are as in (i) and (ii), then $\mu = \nu\mid\mathcal{A}_c(X)$.

Proof.

(i) This is the same as Theorem 54.D of [4].

(ii) Let

$$\theta(f) = \int_X f \, d\mu_0, \quad f \in C_c(X).$$

Since $\theta$ is a positive linear functional on $C_c(X)$, by Theorem 2.2 there exists a unique Radon-regular Borel measure $\nu = \tilde{\mu}_\theta$ such that $\theta(f)$ is also given by the integral in (1) with $\mu_0$ replaced by $\nu$. If $w = \nu\mid\mathcal{A}_0(X)$, then $w$ is a Baire measure and $\theta(f)$ is also given by the integral in (1) with $\mu_0$ replaced by $w$. Thus, by Lemma 3.6
(ii) we conclude that \( w = \mu_0 \) and hence \( \nu \) is a Radon-regular extension of \( \mu_0 \). The uniqueness of \( \nu \) follows from Proposition 3.4 (iv) and Theorem 2.2.

(iii) Follows from the uniqueness part of (i) and Lemma 3.5 (ii).

\[ \square \]

**Theorem 3.8.** Every \( \mathcal{B}_c(X) \)-regular measure \( \mu \) has a unique extension \( \hat{\mu} \) to \( \mathcal{B}(X) \) as a Radon-regular measure.

**Proof.** Apply the proof of Theorem 3.7 (ii) considering

\[ \theta(f) = \int_X f \, d\mu, \quad f \in C_c(X) \]

and using Lemma 3.6 (i) in the place of Lemma 3.6 (ii).

\[ \square \]

**Theorem 3.9.** Let \( \mu \) be a \( \mathcal{D}(X) \)-regular measure. Then:

(i) The unique extension \( \hat{\mu} \) of \( \mu \) to \( \mathcal{B}_c(X) \) as a measure is \( \mathcal{B}_c(X) \)-regular.

(ii) \( \mu \) admits a unique extension \( w \) to \( \mathcal{B}(X) \) as a Radon-regular measure and \( w|\mathcal{B}_c(X) = \hat{\mu} \), where \( \hat{\mu} \) is as in (i).

**Proof.**

(i) Since \( \mu \) is finite on \( \mathcal{D}(X) \), \( \mu \) admits a unique extension \( \hat{\mu} \) to \( \mathcal{I}(\mathcal{D}(X)) = \mathcal{B}_c(X) \) as a measure. Clearly, \( \hat{\mu}(K) < \infty \) for \( K \in \mathcal{K} \). Let \( E \in \mathcal{B}_c(X) \). Then \( E \subseteq \bigcup_{n=1}^{\infty} C_n, C_n \in \mathcal{K} \). Since \( E \cap C_n \in \mathcal{D}(X) \) and \( \mu \) is \( \mathcal{D}(X) \)-regular, given \( \epsilon > 0 \), there exists \( U_n \in \mathcal{U} \cap \mathcal{D}(X) \) such that \( E \cap C_n \subseteq U_n \) and \( \mu(U_n) - \mu(E \cap C_n) < \epsilon/2^n \).

If \( U = \bigcup_{n=1}^{\infty} U_n \), then \( U \in \mathcal{U} \cap \mathcal{B}_c(X) \) and \( E \subseteq U \). If \( \hat{\mu}(E) = \infty \), then \( \hat{\mu}(U) = \infty \).

If \( \hat{\mu}(E) < \infty \), then \( \hat{\mu}(U) - \hat{\mu}(E) \leq \sum_{n=1}^{\infty} (\mu(U_n) - \mu(E \cap C_n)) < \epsilon \). Thus \( \hat{\mu} \) satisfies (ii) of Definition 3.2 with \( \mathcal{B} = \mathcal{B}_c(X) \). Again, letting \( E_n = \bigcup_{k=1}^{n} (E \cap C_k) \) we have \( E_n \in \mathcal{D}(X) \) and \( E_n \uparrow E \) so that

\[ \hat{\mu}(E) = \sup_n \mu(E_n) = \sup_n \sup \{ \mu(C) : C \subseteq E_n, C \in \mathcal{K} \} \]

\[ \leq \sup \{ \mu(C) : C \subseteq E, C \in \mathcal{K} \} \]

\[ \leq \hat{\mu}(E) \]

and hence \( \hat{\mu} \) is \( \mathcal{B}_c(X) \)-regular.

(ii) Taking \( \hat{\mu} \) as in (i) and applying Theorem 3.8 to \( \hat{\mu} \), we obtain a Radon-regular extension \( w \) of \( \hat{\mu} \) and, hence of \( \mu \), to \( \mathcal{B}(X) \). Besides, \( w \) is unique by the uniqueness part of Theorem 3.8.

\[ \square \]
With each $\theta \in \mathcal{K}(X)^*$ we associate a unique complex measure $\mu_\theta$ (vide Definition 4.3) defined canonically on a $\delta$-ring $M_\theta$ which contains $\mathcal{D}(\mathcal{K})$ and call $\mu_\theta$ the complex Radon measure induced by $\theta$. The object of this section is to study the properties of $\mu_\theta$ and $M_\theta$ when $\theta$ is real and when is arbitrary.

We use the notation of $\mu_\theta^+, \mu_\theta^- \text{ and } \tilde{\mu}_\theta$ as in Theorem 2.2. We denote $\mu_\theta^+|\mu_\theta^-$ by $\tilde{\mu}_\theta$ and call $\tilde{\mu}_\theta$ (resp., $\tilde{\mu}_\theta$) the Radon (resp., the Borel-Radon) measure induced by the positive linear functional $\theta$ on $C_c(X)$.

