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A NOTE ON EMBEDDINGS MANIFOLDS INTO TOPOLOGICAL GROUPS PRESERVING DIMENSIONS

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1. INTRODUCTION

In this paper, we assume that all spaces are Tychonoff. It is well-known that topological groups are Tychonoff (e.g., see [7, p. 29]). In [2], Bel'nov proved that every spaces $X$ can be embedded into a homogeneous space $H_X$ such that $\text{ind} \ H_X = X$, $\text{Ind} \ H_X = \text{Ind} \ X$ and $\dim H_X = \dim X$ in the case when the corresponding dimension of $X$ is finite. Also, Bel'nov asked whether every spaces $X$ can be embedded into a topological group $G$ with $\dim G \leq \dim X$ (see [9]). Shakhmatov proved that if $n \neq 0,1,3,7$, then the $n$-dimensional sphere $S^n$ can not be embedded into an $n$-dimensional topological group, and he showed that in the case $\dim X = 0$, the answer to this question is positive [9]. In [6], Kimura proved that if a topological group $G$ contains the bouquet $S^1 \vee S^1$ of two circles, then $\dim G \geq 2$, which implies that in the case $\dim X = 1$, the answer is negative. In [5], the author proved that if $G$ contains the one point union $S^n \vee I$ of the $n$-dimensional sphere $S^n$ and an arc $I$, then $\dim G \geq n + 1 \ (n = 1, 2, \ldots)$, which implies that in the case $\dim X \geq 1$, the question is negative.

Also, in [9, p.182] Shakhmatov asked whether $S^7$ can be embedded into a topological group $G$ with $\dim G = 7$. Note that $S^n (n = 0,1,3)$ is a topological group, $S^7$ is an $H$-space but not a topological group, and $S^n (n \neq 0,1,3,7)$ is not an $H$-space (see [1]). To prove his above result, Shakhmatov essentially used the Adams' theorem that $S^n (n \neq 0,1,3,7)$ is not an $H$-space. Naturally, the following problem will be raised: What kinds of manifolds can be embedded into topological groups preserving dimensions?

In this paper, we prove that if a topological group $G$ contains the one point union $D^n \vee I$ of an $n$-ball $D^n$ and an arc $I$, then $\dim G \geq n + 1$. The case $n = 1$ is a negative answer to a question of Kimura [6, (4.5) Question]. Next, we prove the following theorem: Let $M$ be an $n$-dimensional compact manifold without boundary. Then $M$
can be embedded into an $n$-dimensional topological group if and only if $M$ is itself a topological group. Hence $S^7$ cannot be embedded into a topological group $G$ with $\dim G = 7$ or $\text{ind} G = 7$ or $\text{Ind} G = 7$.

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2. EMBEDDINGS INTO TOPOLOGICAL GROUPS AND DIMENSIONS.

Let $R$ be the real line and let $D^n$ be the $n$-ball $\{ (x_1, x_2, \ldots, x_n) \in R^n \mid \sum_{i=1}^{n} x_i^2 \leq 1 \}$. Let $I$ be the unit interval $[0,1]$ in $R$. Also, let $S^n$ be the $n$-dimensional sphere $\{ (x_1, x_2, \ldots, x_{n+1}) \in R^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1 \}$ and let $p = (1,0,\ldots,0) \in S^n$. By identifying the point $* = (0,0,\ldots,0) \in D^n$ and $0 \in I$, we obtain the one point union $(D^n \vee I, *)$ of $(D^n, *)$ and $(I, 0)$.

Then we have the following theorem.

**Theorem 2.1.** Let $G$ be a topological group. If $G$ contains $D^n \vee I$ ($n \geq 1$), then $\dim G \geq n + 1$.

To prove (2.1), we need the following well-known result (e.g., see [3, (3.2.10) Theorem]).

**Theorem 2.2.** A normal space $X$ satisfies the inequality $\dim X \leq n$ ($n \geq 0$) if and only for every closed subset $A$ of $X$ and each mapping $f: A \to S^n$ there is an (continuous) extension $F: X \to S^n$ of $f$ over $X$.

**Proof of (2.1).** Suppose, on the contrary, that there is a topological group $G$ with contains $D^n \vee I$ and $\dim G \leq n$. Let $h: D^n \vee I \to G$ be an embedding. Since $G$ is homogeneous, we may assume that $h(*) = e$ is the unit element of the group $G$. We may assume that $D^n$ and $I$ are naturally the subsets of $D^n \vee I$. Let $\varphi: D^n \times I \to G$ be the homotopy defined by

$$\varphi(x,t) = h(x) \cdot h(t)$$

for $x \in D^n$ and $t \in I$, where the symbol $\cdot$ denotes the group composition of $G$.

