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ON LATTICE ORDERED PERIODIC SEMIGROUPS

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As in our previous papers [3], [4], [5], by a lattice ordered semigroup, we mean a semigroup S on which we can define an order relation $\leqslant$ such that

– $(S, \leqslant)$ is a distributive lattice; $\wedge$ and $\vee$ are the least upper bound and the greatest lower bound.

– $\forall a\, \forall b\, \forall c \quad a(b \wedge c) = ab \wedge ac$ and $(b \wedge c)a = ba \wedge ca$

– $\forall a\, \forall b\, \forall c \quad a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$.

The purpose of this note is to give some algebraic properties of lattice ordered periodic semigroups and particularly in the finite case.

1. Lattice ordered nilsemigroups. Lattice ordered periodic semigroups

Proposition 1. *Let S be a lattice ordered finite semigroup, generated by the element "a". If the order of S is n, then $\{a^n\}$ is the unique subgroup of S and a^n is a zero of S. Moreover, S is totally ordered.*

Proof. We know, cf. [2], chapter 1, that $S = \langle a\rangle = \{a, a^2, \ldots, a^r, \ldots, a^n\}$, where $K = \{a^r, a^{r+1}, \ldots, a^n\}$ is a cyclic subgroup of S of order $n - r + 1$, with $a^{n+1} = a^r$. Let $a^k = e$ be the idempotent of K, the identity element of K; $k \geqslant r$ and $(e \vee a)^k = (a^i)^k$ for some integer i and consequently $(e \vee a)^k = (a^k)^i = e$. But since S is abelian, we have $e = e \vee ea \vee ea^2 \vee \ldots ea^{k-1} \vee a^k$ and $ea \leqslant e$, $ea^k = e \leqslant ea^{k-1} \ldots \leqslant ea \leqslant e$ and $e = ea (= ae)$; e is the zero of S. Clearly, $K = \{e\}$.

Let us now show that S is totally ordered. If a and a^2 are incomparable, then $a \vee a^2 = a^i$, $i > 2$ and $a \wedge a^2 = a^j$, $j > 2$. From $a \wedge a^2 = a^j$, we deduce $a^{n-1} \wedge a^n = a^{j+n-2} = ea^{j-n} = e = a^n$ and $a^n \lneqq a^{n-1}$ and from $a \vee a^2 = a^i$, we deduce similarly $a^{n-1} \lneqq a^n$, contradicting $a^n \lneqq a^{n-1}$. Hence a and a^2 are comparable and S is totally ordered. □

Proposition 2. *Every lattice ordered nilsemigroup is locally finite.*

Proof. Let S be a such semigroup, of zero 0. Let $a_1, a_2, \ldots, a_p$ be elements of S and denote by A the subsemigroup they generate. We show that A is finite. (We know that this property is true if S is abelian, or if S is totally ordered, cf. [6]). As S is a nilsemigroup, we can suppose $a_1^n = a_2^n = \ldots = a_p^n = 0 = (a_1 \vee a_2 \vee a_3 \ldots \vee a_p)^n = (a_1 \wedge a_2 \wedge \ldots \wedge a_p)^n$, since $a^{n_i} = 0$ implies $a^{kn_i} = 0$ for every integer k, $k \geqslant 1$. Let a be in A: $a = \prod_{i=1}^{N} x_i$, with $x_i \in \{a_1, a_2, \ldots, a_p\}$. Suppose that $N \geqslant n$.

$$\begin{aligned} \text{Then} \quad & a = (x_1 x_2 \ldots x_n) x_{n+1} \ldots x_N, \quad \text{and} \\ & a \leqslant (a_1 \vee a_2 \vee \ldots \vee a_p)^n x_{n+1} \ldots x_N = 0 \quad \text{and} \\ & a \geqslant (a_1 \wedge a_2 \wedge \ldots \wedge a_p)^n x_{n+1} \ldots x_N = 0 \quad \text{since, for each } x_i, \end{aligned}$$

we have $(a_1 \wedge a_2 \wedge \ldots \wedge a_p) \leqslant x_i \leqslant (a_1 \vee a_2 \vee \ldots \vee a_p)$.