**Theorem 4.1.** A measure $\mu$ on $\mathcal{B}(X)$ is Borel-Radon if and only if $\mu$ is Radon-regular.

**Proof.** By Theorem 2.2 the condition is necessary. Conversely, if $\mu$ is Radon-regular, let

$$\theta(f) = \int_X f \, d\mu, \quad f \in C_c(X).$$

Then $\theta$ is a positive linear functional and by the uniqueness part of Theorem 2.2 (vii) and by Proposition 3.4 (iv) we conclude that $\mu = \tilde{\mu}_\theta$. $\square$

**Proposition 4.2.** Let $\theta \in \mathcal{K}(X)^*$ and let $\theta_1 = \text{Re} \, \theta; \theta_2 = \text{Im} \, \theta$. Then:

(i) If $M_\theta = \{A \subseteq X : \tilde{\mu}_\theta^+(A) < \infty, \tilde{\mu}_\theta^-(A) < \infty, j = 1, 2\}$, then $M_\theta$ is a $\delta$-ring and contains $\mathcal{D}(\mathcal{K})$.

(ii) Let $M(\theta) = \mu_{\theta_1}^+ \cap \mu_{\theta_2}^+ \cap \mu_{\theta_1}^- \cap \mu_{\theta_2}^-$. Then $M(\theta)$ is a $\sigma$-algebra and contains $\mathcal{B}(X)$. The members of $M(\theta)$ are called $\theta$-measurable sets.

(iii) Let

$$\mu_\theta(E) = \{ (\tilde{\mu}_\theta^+ - \tilde{\mu}_\theta^-) + i(\tilde{\mu}_\theta^+ - \tilde{\mu}_\theta^-) \}(E), \quad E \in M_\theta.$$  

Then $\mu_\theta$ is a complex measure on $M_\theta$.

(iv) If $\theta$ is real, then $\mu_\theta$ is real.

(v) If $\theta$ is positive, then $\mu_\theta$ is positive and finite.

**Proof.** This is immediate from Theorem 2.2. $\square$

**Definition 4.3.** Let $\theta \in \mathcal{K}(X)^*$. The complex measure $\mu_\theta$ in Proposition 4.2 (iii) is called the **complex Radon measure induced by $\theta$** and the $\delta$-ring $M_\theta$ is called the **domain of $\mu_\theta$**.

**Notation 4.4.** Let $\mu_\theta$ be the complex Radon measure induced by $\theta, \theta \in \mathcal{K}(X)^*$. We denote $\nu(\mu_\theta, M_\theta)$ by $|\mu_\theta|$. If $\nu$ is a real measure on a $\delta$-ring $\mathcal{R}_2$ and $\mathcal{R}_1$ is another $\delta$-ring such that $\mathcal{R}_2 \supseteq \mathcal{R}_1$, then

$$(\nu, \mathcal{R}_1, \mathcal{R}_2)^+(E) = \sup \{ \nu(F) : F \subseteq E, \ F \in \mathcal{R}_1 \}$$
and
\[
(\nu, \mathcal{A}_1, \mathcal{A}_2)^-(E) = -\inf \{ \nu(F) : F \subset E, \ F \in \mathcal{A}_1 \}
\]
for \( E \in \mathcal{A}_2 \). If \( \nu \) is real or complex on \( \mathcal{A}_2 \), then
\[
v(\nu, \mathcal{A}_1, \mathcal{A}_2)(E) = \sup \left\{ \sum_{i=1}^{n} |\nu(E_i)| : \{E_i\}_1^n \subset \mathcal{A}_1, \ E_i \cap E_j = \emptyset, \ i \neq j, \ \text{and} \ \bigcup_{1}^{n} E_i \subset E \right\}
\]
for \( E \in \mathcal{A}_2 \).

**Theorem 4.5.** Let \( \theta \in \mathcal{K}(X)^*, \ \theta \) real. Then the following assertions hold:

(i) \( \mu_\theta \) is \( M_\theta \)-regular.

(ii) \( \mu_\theta \) is of finite variation on \( M_\theta \).

(iii) \( \mu_\theta^+ \leq \mu_\theta^+ \) and \( \mu_\theta^- \leq \mu_\theta^- \) in \( M_\theta \), where \( \mu_\theta = \mu_\theta^+ - \mu_\theta^- \) is the Jordan decomposition of \( \mu_\theta \) in \( M_\theta \).

(iv) \( |\mu_\theta|, \ |\mu_\theta^+| \) and \( |\mu_\theta^-| \) are \( M_\theta \)-regular.

(v) \( \mu_\theta^+ |\mathcal{D}(\mathcal{X}) = (\mu_\theta |\mathcal{D}(\mathcal{X}))^+; \mu_\theta^- |\mathcal{D}(\mathcal{X}) = (\mu_\theta |\mathcal{D}(\mathcal{X}))^- \) and \( v(\mu_\theta |\mathcal{D}(\mathcal{X}), \ \mathcal{D}(\mathcal{X})) = |\mu_\theta| |\mathcal{D}(\mathcal{X})| \).

(vi) \( |\mu_\theta||\mathcal{D}(\mathcal{X}), \ |\mu_\theta^+| |\mathcal{D}(\mathcal{X}), \ |\mu_\theta^-| |\mathcal{D}(\mathcal{X}) \) and \( \mu_\theta |\mathcal{D}(\mathcal{X}) \) are \( \mathcal{D}(\mathcal{X}) \)-regular.

(vii) (a) \( (\mu_\theta, \mathcal{D}(\mathcal{X}), M_\theta)^+(E) = \mu_\theta^+(E), \ E \in M_\theta \).

(b) \( (\mu_\theta, \mathcal{D}(\mathcal{X}), M_\theta)^-(E) = \mu_\theta^-(E), \ E \in M_\theta \).