Choose a neighborhood $U$ of $\varphi(\partial D^n \times \{0\})$ in $G$ and a neighborhood $V$ of $\varphi(*,0)$ ($= h(*)$) in $G$ such that $U \cap V = \emptyset$, where $\partial D^n$ denotes the manifold boundary. Then, take a sufficiently small positive number $t$ such that $\varphi(\partial D^n \times [0,t]) \subset U$ and $\varphi(*) \times [0,t]) \subset V$. Since $\varphi(D^n \times \{t\})$ is an $n$-ball and $\varphi(*,t)$ is not contained in $\varphi(D^n \times \{0\})$, we can choose a small $n$-ball $B$ in $\varphi(D^n \times \{t\}) \cap V$ such that $\varphi(*,t) \in B$ and $B \cap \varphi(D^n \times \{0\}) = \emptyset$. Note that $B \cap (\varphi(\partial D^n \times [0,t]) \cup \varphi(D^n \times \{0\})) = \emptyset$. Define
a map \( f : \varphi(D^n \times \{0,t\}) \cup \varphi(\partial D^n \times [0,t]) \rightarrow S^n \) as follows: If \( x \) is not contained in \( B \), \( f(x) = p \), and \( f|B: (B, \partial B) \rightarrow (B/\partial B, *) \equiv (S^n, p) \) is the natural quotient map which is obtained from \( B \) by shrinking the boundary \( \partial B \) to a point \( *' \). Note that \( \varphi(D^n \times I) \) is compact metrizable. Since \( \dim G \leq n \), \( \dim \varphi(D^n \times [0,t]) \leq n \).

By (2.2), we have an extension \( F : \varphi(D^n \times [0,t]) \rightarrow S^n \) of \( f \). Put \( H' = F\varphi: D^n \times I \rightarrow S^n \). Note that \( H'(\partial D^n \times [0,t]) = * \). Hence we obtain a homotopy \( H : S^n \times [0,t] \rightarrow S^n \) induced by \( H' \) such that \( H_0 \) is a constant map and \( H_t \) is homotopic to the identity map of \( S^n \), where \( H_s(x) = H(x, s) \) for \( 0 \leq s \leq t \) and \( x \in S^n \). Since \( S^n \) is not contractible, this is a contradiction.

\[ \square \]

**Remark 2.3.** By (2.1), the one point union \( D^n \vee I(n \geq 1) \) can not be embedded into an \( n \)-dimensional topological group and \( \dim(D^n \vee I) = n \). Hence the one point union \( D^n \vee I(n \geq 1) \) is the simplest example which gives a negative answer to the question of Bel'nov. The case \( n = 1 \) is a negative answer to the question of Kimura [6, (4.5) Question]. Also, in the proof of (2.1) we get a contradiction to assume that \( \dim \varphi(D^n \times I) \leq n \). Since \( \varphi(D^n \times I) \) is compact and metrizable, we can conclude that \( D^n \vee I \) can not be embedded into a topological group \( G \) such that \( \text{ind } G \leq n \) or \( \text{Ind } G \leq n \) or \( \dim G \leq n \).

The following lemma is trivial.

**Lemma 2.4.** Let \( G \) be a topological group. If \( P \) is the path component containing the unit element of \( G \), then \( P \) is a subgroup of \( G \).

The following is the main theorem of this paper.

**Theorem 2.5.** Let \( M \) be a \( n \)-dimensional compact manifold without boundary \( (n \geq 1) \). Then \( M \) can be embedded into an \( n \)-dimensional topological group \( G \) if and only if \( M \) is itself a topological group.

**Proof.** Suppose that \( M \) is not a topological group. We may assume that \( M \) is path connected and \( M \) contains the unit element \( e \) of \( G \). Suppose, on the contrary, that \( G \) is an \( n \)-dimensional topological group \( G \) containing \( M \). Let \( P \) be the path component of \( G \) containing the unit element \( e \) of \( G \). Then \( P \) is also a topological group (see (2.4)) and \( P \supset M \). Since \( M \) is not a topological group, \( P - M \neq \emptyset \). Take a point \( x_0 \in P - M \). Since \( P \) is path connected, there is an arc \( A \) in \( P \) from \( x_0 \) to a point \( y_0 \) of \( M \) such that \( A \cap M = \{y_0\} \). Since \( M \) is an \( n \)-dimensional manifold without boundary, there is a subset \( K \) of \( P \) which is homeomorphic to \( D^n \vee I \). Let \( \varphi: D^n \times I \rightarrow P \) be the homotopy as in the proof of (2.1). Then we see that \( \varphi(D^n \times I) \) is compact and metrizable with \( \dim \varphi(D^n \times I) \geq n + 1 \). Hence \( \dim G \geq n + 1 \). This is a contradiction. The converse assertion is obvious. \[ \square \]
Remark 2.6. It is well known that an $n$-dimensional sphere $S^n$ is a topological group if and only if $n = 0, 1, 3$. Hence $S^n(n \neq 0, 1, 3)$ can not be embedded into an $n$-dimensional topological group. The case $n = 7$ is a negative answer to a question of Shakhmatov [9, p. 182]. Also, by the proof of (2.5), we can conclude that if $M$ is an $n$-dimensional compact manifold without boundary which is contained in an $n$-dimensional topological group $G$, then $M$ is a path component of $G$.

In the theory of topological groups (see [7]), the structure of locally compact topological groups has been studied by many mathematicians. Especially, the following is well known as a positive answer to Hilbert’s fifth problem: A locally compact topological group which is finite-dimensional and locally (path) connected is a Lie group, in particular, a manifold without boundary (see [7, (4.10.1) Theorem]). In the theory of locally compact topological groups, the property of being locally compact is essential (see [7] and [4]).

Now, we will give the proof of the following without the assumption of locally compactness.

Corollary 2.7. If $G$ is an $n$-dimensional topological group which contains an $n$-ball and $G$ is locally path connected, then $G$ is a Lie group.

Proof. By (2.1), we know that $G$ does not contain $D^n \vee I$. Since $G$ is homogeneous and locally path connected, we can see that $G$ is an $n$-dimensional manifold, which implies that $G$ is locally compact. Hence $G$ is a Lie group. $\square$

References


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