Finally $a = 0$, and every element of $A \neq 0$ is a product of at most $n - 1$ elements, chosen among p elements. Therefore A is finite and S is locally finite. □

Theorem 1. *Let S be a periodic ordered semigroup, and suppose that the idempotents of S form a bisimple semigroup of S. Then every spindle F_e is a subsemigroup of S, convex sublattice of S, nilsemigroup of zero e.*

"Let us recall that in a periodic semigroup S we can define the equivalence relation $\mathscr{F}$ by

$$a \equiv b \mathscr{F} \Leftrightarrow \exists e \in S, \quad e = e^2 \quad \text{and} \quad \exists n \in \mathbb{N}^*, \quad a^n = b^n = e.$$

Every class is called a spindle and will be denoted by F_e, where e is the idempotent of this class. It is well known, cf. [6] that if S is totally ordered, F_e is a subsemigroup of S."

Proof. In a first time, we show that e is zero of F_e. Let x be in F_e; $x^n = e$ for some integer n. As

$$\begin{aligned} xe = ex = x^{n+1}, \quad (x \vee e)^n &= x^n \vee x^{n-1}e \vee x^{n-2}e \ldots \vee xe \vee e \\ &= x^{n-1}e \vee x^{n-2}e \ldots \vee xe \vee e \end{aligned}$$

and $x(x \vee e)^n = x^n e \vee x^{n-1}e \vee \ldots \vee x^2 e \vee xe = e \vee x^{n-1}e \vee x^{n-2}e \ldots \vee x^2 e \vee xe$. Then $x(x \vee e)^n = (x \vee e)^n$ and $x^k (x \vee e)^n = (x \vee e)^n$ for any integer k. Finally $(x \vee e)(x \vee e)^n = (x \vee x^n)(x \vee e)^n = x(x \vee e)^n \vee x^n (x \vee e)^n = (x \vee e)^n$ and $(x \vee e)^n = f = f^2$. $(x \vee e)^n = f$ is an idempotent such that $xf = f = fx$ by symmetry. We deduce $ef = fe = f$. But $efe = e$, $fef = f$ since the idempotents of S form a bisimple subsemigroup. Hence $e = f$ and $x^n = (x \vee e)^n = e$. Similarly, $(x \wedge e)^n = e$.

From $(x \vee e)^n = e$, we deduce $xe \leqslant e$ and $x^n e = e \leqslant x^{n-1}e \leqslant xe \leqslant e$. In conclusion, e is zero of the spindle F_e.

Now let x and y be two elements of F_e: $x^n = y^n = e$. From $xe = ex = ey = ye = e$, we find $(x \vee y)e = e$, and if $x \vee y$ belongs to F_g, with $g = g^2$, $ge = eg = e$. But $ege = e$, $geg = g$, and $g = e$. And we have $x \vee y \in F_e$ and similarly $x \wedge y \in F_e$. Therefore F_e is a sublattice of S, evidently a convex sublattice.

From the inequality $(a \wedge b)^2 \leqslant ab \leqslant (a \vee b)^2$, we deduce that F_e is a subsemigroup of S. □

2. Weakly negative lattice ordered periodic semigroups

Definition. An ordered semigroup is said to be weakly negative if for all x, $x^2 \leqslant x$.

Lemma 1. *In a weakly negative lattice ordered periodic semigroup, every spindle F_e is a subset of zero e and e is the least element of F_e.*

It is routine to prove these properties. We note that generally F_e is not a subsemigroup.

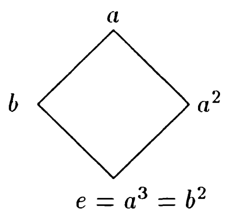
In the following, S is a weakly negative lattice ordered periodic semigroup. The definition of "height" is given in [1]. We suppose that S is a distributive lattice of finite length.

Lemma 2. *Let a be an element of height 2 in a spindle F_e of S. Then a permute with all elements b of height 1 of F_e which are comparable with a, and we have $ab = ba = e$ or $ab = ba = a^2$.*

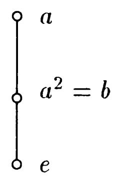
Proof. Suppose $e < b < a$ with a of height 2 and b of height 1. Necessarily $b^2 = e$. We have $e \leqslant ab \leqslant (a \vee b)^2 \leqslant a \vee b$. But $ab = a$ is impossible since $ab = a$ implies $ab^2 = ae = ab = a = e$. Therefore $ab = e$ or $e \lneqq ab \lneqq a$ with $ab \neq b$ $(ab = b \Rightarrow a^i b = b = eb = e)$.