(c) \( v(\mu_\theta, \mathcal{D}(\mathcal{X}), M_\theta)(E) = |\mu_\theta||(E), \ E \in M_\theta \).

(viii) Given \( E \in M_\theta \), there exist \( A, B \) in \( \mathcal{B}(X) \cap M_\theta \) such that \( A \subset E \subset B \) and \( \mu_\theta^+(B \setminus A) = \mu_\theta^-(B \setminus A) = 0 \). Then \( \mu_\theta^+(E) = \mu_\theta^+(A), \mu_\theta^-(E) = \mu_\theta^-(A) \), and \( \mu_\theta \) (resp., \( M_\theta \)) is the Lebesgue completion of \( \mu_\theta |\mathcal{A} \) (resp., of \( \mathcal{A} \)) with respect to \( \mathcal{A} \) (resp., to \( \mu_\theta |\mathcal{A} \)), where \( \mathcal{A} = \mathcal{B}(X) \cap M_\theta \).

**Proof.** (i) Since \( \mu_\theta^+ \) and \( \mu_\theta^- \) are \( M_\theta \)-regular by Theorem 2.2, the assertion (i) holds.

(ii) This is immediate from the fact that \( M_\theta \) is a \( \delta \)-ring. (Vide Corollary 2 on p. 48 of [2]).

(iii) Since \( \mu_\theta = \mu_\theta^+ - \mu_\theta^- \) in \( M_\theta \) and \( M_\theta \) is a \( \delta \)-ring, (iii) follows from the proof of Proposition 16, §3 of [2].

(iv) Since \( \mu_\theta^+ \) and \( \mu_\theta^- \) are \( M_\theta \)-regular by Theorem 2.2, the result follows from (iii).

(v) As in the proof of (iii) we have \( (\mu_\theta |\mathcal{D}(\mathcal{X}))^+ \leq \mu_\theta^+ |\mathcal{D}(\mathcal{X}) \). Let \( E \in \mathcal{D}(\mathcal{X}) \) and \( \varepsilon > 0 \). By (iv), given \( F \in M_\theta \), there exists \( \mathcal{F}_F \in \mathcal{K} \) such that \( \mathcal{F}_F \subset F \) and

\[
\forall x \in \text{supp}(\mu_\theta) \quad \exists G \in \mathcal{F}_F \quad \text{s.t.} \quad G \subset E \quad \text{and} \quad |(\mu_\theta |\mathcal{D}(\mathcal{X}))^+(E) - (\mu_\theta |\mathcal{D}(\mathcal{X}))^+(G)| < \varepsilon.
\]
\[ |\mu_\theta|(F \setminus C_F) < \varepsilon. \] Consequently, we have
\[
(\mu_\theta|\mathcal{D}(\mathcal{X}))^+(E) = \sup \{\mu_\theta(F): F \subset E, F \in \mathcal{D}(\mathcal{X})\}
\geq \sup \{\mu_\theta(C_F): F \subset E, F \in \mathcal{M}_\theta\}
\geq \sup \{\mu_\theta(F) - \varepsilon: F \subset E, F \in \mathcal{M}_\theta\}
= \mu_\theta^+(E) - \varepsilon.
\]
Thus \((\mu_\theta|\mathcal{D}(\mathcal{X}))^+ \geq \mu_\theta^+|\mathcal{D}(\mathcal{X})\) so that \((\mu_\theta|\mathcal{D}(\mathcal{X}))^+ = \mu_\theta^+|\mathcal{D}(\mathcal{X})\). Similarly, it follows that \((\mu_\theta|\mathcal{D}(\mathcal{X}))^- = \mu_\theta^-|\mathcal{D}(\mathcal{X})\) and consequently,
\[
|\mu_\theta|\mathcal{D}(\mathcal{X}) = (\mu_\theta^++\mu_\theta^-)|\mathcal{D}(\mathcal{X}) = v(\mu_\theta|\mathcal{D}(\mathcal{X}),\mathcal{D}(\mathcal{X})).
\]

(vi) Let \(E \in \mathcal{D}(\mathcal{X})\) and \(\varepsilon > 0\). By (iv) there exist \(U \in \mathcal{U} \cap \mathcal{M}_\theta\) and \(K \in \mathcal{X}\) such that \(K \subset E \subset U\) and \(|\mu_\theta|(U \setminus K) < \varepsilon\). As \(\overline{E} \in \mathcal{X}\), by Proposition 11, §14 of [2] there exists \(U_0 \in \mathcal{U} \cap \mathcal{D}(\mathcal{X})\) such that \(\overline{E} \subset U_0\). If \(V = U \cup U_0\), then \(E \subset V, V \in \mathcal{U} \cap \mathcal{D}(\mathcal{X})\) and \(|\mu_\theta|(V \setminus K) < \varepsilon\). Thus \(|\mu_\theta|\mathcal{D}(\mathcal{X})\) is \(\mathcal{D}(\mathcal{X})\)-regular and consequently, (vi) holds.

(vii) Let \(E \in \mathcal{M}_\theta\). By (iv) and (v) we have
\[
\mu_\theta^+(E) = \sup \{\mu_\theta^+(C): C \subset E, C \in \mathcal{X}\}
= \sup \sup \{\mu_\theta(F): F \subset C, F \in \mathcal{D}(\mathcal{X})\}
\leq \sup \{\mu_\theta(F): F \subset E, F \in \mathcal{D}(\mathcal{X})\}
\leq \sup \{\mu_\theta(F): F \subset E, F \in \mathcal{M}_\theta\}
= \mu_\theta^+(E)
\]
and thus (a) holds. Similarly, (b) is proved.

