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COHERENCE AND WEAK COHERENCE IN THE SQUARE OF ALGEBRAS

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1. PRELIMINARIES

The well-known theorem of H. Werner [12] asserts that a variety V is permutable iff any diagonal (symmetric) subalgebra of the square $A \times A$ is a congruence on A, $A \in V$. A similar characterization of permutable and regular varieties was obtained by means of regular diagonal (symmetric) subalgebras of the square, see [6]. The present paper shows that also coherence, see [9] for this concept, of diagonal (symmetric) subalgebras and some congruences on the square, gives a description of permutable and regular varieties. This result clarifies the relationship between coherent varieties and varieties with permutable and regular congruences, a question discussed in [1], [3], [5], [9] and [11]. Finally, we show that analogous results hold for weak coherence, see [2], permutability and weak regularity.

Notation. Let A be an algebra. The symbol $\omega_A(\iota_A)$ denotes the *least* (the *greatest*, resp.) congruence on A.

Definition 1. Let A be an algebra, B a subalgebra of $A \times A$. B is called a *diagonal subalgebra* of $A \times A$ whenever the inclusion $\omega_A \subseteq B$ holds.

Definition 2. Let A be an algebra. We say that a congruence Θ on $A \times A$ has *factorable blocks* whenever any Θ-block B is of the form $B = C \times D$ for some subsets C, D of A.

A congruence Θ on $A \times A$ is called *factorable* whenever $\Theta = \Psi \times \Phi$ for some congruences Ψ, Φ on A.

A congruence Θ on $A \times A$ is called *subfactor* whenever $\Theta \subseteq \omega_A \times \iota_A$ or $\Theta \subseteq \iota_A \times \omega_A$ holds.

Definition 3. Let A be an algebra. We say that A is *permutable* whenever $\Psi \circ \Phi = \Psi \circ \Phi$ holds for any congruences Ψ, Φ on A.

A variety V is called *permutable* whenever any V-algebra has this property.

2. Coherence in the square of algebras

Definition 4. Let A be an algebra. We say that A is *regular* if any two congruences on A coincide whenever they have a class in common.

A variety V is called *regular* whenever any V-algebra has this property.

Definition 5. Let A be an algebra. We say that a subalgebra B of A is *coherent* with a congruence Θ on A whenever the assumption $[b]\Theta \subseteq B$ for some $b \in B$ implies $[x]\Theta \subseteq B$ for every $x \in B$.

Theorem 1. *For a variety V, the following conditions are equivalent:*

(1) *any diagonal subalgebra of $A \times A$ is coherent with congruences on $A \times A$ having factorable blocks, $A \in V$;*

(1′) *any diagonal subalgebra of $A \times A$ is coherent with factorable congruences on $A \times A$, $A \in V$;*

(1″) *any diagonal subalgebra of $A \times A$ is coherent with subfactor congruences on $A \times A$, $A \in V$;*

(1‴) *any diagonal subalgebra of $A \times A$ is coherent with factorable subfactor congruences on $A \times A$, $A \in V$;*

(2) *any diagonal symmetric subalgebra of $A \times A$ is coherent with congruences on $A \times A$ having factorable blocks, $A \in V$;*

(2′) *any diagonal symmetric subalgebra of $A \times A$ is coherent with factorable congruences on $A \times A$, $A \in V$:*

(2″) *any diagonal symmetric subalgebra of $A \times A$ is coherent with subfactor congruences on $A \times A$ is coherent with subfactor congruences on $A \times A$, $A \in V$;*

(2‴) *any diagonal symmetric subalgebra of $A \times A$ is coherent with factorable subfactor congruences on $A \times A$, $A \in V$;*

(3) *V is permutable and regular.*

P r o o f. It suffices to verify the implications (2‴) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1).