If $a^2 = e$, then $ab = ba = e$ since $e \leqslant ab \leqslant a^2$, $e \leqslant ba \leqslant a^2$ by isotony.

If $a^2 \neq e$, $e < a^2 < a$ and $a^3 = e$, a^2 is of height 1. We have then two possibilities:



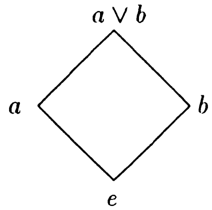
or



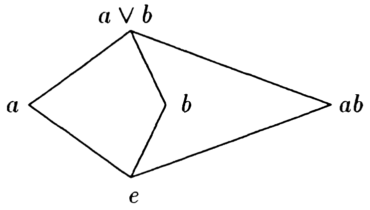
In the first case, $a^2 \vee b = a$ which implies $a^3 \vee ab = a^2 = e \vee ab = ab$ and similarly $a^3 \vee ba = a^2 = e \vee ba$ and $ab = ba = a^2$.

In the second case, $a^2 = b$ which implies $ab = ba = a^3 = e$. Finally in all cases $ab = ba$. □

Lemma 3. *If two elements a and b are of height 1 in a spindle F_e, then $ab = ba = e$.*



If $a \neq b$, we have $a \wedge b = e$ and $ab \leqslant (a \vee b)^2 \leqslant a \vee b$. The equality $ab = a \vee b$ is impossible, as $a \leqslant a \vee b = ab$ implies $e \lneqq a \leqslant ab \leqslant ab^2 = e$ by isotony. Therefore, $ab < a \vee b$. But a covers $a \wedge b = e$, b covers $a \wedge b = e$, therefore $a \vee b$ covers a and b, and $a \vee b$ is of height 2. Lemma 2 implies $a(a \vee b) = (a \vee b)a$ e.g. $e \vee ab = e \vee ba(a^2 = e)$, and $ab = ba$. But, from $e \leqslant ab \lneqq a \vee b$ we deduce $ab = e$ or ab is of height 1. Suppose that $ab = ba$ is of height 1: then, $a \wedge ab = e = a \wedge b = b \wedge ab$ ($ab \neq a$, $ab \neq b$ otherwise $a = e$, $b = e$) and $ab \vee a$, $ab \vee b$ are of height 2. But $ab < a \vee b$ implies $a \vee ab \leqslant a \vee b$, $b \vee ab \leqslant a$; as $a \vee ab$, $b \vee ab$, $a \vee b$ are of the same height 2, we will have in this case a lattice of type:

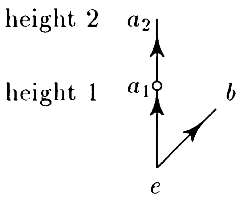


with $a \vee b = a \vee ab = b \vee ab$. But this lattice, sublattice of S, is not distributive.
Then, $ab = ba = e$.

Lemma 4. *In a spindle F_e, the product of an element of height 2 by an element of height 1 is an element of height 1 or is egal to e (height 0).*

If $e < a < b$ with a of height 1 and b of height 2, we have seen, in lemma 2, that $ab = ba = e$ or b^2. As b^2 is of height 1 or $b^2 = e$, we have the result.

We consider now the following case:

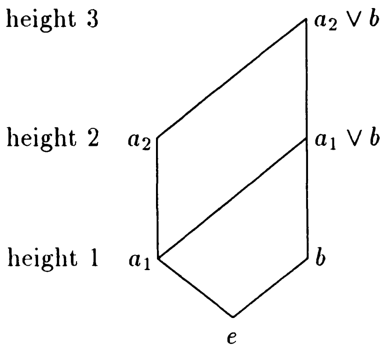


and we examine the product $a_2 b$ with $b \nless a_2$.

$a_1 \wedge b = e$, a_1 and b cover e, then $a_1 \vee b$ covers a_1 and b; therefore $a_1 \vee b$ is of height 2.