For \(E \in \mathcal{M}_\theta\), given \(\varepsilon > 0\), there exists a partition \(\{E_i\}_1^n\) of \(E\) in \(\mathcal{M}_\theta\) such that
\[
\sum_{i=1}^n |\mu_\theta(E_i)| > |\mu_\theta|(E) - \frac{1}{2}\varepsilon.
\]
By (i), for each \(i\) there exists \(C_i \in \mathcal{X}\) such that \(C_i \subset E_i\) and \(|\mu_\theta(G)| < \frac{\varepsilon}{2n}\) for \(G \in \mathcal{M}_\theta\) with \(G \subset E_i \setminus C_i\). Then \(\sum_{i=1}^n |\mu_\theta(C_i)| \geq \sum_{i=1}^n |\mu_\theta(E_i)| - \varepsilon/2 > |\mu_\theta|(E) - \varepsilon\).
Thus \(v(\mu_\theta, \mathcal{D}(\mathcal{X}), \mathcal{M}_\theta)(E) \geq |\mu_\theta|(E)\).

Since the reverse inequality is obvious, (c) holds.

(viii) Since \(E \in \mathcal{M}_\theta \subset M_\theta^+ \cap M_\theta^+\) and \(\mu_\theta^+(E)\) and \(\mu_\theta^-(E)\) are finite, by Theorem 2.2 (vi) there exist \(A, B \in \mathcal{A}(\mathcal{X})\) with \(A \subset E \subset B\) and \(\mu_\theta^+(B \setminus A) = \mu_\theta^-(B \setminus A) = 0\). Now, by (vii) (c), for \(F \in \mathcal{A}\) we have
\[
|\mu_\theta|(F) = v(\mu_\theta, \mathcal{D}(\mathcal{X}), \mathcal{M}_\theta)(F) \leq v(\mu_\theta, \mathcal{A}, \mathcal{A})(F) \leq v(\mu_\theta, \mathcal{M}_\theta)(F) = |\mu_\theta|(F)
\]
and hence from (iii) it follows that

\[ v(\mu_\theta | D, D)(B \setminus A) = |\mu_\theta|(B \setminus A) \leq (\mu_\theta^+ + \mu_\theta^-)(B \setminus A) = 0. \]

This proves (viii). □

**Theorem 4.6.** Let \( \theta \in \mathcal{K}(X)^* \), \( \theta \) real. Let \( \mu_\theta^+ \) and \( \mu_\theta^- \) be as in Theorem 4.5 (iii). Then:

(i) \( \mu_\theta^+ | D(X) \) and \( \mu_\theta^- | D(X) \) admit unique extensions \( \tilde{\mu}_\theta^+ \) and \( \tilde{\mu}_\theta^- \), respectively, to \( D(X) \) as Radon-regular measures and \( \tilde{\mu}_\theta^+(E) = \mu_\theta^+(E), \tilde{\mu}_\theta^-(E) = \mu_\theta^-(E) \) for \( E \in \mathcal{B}_c(X) \cap M_\theta \).

(ii) If \( \nu = |\mu_\theta||D(X) \), then \( \nu \) has a unique extension \( \nu^* \) to \( D(X) \) as a Radon-regular measure and \( \nu^*(E) = |\mu_\theta|(E) \) for \( E \in B_c(X) \cap M_\theta \).

(iii) There exists a positive linear functional \( \Psi \) on \( C_c(X) \) such that \( \nu = \tilde{\mu}_\Psi \), where \( \nu^* \) is as in (ii).

(iv) \( |\theta| \leq \Psi \), where \( \Psi \) is as in (iii).

(v) \( \mu_\theta^+ = \mu_\theta^+ \) and \( \mu_\theta^- = \mu_\theta^- \) in \( M_\theta \).

(vi) \( M_\theta = M_{|\theta|} \) and \( |\mu_\theta| = |\mu_{|\theta|}| \).

**Proof.** (i) By Theorems 4.5 (vi) and 3.9 (ii), \( \mu_\theta^+ | D(X) \) has a unique extension \( \tilde{\mu}_\theta^+ \) to \( D(X) \) as a Radon-regular measure. Then by Lemma 3.5 (ii), \( \tilde{\mu}_\theta^+ | D_c(X) \) is \( D_c(X) \)-regular and hence by Theorem 4.5 we have

\[ \tilde{\mu}_\theta^+(E) = \sup \{ \tilde{\mu}_\theta^+(C) : C \subseteq E, C \in \mathcal{K} \} \]

\[ = \sup \{ \mu_\theta^+(C) : C \subseteq E, C \in \mathcal{K} \} \]

\[ = \mu_\theta^+(E) \]

for \( E \in D_c(X) \cap M_\theta \). Similarly, the other result holds.

(ii) By Theorems 4.5 (vi) and 3.9 (ii) such an extension \( \nu^* \) of \( |\mu_\theta||D(X) \) exists uniquely. An argument similar to that in the proof of (i) shows that \( \nu^*(E) = |\mu_\theta|(E) \) for \( E \in D_c(X) \cap M_\theta \).

(iii) This is immediate from Theorem 4.1.