(2‴) $\Rightarrow$ (3) *Permutability*: Let Ψ, Φ be congruences on $A \in V$. Then $T = \Psi \circ \Phi \cap \Phi \circ \Psi$ is evidently a diagonal symmetric subalgebra of $A \times A$. Since $[\langle a, a \rangle]\Psi \times \omega_A \subseteq \Psi \subseteq T$ the assumption of coherence yields that T is a union of $\Psi \times \omega_A$-blocks. By the same argument T is a union of $\omega_A \times \Phi$-blocks. Then T is a union of $(\Psi \times \omega_A) \vee (\omega_A \times \Phi) =$

($\Psi \times \Phi$-blocks, i.e. $T = \bigcup\{[\langle x, y\rangle]\Psi \times \Phi;\ \langle x, y\rangle \in T\} = \Psi \circ T \circ \Phi \supseteq \Psi \circ \Phi$. The inclusion $\Psi \circ \Phi \cap \Phi \circ \Psi \supseteq \Psi \circ \Phi$ establishes the permutability of the congruences Ψ, Φ.

Regularity: Let Ψ, Φ be congruences on A such that $[a]\Psi = [a]\Phi$ for some $a \in A$. Then $[\langle a, a\rangle]\Psi \times \omega_A = [a]\Psi \times \{a\} = [a]\Phi \times \{a\} \subseteq \Phi$. By coherence, the diagonal symmetric subalgebra Φ of $A \times A$ is a union of $\Phi \times \omega_A$-blocks, i.e. $\Phi = \bigcup\{[\langle x, y\rangle]\Psi \times \omega_A;\ \langle x, y\rangle \in \Phi\} = \Psi \circ \Phi \circ \omega_A \supseteq \Psi$. The opposite inclusion follows by a symmetrical argument.

(3) $\Rightarrow$ (1): Let Θ be a diagonal subalgebra of $A \times A$. Then the congruence permutability of V yields that Θ is a congruence on A, see [12]. Further, let Ψ be a congruence on $A \times A$ having factorable blocks. Suppose that $[\langle a, b\rangle]\Psi \subseteq \Theta$. By hypothesis, $[\langle a, b\rangle]\Psi = C \times D$ for some subsets C, D of A, and thus the assumption $\langle c, d\rangle \in [\langle a, b\rangle]\Psi$ implies $\langle c, b\rangle \in C \times D \subseteq \Theta$. Using the transitivity of Θ we get that $\langle c, a\rangle \in \Theta$. Analogously $\langle d, b\rangle \in \Theta$ can be obtained. In this way we have verified the inclusion $[\langle a, b\rangle]\Psi \subseteq [a]\Theta \times [b]\Theta = [\langle a, b\rangle]\Theta \times \Theta$. Now $\Psi \subseteq \Theta \times \Theta$ follows from the regularity and so Θ is a union of Ψ-blocks. □

For transitive subalgebras of the square we have

Theorem 2. *For a variety V, the following conditions are equivalent:*

(1) *any diagonal transitive subalgebra of $A \times A$ is coherent with congruences on $A \times A$ having factorable blocks, $A \in V$;*

(1′) *any diagonal transitive subalgebra of $A \times A$ is coherent with factorable congruences on $A \times A$, $A \in V$;*

(1″) *any diagonal transitive subalgebra of $A \times A$ is coherent with subfactor congruences on $A \times A$, $A \in V$;*

(1‴) *any diagonal transitive subalgebra of $A \times A$ is coherent with factorable subfactor congruences on $A \times A$, $A \in V$;*

(2) *any congruence on A is coherent with congruences on $A \times A$ having factorable blocks, $A \in V$;*

(2′) *any congruence on A is coherent with factorable congruences on $A \times A$, $A \in V$;*

(2″) *any congruence on A is coherent with subfactor congruences on $A \times A$, $A \in V$;*

(2‴) *any congruence on A is coherent with factorable subfactor congruences on $A \times A$, $A \in V$;*

(3) *V is regular.*

Proof. (2‴) $\Rightarrow$ (3): See part (2‴) $\Rightarrow$ (3) from the proof of Theorem 1.

(3) $\Rightarrow$ (1): By [10], regular varieties are n-permutable for some integer $n > 1$. Then any diagonal transitive subalgebra of the square is a congruence, see [10] again. The rest of our proof is the same as in part (3) $\Rightarrow$ (1) from the proof of Theorem 1. □

3. Weak coherence in the square of algebras

Definition 6. Let A be an algebra with a nullary operation 0. We say that A is *weakly regular* if any two congruences Ψ, Φ on A coincide whenever $[0]\Psi = [0]\Phi$.