$a_2 \wedge b = e$ is covered by b, therefore $a_2 \vee b$ covers a_2 and $a_2 \vee b$ is of height 3.

$b \not< a_2$ implies $a_1 \vee b \neq a_2$. Therefore $a_1 \vee b$ and a_2 are of same height and incomparable. So, we have an ordered set of the following type:



But in a spindle F_e containing x and y, we have always $xy \leqslant (x \vee y)^2 \leqslant x \vee y$ and the equality $xy = x \vee y$ is impossible if $x \neq e$, $y \neq e$ because it implies $x^2y = x^2 \vee xy = x^2 \vee x \vee y = x \vee y = xy$ and $x \vee y = x^2y = \ldots = x^n y = e$ which is not. Therefore, here, $a_2b \lneqq a_2 \vee b$, $ba_2 \lneqq a_2 \vee b$ and also $a_2b \neq a_2$, $a_2b \neq b$, $ba_2 \neq a_2$, $ba_2 \neq b$.

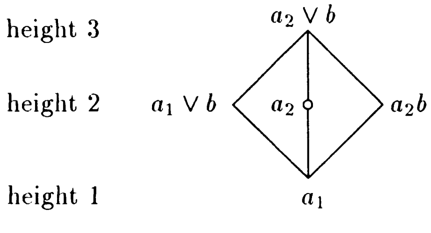
Suppose now a_2b is of height 2.

If $b < a_2b$, then $a_2b \leqslant a_2^2b \leqslant a_2b$ and $a_2b = a_2^2b \ldots = a_2^kb = e$ which is not.

Therefore $b \not< a_2b$ and of course $a_1 \vee b \neq a_2b$, $b \wedge a_2b = e$.

Suppose, moreover, that $a_1 < a_2b$.

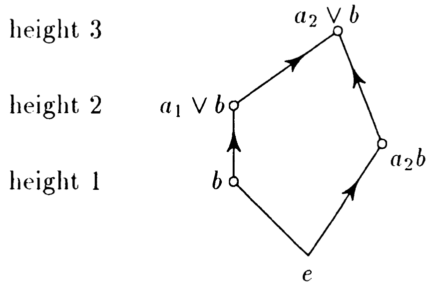
In this case, we have:



$a_1 \vee b \vee a_2 = a_2 \vee b$; $a_2 \vee a_2b = a_2 \vee b$ necessarily because $a_2 < a_2 \vee b$, $a_2b < a_2 \vee b$ and the heights are 2 for a_2, a_2b, 3 for $a_2 \vee b$; $(a_1 \vee b) \vee a_2b = a_2 \vee b$ for the same reasons.

But this is impossible, as this sublattice is not distributive.

Therefore $a_1 \not< a_2b$ and necessarily we have a scheme of this following type:



Effectively, $(a_1 \vee b) \vee (a_2 b) = a_2 \vee b$, because $a_2 b < a_2 \vee b$, $a_1 \vee b < a_2 \vee b$ and the heights of $a_1 \vee b$, $a_2 b$ are 2, the height of $a_2 \vee b$ is 3.

$(a_1 \vee b) \wedge a_2 b = (a_1 \wedge a_2 b) \vee (b \wedge a_2 b)$. But $a_1 \not< a_2 b$, $b \not< a_2 b$, a_2 and b are of height 1. Therefore $a_1 \wedge a_2 b = e$, $b \wedge a_2 b = e$, and we have $(a_1 \vee b) \wedge a_2 b = b \wedge a_2 b = e$. But this sublattice cannot exist: This lattice is not modular! ...

Consequently $a_2 b$ (and $b a_2$) are of height 1 or 0.

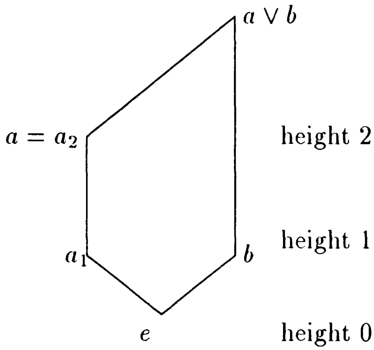
Theorem 2. *Let S be a finite weakly negative lattice ordered semigroup and let F_e be a spindle. If a, element of F_e is of height 2 and if b, element of F_e, is of height 1, there are two possibilities:*

either $ab = ba$ is an element of height 1 or 0

or $ab \neq ba$, and one of these two elements is of height 1, the other being of height 0.