(iv) Let \( f \in C_c(X) \) with \( \text{supp} f = K \). Then \( B(K) \subseteq D(X) \). As \( \theta(f) = \theta^+(f) - \theta^-(f) \), by Theorem 2.2 we have

\[ \theta(f) = \int_X f d\tilde{\mu}_\theta^+ - \int_X f d\tilde{\mu}_\theta^- = \int_K f d(\mu_\theta | D(K)) \]

since \( f(x) = 0 \) for \( x \in X \setminus K \), \( B(K) = B(X) \cap K \) and \( \mu_\theta \) is a real measure on \( B(K) \). Thus,

\[ |\theta(f)| \leq \int_K |f| d\nu(\mu_\theta | D(K), D(K)) \]

\[ \leq \int_X |f| d\nu^* = \int_X |f| d\tilde{\mu}_\Psi = \Psi(||f||) \]

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since \( v(\mu_\theta | \mathcal{B}(K), \mathcal{B}(K)) \leq v(\mu_\theta | \mathcal{D}(\mathcal{X}), \mathcal{D}(\mathcal{X})) = |\mu_\theta| |\mathcal{D}(\mathcal{X})| \) by Theorem 4.5 (v). Thus \(|\theta| \leq \Psi\).

(v) By (iv) and by Proposition 15, §1, Chapter IV of [1]

\[
\mu_{|\theta|} \leq \mu_{\Psi}.
\]

By the same proposition of [1] we have

(2)

\[
\mu_{|\theta|} = \mu_{\theta^+} + \mu_{\theta^-} \text{ in } \mathcal{B}(\mathcal{X}) \cap M_{\theta}.
\]

Consequently, from Theorem 4.5 (iii) and from (ii), (1) and (2) it follows that

\[
|\mu_{\theta}|(E) = (\mu_{\theta^+} + \mu_{\theta^-})(E) \leq (\mu_{\theta^+} + \mu_{\theta^-})(E) = \mu_{|\theta|}(E) \leq \mu_{\Psi}(E) = |\mu_{\theta}|(E)
\]

for \( E \in \mathcal{B}(\mathcal{X}) \cap M_{\theta} \). Thus

(3)

\[
|\mu_{\theta}|(E) = \mu_{|\theta|}(E), \quad E \in \mathcal{B}(\mathcal{X}) \cap M_{\theta}.
\]

Since \( \mu_{\theta^+} \leq \mu_{\theta^+} \) and \( \mu_{\theta^-} \leq \mu_{\theta^-} \) by Theorem 4.5 (iii), we conclude from (2) that

(4)

\[
\mu_{\theta^+}(E) = \mu_{\theta^+}(E) \quad \text{and} \quad \mu_{\theta^-}(E) = \mu_{\theta^-}(E)
\]

for \( E \in \mathcal{B}(\mathcal{X}) \cap M_{\theta} \).

Now, let \( E \in M_{\theta} \). By Theorem 4.5 (viii) there exist \( A, B \in \mathcal{B}(\mathcal{X}) \cap M_{\theta} \) such that \( A \subseteq E \subseteq B \) with \( \mu_{\theta^+}(B \setminus A) = \mu_{\theta^-}(B \setminus A) = 0 \). Since \( E \in M_{\theta} \), by Theorem 2.2 (vi) we can assume \( A \) to be \( \sigma \)-compact so that \( A \in \mathcal{B}(\mathcal{X}) \cap M_{\theta} \). Then, by Proposition 15, §1, Chapter IV of [1] we have \( \mu_{|\theta|}(B \setminus A) = (\mu_{\theta^+} + \mu_{\theta^-})(B \setminus A) = 0 \) and hence \( \mu_{|\theta|}(B \setminus A) = 0 \). Consequently, \( E \setminus A \in M_{|\theta|} \). Besides, by the same proposition of [1] we have \( \mu_{|\theta|}(A) = (\mu_{\theta^+} + \mu_{\theta^-})(A) < \infty \) and as \( A \in \mathcal{B}(\mathcal{X}) \), we conclude that \( A \in M_{|\theta|} \). Therefore, \( E = A \cup (E \setminus A) \in M_{|\theta|} \) and thus we have shown that

(5)

\[
M_{\theta} \subseteq M_{|\theta|}.
\]

Besides, by (4)

\[
\mu_{|\theta|}(E) = \mu_{|\theta|}(A) = (\mu_{\theta^+} + \mu_{\theta^-})(A) = (\mu_{\theta^+} + \mu_{\theta^-})(A) = (\mu_{\theta^+} + \mu_{\theta^-})(E) = |\mu_{\theta}|(E)
\]

so that by (5) we have

(6)

\[
\mu_{|\theta|} |M_{\theta}| = |\mu_{\theta}|.
\]

Also, from (4) it follows that \( \mu_{\theta^+}(E) = \mu_{\theta^+}(A) = \mu_{\theta^+}(A) = \mu_{\theta^+}(E); \mu_{\theta^-}(E) = \mu_{\theta^-}(A) = \mu_{\theta^-}(E) \) and hence \( \mu_{\theta^+} = \mu_{\theta^+} \) and \( \mu_{\theta^-} = \mu_{\theta^-} \) in \( M_{\theta} \).
In view of (5) and (6), it suffices to show that $M_{|\theta|} \subset M_{\theta}$. Let $E \in M_{|\theta|}$. Then by Theorem 2.2 (vi) there exist $A, B \in A_c(X) \cap M_{|\theta|}$ such that $A \subset E \subset B$, $\mu_{|\theta|}(B \setminus A) = 0$ and $\mu_{|\theta|}(E) = \mu_{|\theta|}(A)$. Since $\hat{\mu}_{\theta\pm} \leq \hat{\mu}_{|\theta|}$ on $A_c(X)$ by proposition 15, §1, Chapter IV of [1], it follows that $A \in A_c(X) \cap M_{\theta}$. Besides, as $\mu_{|\theta|}(B \setminus A) = 0$, by the same proposition of [1], $\hat{\mu}_{\theta^+}(B \setminus A) = \hat{\mu}_{\theta^-}(B \setminus A) = 0$ so that $\mu_{\theta^+}(E \setminus A) = \mu_{\theta^-}(E \setminus A) = 0$. Thus $E \setminus A \in M_{\theta}$ and hence $E \in M_{\theta}$.