A variety V with a nullary operation 0 is called *weakly regular* whenever any V-algebra has this property.

Definition 7. Let A be an algebra with a nullary operation 0. We say that a subalgebra B of A is *weakly coherent* with a congruence Θ on A whenever the assumption $[0]\Theta \subseteq B$ implies $[x]\Theta \subseteq B$ for every $x \in B$.

Theorem 3. *Let V be a variety with a nullary operation 0. The following conditions are equivalent:*

(1) *any diagonal subalgebra of $A \times A$ is weakly coherent with congruences on $A \times A$ having factorable blocks, $A \in V$;*

(1′) *any diagonal subalgebra of $A \times A$ is weakly coherent with factorable congruences on $A \times A$, $A \in V$;*

(1″) *any diagonal subalgebra of $A \times A$ is weakly coherent with subfactor congruences on $A \times A$, $A \in V$;*

(1‴) *any diagonal subalgebra of $A \times A$ is weakly coherent with factorable subfactory congruences on $A \times A$, $A \in V$;*

(2) *any diagonal symmetric subalgebra of $A \times A$ is weakly coherent with congruences on $A \times A$ having factorable blocks, $A \in V$;*

(2′) *any diagonal symmetric subalgebra of $A \times A$ is weakly coherent with factorable congruences on $A \times A$, $A \in V$;*

(2″) *any diagonal symmetric subalgebra of $A \times A$ is weakly coherent with subfactor congruences on $A \times A$, $A \in V$;*

(2‴) *any diagonal symmetric subalgebra of $A \times A$ is weakly coherent with factorable subfactor congruences on $A \times A$, $A \in V$;*

(3) *V is permutable and weakly regular.*

P r o o f. (2‴) $\Rightarrow$ (3): Put $a = 0$ and replace the word "coherence" ("regularity") by the term "weak coherence" ("weak regularity", resp.) in part (2‴) $\Rightarrow$ (3) of the proof of Theorem 1.

(3) $\Rightarrow$ (1): Put $a = b = 0$ and replace the word "regularity" by the term "weak regularity" in part (3) $\Rightarrow$ (1) of the proof of Theorem 1. □

Theorem 4. *Let V be a variety with a nullary operation 0. The following conditions are equivalent:*

(1) *any diagonal transitive subalgebra of* $A \times A$ *is weakly coherent with congruences on* $A \times A$ *having factorable blocks,* $A \in V$;

(1′) *any diagonal transitive subalgebra of* $A \times A$ *is weakly coherent with factorable congruences on* $A \times A$, $A \in V$;

(1″) *any diagonal transitive subalgebra of* $A \times A$ *is weakly coherent with subfactor congruences on* $A \times A$, $A \in V$;

(1‴) *any diagonal transitive subalgebra of* $A \times A$ *is weakly coherent with factorable subfactor congruences on* $A \times A$, $A \in V$;

(2) *any congruence on* A *is weakly coherent with congruences on* $A \times A$ *having factorable blocks,* $A \in V$;

(2′) *any congruence on* A *is weakly coherent with factorable congruences on* $A \times A$, $A \in V$;

(2″) *any congruence on* A *is weakly coherent with subfactor congruences on* $A \times A$, $A \in V$;

(2‴) *any congruence on* A *is weakly coherent with factorable subfactor congruences on* $A \times A$, $A \in V$;

(3) V *is weakly regular.*

Proof. (2‴) $\Rightarrow$ (3): Put $a = 0$ and replace the word "coherence" ("regularity") by the term "weak coherence" ("weak regularity", resp.) in the second part of the implication (2) $\Rightarrow$ (3) from the proof of Theorem 1.

(3) $\Rightarrow$ (1): By [10], weakly regular varieties are n-permutable for some integer $n > 1$. Hence any diagonal transitive subalgebra of the square is a congruence and so it remains to put $a = 0$ and replace the word "coherence" ("regularity") by the term "weak coherence" ("weak regularity", resp.) in the implication (3) $\Rightarrow$ (1) from the proof of Theorem 1. □

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