1°) If $e < b < a$, then, from lemma 2, we deduce $ab = ba$, and $ab = ba = e$ or $ab = ba = a^2$, which is of height 1.

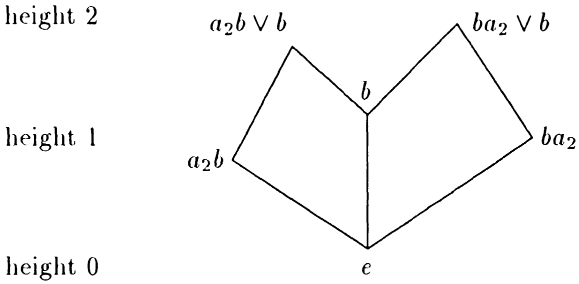
2°) Now, we suppose that a and b are incomparable; we put $a = a_2$, and of course we have a diagram of this type:



From lemma 4, we know that $a_2 b$ and $b a_2$ are of height 1 or 0.

If we suppose $a_2 b \neq b a_2$, and if we suppose moreover that $a_2 b$ and $b a_2$ are both of height 1, then we have the following properties:

a_2b and b_2a are distinct of $b(a_2b = b \Rightarrow a_2^n b = b = e)$; therefore $a_2b \wedge b = ba_2 \wedge b = e$, $a_2b \wedge ba_2 = e$ too, since a_2b and ba_2 are of height 1 and different. As the double equality $a_2b \vee b = ba_2 \vee b$, $a_2b \wedge b = ba_2 \wedge b$ implies $a_2b = ba_2$ in a distributive lattice, we necessarily have $a_2b \vee b \neq ba_2 \vee b$. Moreover a_2b and b cover $a_2b \wedge b = e$, then $a_2b \vee b$ covers a_2b and b; similarly $ba_2 \vee b$ covers ba_2 and b. So, $a_2b \vee b$ and $ba_2 \vee b$ are of height 2. And we finally obtain the diagram



Consequently, $a_2b \vee b$ and $ba_2 \vee b$ being of the same height 2 and incomparable, $a_2b \vee b \vee ba_2$ is of height $\geqslant 3$.

But $a_2b \vee b \leqslant a_2 \vee b$, $ba_2 \vee b \leqslant a_2 \vee b[a_2b \leqslant (a_2 \vee b)^2 \leqslant a_2 \vee b]$ and $a_2 \vee b$ is of height 3. (In a finite distributive lattice, $h[x] + h[y] = h[x \vee y] + h(x \wedge y])$. Therefore,

$$a_2b \vee b \vee ba_2 = a_2 \vee b = (a_2b \vee ba_2) \vee b.$$

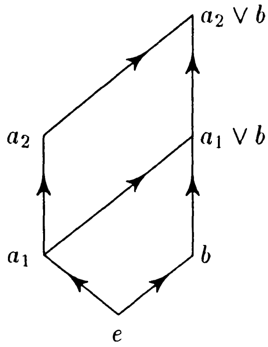
Elsewhere, $(a_2b \vee ba_2) \wedge b = (a_2b \wedge b) \vee (ba_2 \wedge b) = e = a_2 \wedge b$.

$$\text{And finally, we obtain } \begin{cases} (a_2b \vee ba_2) \vee b = a_2 \vee b \\ (a_2b \vee ba_2) \wedge b = a_2 \wedge b \end{cases}$$

and, as S is a distributive lattice $a_2 = a_2b \vee ba_2$. From $ba_2 \vee a_2b = a_2$, we deduce $ba_2b \vee a_2b^2 = a_2b$, and $ba_2b \vee e = a_2b = ba_2b$; now $a_2b = ba_2b$ implies $b^2(a_2b) = ba_2b = a_2b = e$, which is impossible. [$a_2b$ is of height 1).

Therefore $a_2b \neq ba_2$ implies that one of the two elements a_2b, ba_2 is of height 0, e.g. is e.