Now we pass on to the study of the properties of $\mu_{\theta}$ when $\theta$ is arbitrary in $\mathcal{K}(X)^\ast$.

**Theorem 4.7.** Let $\theta \in \mathcal{K}(X)^\ast$. Then the following assertions hold:

(i) $\mu_{\theta}$ is $M_{\theta}$-regular.

(ii) $\mu_{\theta}$ is of finite variation in $M_{\theta}$.

(iii) $|\mu_{\theta}|$ is $M_{\theta}$-regular.

(iv) $\nu(\mu_{\theta}|A_c(X), A_c(X)) = |\mu_{\theta}|A_c(X)$.

(v) $|\mu_{\theta}|(A_c(X))$ is $A_c(X)$-regular and hence $\nu(\mu_{\theta}|A_c(X))$ is $A_c(X)$-regular.

(vi) $|\mu_{\theta}|(A_c(X), M_{\theta}) = |\mu_{\theta}|$.

(vii) Given $E \in M_{\theta}$, there exist $A, B \in A_c(X) \cap M_{\theta}$ such that $A \subset E \subset B$ and $|\mu_{\theta}|(B \setminus A) = 0$. Consequently, $\mu_{\theta}(E) = \mu_{\theta}(A)$ and $\mu_{\theta}$ (resp., $M_{\theta}$) is the Lebesgue completion of $\mu_{\theta}|A_c(X)$ (resp., of $A_c(X)$) with respect to $A_c(X)$ (resp., to $\mu_{\theta}|A_c(X)$), where $\mathcal{A} = A_c(X) \cap M_{\theta}$.

(viii) If $\nu = |\mu_{\theta}|A_c(X)$, then $\nu$ has a unique extension $\hat{\nu}$ to $A_c(X)$ as a Radon-regular measure and $\hat{\nu}(E) = |\mu_{\theta}|(E)$ for $E \in A_c(X) \cap M_{\theta}$.

(ix) There exists a positive linear functional $\psi$ on $C_c(X)$ such that $\hat{\nu} = \hat{\mu}_\psi$, where $\nu$ is as in (viii). Besides $|\theta| \leq \psi$.

**Proof.** Let $\theta_1 = \Re \theta$ and $\theta_2 = \Im \theta$.

(i) As $M_{\theta} = M_{\theta_1} \cap M_{\theta_2}$ and $U_1 \cap U_2 \in M_{\theta}$ for $U_i \in \mathcal{U} \cap M_{\theta_i}$, $i = 1, 2$, (i) is immediate from Theorem 4.5 (i).

(ii) The proof is similar to that of Theorem 4.5 (ii).

(iii) This is an immediate consequence of (i) and the inequality

$$|\mu_{\theta}|(E) \leq \sup \{|\mu_{\theta}(F)| : F \subset E, F \in M_{\theta}\}.$$  

(vi) The proof of Theorem 4.5 (vii) (c) holds here verbatim.

(iv) This is immediate from (vi).

(v) The proof of the $A_c(X)$-regularity of $|\mu_{\theta}|A_c(X)$ in Theorem 4.5 (vi) holds here verbatim.

(vii) Let $E \in M_{\theta}$. Since $\mu_{\theta^+}(E) < \infty$ and $\mu_{\theta^-}(E) < \infty$ for $j = 1, 2$, by Theorem 2.2 there exist $A, B \in A_c(X) \cap M_{\theta}$ such that $A \subset E \subset B$, $A \sigma$-compact and $\mu_{\theta_j^+}(B \setminus A) = \mu_{\theta_j^-}(B \setminus A) = 0$ for $j = 1, 2$. Thus $|\mu_{\theta}|(B \setminus A) = 0$. As by (vi) $|\mu_{\theta}|(F) = \nu(\mu_{\theta}|A_c(X), A_c(X))(F)$ for $F \in A_c(X)$, the result holds.
(viii) The proof is the same as that of Theorem 4.6 (ii).
(ix) The first part is due to Theorem 4.1. As in the proof of Theorem 4.6 (iv) we have

\[ |\theta(f)| = \left| \int_K f \, d(\mu_1, |\mathcal{H}(K)|) + i \int_K f \, d(\mu_2, |\mathcal{H}(K)|) \right| \]
\[ \leq \int_K |f| \, d\nu(\mu_0, |\mathcal{H}(K)|, \mathcal{H}(K)) \]
\[ \leq \int_X |f| \, d\tilde{\nu} = \int_X |f| \, d\hat{\mu}_\psi = \psi(|f|) \]

for \( f \in C_c(X) \) with \( \text{supp} \, f = K \) since \( v(\mu_0, |\mathcal{H}(K)|, \mathcal{H}(K)) \leq v(\mu_0, |\mathcal{D}(X)|, \mathcal{D}(X)) = |\mu_0||\mathcal{D}(X)| \) by (iv). Therefore, \( |\theta| \leq \psi \).