E x a m p l e. We built a finite weakly negative lattice ordered semigroup, which is a nilsemigroup (e.g. it is reduced to an unique spindle). The diagram of the order relation is the following:



If we put $a_2 b = a_1$, $ba_2 = e$, we obtain the following multiplication table, which is effectively the one of a semigroup

	e	a_1	a_2	b	$a_1 \vee b$	$a_2 \vee b$
e	e	e	e	e	e	e
a_1	e	e	e	e	e	e
a_2	e	e	e	a_1	a_1	a_1
b	e	e	e	e	e	e
$a_1 \vee b$	e	e	e	e	e	e
$a_2 \vee b$	e	e	e	a_1	a_1	a_1

Lemma 5. *Let S be a lattice ordered periodic semigroup. If e is a maximal idempotent among the idempotents, then e is the greatest of idempotents.*

Let e be a maximal idempotent and let f be in S so that $f = f^2$; $e \vee f \in S$ and $e \leqslant e \vee f$. As $e^n = e$ for all integers n, $e \leqslant (e \vee f)^n$ too. As S is a periodic semigroup, there exists $p \in \mathbb{N}^*$ so that $(e \vee f)^p$ is idempotent and $e = (e \vee f)^p$. If we develop the product $(e \vee f)^p$ we find an expression of the type $e \vee f \vee x$ and consequently $e \vee f \vee x = e \geqslant f$.

Corollary 1. *Let S be a lattice ordered periodic semigroup. If e is a maximal idempotent, among the idempotents, then ef and fe are idempotents, for any idempotent f of S.*

From lemma 5, we deduce $f \leqslant e$ for every idempotent f. And it is well known that if two idempotents are comparable, their product is an idempotent.

N o t a t i o n . In the following we say that b covers a (and we note $b \succ a$ (or $a \prec b$)) if there is no such element c that $a \lneqq c \lneqq b$.

Lemma 6. *Let S be a finite weakly negative lattice ordered semigroup and let e be the greatest idempotent of S.*

If $f = f^2$ and if $f \prec e$ (in the ordered subset of idempotents), then for all integers k, $k \neq 0$, and for all b in F_f $be \leqslant e$, $eb \leqslant e$, and $(e \vee b)^k = e \vee b^k$.

P r o o f. For some integer $n \in \mathbf{N}^*$, $(e \vee b)^n = e$; from this equality we deduce $e = e \vee b^n \vee eb \vee be \vee y$, $y \in S$, and we obtain $eb \leqslant e$, $be \leqslant e$ and $(e \vee b)^k = e \vee b^k$. □

N o t a t i o n. If F_e and F_f are two spindles, we put $F_f < F_e$ if: $\forall x \in F_f$, $\forall y \in F_e \Rightarrow x < y$.

Theorem 3. *Let S be a weakly negative lattice ordered periodic semigroup. Let e and f be two idempotents such that e covers f in the ordered subset of idempotents, $F_f < F_e$, and $(F_f)^2 \neq \{f\}$.*

Then $ef = e$ if and only if $fe = e$ and in this case, $F_e F_f = F_f F_e = e$.

P r o o f. Suppose for example that $ef = e$. If $a \in F_f$, and if $b \in F_e$, from the hypothesis and from Lemma 1, we deduce $f \leqslant a \lneqq e \leqslant b$. Consequently, we obtain $ef = e \leqslant ba \leqslant be = e$ and $ba = e$.

And we have $F_e F_f = e$. Moreover, as $f < e$, fe is an idempotent between e and f and as e covers f, $fe = e$ or $fe = f$.

We suppose now that $fe = e$. Let be $x \in F_f$; $f \leqslant x < e$. Then $f \leqslant x^2 \leqslant xe \leqslant e$, $f \leqslant (xe)^k \leqslant e$ for each integer k.