**Lemma 4.8.** If \( \theta \in \mathcal{H}(X)^* \), then \( |\mu_0(K)| \leq |\mu_{|\theta|}(K), K \in \mathcal{H} \).

**Proof.** Let \( \theta_1 = \text{Re} \theta \) and \( \theta_2 = \text{Im} \theta \). Then, by Theorem 2.2, \( \mu_{\theta_j}^+ |M_\theta \) and \( \mu_{\theta_j}^- |M_\theta \) are \( M_\theta \)-regular for \( j = 1, 2 \). Therefore, given \( n \in \mathbb{N} \), there exists \( U_n \in \mathcal{U} \cap M_\theta \) such that \( K \subset U_n \), \( \mu_{\theta_j}^+(U_n \setminus K) < \frac{1}{n} \), \( \mu_{\theta_j}^-(U_n \setminus K) < \frac{1}{n} \) for \( j = 1, 2 \). Let \( W_n = \bigcap_{k=1}^n U_k \) and \( W = \bigcap_{k=1}^\infty W_n \). Then \( W \in M_\theta \), as \( M_\theta \) is a \( \delta \)-ring. Besides, \( K \subset W \) and \( \mu_{\theta_j}^+(W \setminus K) = \mu_{\theta_j}^-(W \setminus K) = 0 \) for \( j = 1, 2 \). By Urysohn’s lemma there exists \( f_n \in C_c^+(X) \) with \( \chi_K \leq f_n \leq \chi W_n \). If \( g_n = \bigwedge f_i \), then \( g_n \downarrow \chi_K \) a.e. with respect to \( \mu_{\theta_j}^+ \) and \( \mu_{\theta_j}^- \) for \( j = 1, 2 \). If \( K_1 = \text{supp} \, g_1 \), then by the Lebesgue dominated convergence Theorem we have \( \lim_n \int_X g_n \, d\hat{\mu}_{\theta_j}^\pm = \lim_n \int_{K_1} g_n \, d\hat{\mu}_{\theta_j}^\pm = \mu_{\theta_j}^\pm(K) \) and \( \lim_n \int_X g_n \, d\hat{\mu}_{\theta_j}^- = \lim_n \int_{K_1} g_n \, d\hat{\mu}_{\theta_j}^- = \mu_{\theta_j}^-(K) \) for \( j = 1, 2 \), where we consider the restriction \( \hat{\mu}_{\theta_j}^\pm |\mathcal{H}(K_1) = \mu_{\theta_j}^\pm |\mathcal{H}(K_1) \) and \( \hat{\mu}_{\theta_j}^- |\mathcal{H}(K_1) = \mu_{\theta_j}^- |\mathcal{H}(K_1) \). Then as \( g_n \to \chi_K \mu_{|\theta|} \)-a.e., by the same theorem

\[ |\mu_\theta(K)| = \left| \lim_n \int_{K_1} g_n \, d\mu_\theta \right| = |\lim_n \theta(g_n)| \leq |\lim_n \theta|(g_n) \]
\[ = \lim_n \int_X g_n \, d\mu_{|\theta|} = |\mu_{|\theta|}(K)|. \]

**Lemma 4.9.** For \( \theta \in \mathcal{H}(X)^* \) and \( A \in \mathcal{H}(X) \cap M_\theta \), \( |\mu_\theta|(A) = 0 \) if and only if \( |\mu_{|\theta|}|(A) = 0 \). Consequently, \( M_\theta \subset M_{|\theta|} \).
Proof. Suppose \( \mu_{|\theta|}(A) = 0 \). Let \( \theta_1 = \text{Re} \theta \) and \( \theta_2 = \text{Im} \theta \). Thus, by Theorem 4.6 (vi) and by proposition 15, §1, Chapter IV of [1] we have

\[
\mu_{|\theta|}(A) \leq (\mu_{\theta_1} + \mu_{\theta_1})(A) = v(\mu_{\theta_1}, M_{\theta_1})(A) + v(\mu_{\theta_2}, M_{\theta_2})(A)
\]

\[
= \sup \{ v(\mu_{\theta_1}, M_{\theta_1})(C) : C \subseteq A, \ C \in \mathcal{K} \}
\]

\[
+ \sup \{ v(\mu_{\theta_2}, M_{\theta_2})(C) : C \subseteq A, \ C \in \mathcal{K} \}
\]

\[
= \sup \{ v(\mu_{\theta_1} \mid \mathcal{D}(\mathcal{K}), \mathcal{D}(\mathcal{K}))(C) : C \subseteq A, \ C \in \mathcal{K} \}
\]

\[
+ \sup \{ v(\mu_{\theta_2} \mid \mathcal{D}(\mathcal{K}), \mathcal{D}(\mathcal{K}))(C) : C \subseteq A, \ C \in \mathcal{K} \}
\]

\[
\leq 2 \sup \{ v(\mu_{\theta} \mid \mathcal{D}(\mathcal{K}), \mathcal{D}(\mathcal{K}))(C) : C \subseteq A, \ C \in \mathcal{K} \} = 2|\mu_{\theta}|(A) = 0
\]

since \( v(\mu_{\theta_1}, M_{\theta_1}) \) is \( M_{\theta_1} \)-regular and \( v(\mu_{\theta_2} \mid \mathcal{D}(\mathcal{K}), \mathcal{D}(\mathcal{K})) = v(\mu_{\theta_1}, M_{\theta_1}) \mid \mathcal{D}(\mathcal{K}) \) by (iv) and (v) of Theorem 4.5 for \( j = 1, 2 \) and \( v(\mu_{\theta} \mid \mathcal{D}(\mathcal{K}), \mathcal{D}(\mathcal{K})) = |\mu_{\theta}| \mid \mathcal{D}(\mathcal{K}) \) and \( |\mu_{\theta}| \) is \( M_{\theta} \)-regular by (iv) and (iii) of Theorem 4.7.

Conversely, let \( |\mu_{\theta}|(A) = 0 \). Then \( \mu_{|\theta|,j}(A) \leq \mu_{|\theta|}(A) \) for \( j = 1, 2 \) by the said proposition of [1] and consequently, by Theorem 4.6 (vi) we have

\[
|\mu_{\theta}|(A) \leq v(\mu_{\theta_1}, M_{\theta_1})(A) + v(\mu_{\theta_2}, M_{\theta_2})(A) = \mu_{|\theta|,1}(A) + \mu_{|\theta|,2}(A) = 0
\]

and hence \( |\mu_{\theta}|(A) = 0 \).

Now, let \( E \in M_{\theta} \). Then, by Theorem 4.7 (vii), \( E = A \cup (E \setminus A) \) with \( A \subseteq E \subseteq B \), \( A, B \in \mathcal{A}(\mathcal{K}) \cap M_{\theta} \) and \( |\mu_{\theta}|(B \setminus A) = 0 \). Consequently, \( \mu_{|\theta|}(B \setminus A) = 0 \) and thus \( \mu_{|\theta|}^+(B \setminus A) = 0 \). This shows that \( E \setminus A \in M_{|\theta|} \) and hence \( E \in M_{|\theta|} \) since \( A \in \mathcal{A}(\mathcal{K}) \) and \( \mu_{\theta,j^*}(A) < \infty \) and \( \mu_{\theta,j^-}(A) < \infty \) for \( j = 1, 2 \).

Lemma 4.10. For \( \theta \in \mathcal{K}(\mathcal{X})^\ast \), \( M_{\theta} = M_{|\theta|} \).

Proof. In fact, in the light of Lemma 4.9 it suffices to show that \( M_{|\theta|} \subseteq M_{\theta} \). Let \( E \in M_{|\theta|} \). Then by Theorem 2.2 there exist \( A, B \in \mathcal{A}(\mathcal{K}) \) such that \( A \subseteq E \subseteq B \) and \( \mu_{|\theta|,j}(B \setminus A) = 0 \). Thus \( \mu_{|\theta|}(E) = \mu_{|\theta|}(A) \). If \( \theta_1 = \text{Re} \theta \) and \( \theta_2 = \text{Im} \theta \), then by Proposition 15, §1, Chapter IV of [1], \( \mu_{|\theta|,j}(B \setminus A) = 0, \mu_{|\theta|,j}(A) \leq \mu_{|\theta|}(A) < \infty \) and

\[
\mu_{|\theta|,j^*} = \mu_{\theta,j^*}^* + \mu_{\theta,j^-}^* \quad \text{for} \quad j = 1, 2.
\]

Thus \( \mu_{\theta,j^*}(B \setminus A) = \mu_{\theta,j^-}(B \setminus A) = 0 \) for \( j = 1, 2 \). Consequently, \( \mu_{\theta,j^*}(E \setminus A) = \mu_{\theta,j^-}(E \setminus A) = 0 \) for \( j = 1, 2 \) and hence \( E \in M_{\theta} \).

Theorem 4.11. If \( \theta \in \mathcal{K}(\mathcal{X})^\ast \), then \( M_{\theta} = M_{|\theta|} \) and \( |\mu_{\theta}| = |\mu_{|\theta|}| \).

Proof. Let \( E \in M_{\theta} \) and \( \epsilon > 0 \). By Theorem 4.7 (vi) there exists \( \{E_i\}_i \subseteq \mathcal{A}(\mathcal{K}) \) with \( E_i \cap E_j = \emptyset \) for \( i \neq j \) and \( \bigcup_{i=1}^n E_i \subseteq E \) such that

\[
\sum_{i=1}^n |\mu_{\theta}(E_i)| > |\mu_{\theta}|(E) - \frac{1}{2}\epsilon.
\]
As \( \mu_\theta \mid \mathcal{D}(\mathcal{X}) \) is \( \mathcal{D}(\mathcal{X}) \)-regular by Theorem 4.7 (v), there exists \( C_i \in \mathcal{X} \) such that \( C_i \subseteq E_i \) and \( |\mu_\theta(E_i) - \mu_\theta(C_i)| < \frac{\varepsilon}{2^n} \) for \( i = 1, \ldots, n \). Consequently, by Lemma 4.8 we have

\[
\mu_{|\theta|}(E) \geq \mu_{|\theta|}\left(\bigcup_{1}^{n} C_i\right) = \sum_{1}^{n} \mu_{|\theta|}(C_i) \geq \sum_{1}^{n} |\mu_\theta(C_i)|
\]

\[
> \sum_{1}^{n} |\mu_\theta(E_i)| - \frac{1}{2}\varepsilon > |\mu_\theta|(E) - \varepsilon.
\]

Thus

\[
(1) \quad \mu_{|\theta|}(E) \geq |\mu_\theta|(E) \quad \text{for} \quad E \in M_\theta.
\]

If \( \psi \) is as in Theorem 4.7 (ix), then as \( |\theta| \leq \psi \), from (viii) and (ix) of Theorem 4.7 we have

\[
\mu_{|\theta|}(E) \leq \tilde{\mu}_\psi(E) = |\mu_\theta|(E)
\]

for \( E \in \mathcal{B}_c(X) \cap M_\theta \). Consequently, by (1)

\[
(2) \quad \mu_{|\theta|}(E) = |\mu_\theta|(E)
\]

for \( E \in \mathcal{B}_c(X) \cap M_\theta \).

If \( E \in M_\theta \), then by Theorem 2.2 there exist \( A, B \in \mathcal{B}(X) \) with \( A \subseteq E \subseteq B \) and \( A \) \( \sigma \)-compact, \( |\mu_\theta|(B \setminus A) = 0 \) and \( |\mu_\theta|(E) = |\mu_\theta|(A) \) (vide the proof of Theorem 4.7 (vii)). Hence by (2) and by Lemma 4.9 we have

\[
|\mu_\theta|(E) = |\mu_\theta|(A) = \mu_{|\theta|}(A) = \mu_{|\theta|}(E \setminus A) = \mu_{|\theta|}(E).
\]

Since \( M_\theta = M_{|\theta|} \) by Lemma 4.10, the theorem is established.

\[\square\]

References


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