As $f \prec e$ (in the ordered subset of idempotents) and as $F_f < F_e$, $xe \in F_e$ or $xe \in F_f$. If $xe = a \in F_e$, we have $xe^2 = ae = xe = e$. But, from $xe = e$, it results $fe = e$, which is not. Therefore, $xe = y \in F_f$ and we obtain $(xe)(xe) = y^2 = x(ex)e = xe = y$ since $F_e F_f = e$. But f is the idempotent of F_f and $y = f$, and finally we obtain $F_f \cdot e = f$. As we have supposed $(F_f)^2 \neq \{f\}$, there exists two elements r and s of F_f so that $f \lneqq r \lneqq e$, $f \lneqq s \lneqq e$ with $f \neq rs$. By isotony, we obtain

$$f = fs \leqslant rs \leqslant re = f. \quad \text{Contradiction.}$$

So $ef = e$ implies $fe = e$, and $F_e F_f = F_f F_e = e$. Conversely, if $fe = e$ we obtain $ef = e$ by symmetry. □

Theorem 4. *Let S be a weakly negative lattice ordered periodic semigroup. Let e and f be two such idempotents that e covers f (in the ordered subset of idempotents) and $F_f < F_e$.*

Then, F_f is a nilsemigroup, with f as zero.

P r o o f. If $\{F_f)\}^2 = f$, it is trivial.

If $\{F_f\}^2 \neq f$, we can apply Theorem 3.

Let x and y be two elements of F_f: $f \leqslant x \lneqq e$, $f \leqslant y \lneqq e$. □

Therefore $f \leqslant xy \leqslant e$, $f \leqslant (xy)^n \leqslant e$ for any integer n, and $xy \in F_f \cup F_e$. If $xy \in F_e$, $xy = e$, because e is the least element of F_e. If $x^n = y^n = f$, we have $f = x^{n+1}y^{n+1} = x^n e \cdot y^n = fef$. Consequently, $ef = e = fe$ is impossible and necessarily, $ef = f = fe$. But from $x < e$, $y < e$, we deduce, by isotony, $xy \leqslant ey \leqslant e^2 = e$, and $xy = e$ implies $ey = e$, $ef = e$ $(= fe)$. Contradiction.

So, xy belongs to F_f, which is a subsemigroup of S, and of course a nilsemigroup of zero f.

Remark. With the same hypothesis, as in theorem 4, if $(F_f)^2 = \{f\}$ it is possible to have $ef \neq fe$. We can give an example.

S	f	b	b'	e	a^2	a
f	f	f	f	f	f	f
b	f	f	f	f	f	f
b'	f	f	f	f	f	f
e	e	e	e	e	e	e
a^2	e	e	e	e	e	e
a	e	e	e	e	e	a^2

ordered by $f < b < b' < e < a^2 < a$.

3. Construction of periodic weakly negative lattice ordered semigroups

Let $F_1, F_2, \ldots, F_n$ be n nilsemigroups whose zeros are respectively $e_1, e_2, \ldots, e_n$. Suppose each F_i is a weakly negative lattice, ordered by order relation $\leqslant$ and e_i is the least element of each F_i. We put $S = \bigcup_{i=1}^{n} F_i$ and we define in S the product $x_i \cdot y_j$ where $x_i \in F_i$, $y_j \in F_j$ by

$$\begin{aligned} x_i \cdot y_j = x_i y_j &= \text{ product of } x_i \text{ and } y_j \text{ in } F_i \text{ if } i = j \\ &= e_j \text{ if } i < j \\ &= e_i \text{ if } j < i. \end{aligned}$$

In particular, $e_i e_j = e_j e_i = e_j$ if $i < j$
$= e_i$ if $j < i$.
Then we define on S an order relation by

$$x_i \leqslant y_j \Leftrightarrow i = j \text{ and } x_i \leqslant y_j \text{ in } F_i \text{ or } i < j$$

$(S, ., \leqslant)$ becomes an ordered semigroup. It is easy to see that $x_i \cdot (y_j \cdot z_k) = (x_i \cdot y_j) \cdot z_k = x_i y_j z_k$ if $i = j = k$ and that $x_i \cdot (y_j \cdot z_k) = x_i \cdot (y_j \cdot z_k) = e_{\sup(i,j,k)}$ if the cardinality of $\{i, j, k\}$ is greater that 2. In each F_i, $e_i \leqslant x$ and $x_i^2 \leqslant x_i$ by hypothesis. So S is a weakly negative lattice ordered periodic semigroup,

Conversely, suppose that S is a periodic weakly negative lattice ordered semi-group and that moreover, if F_{e_1}, F_{e_2}, ..., F_{e_n} design the spindles of S, $F_{e_1} < F_{e_2} < F_{e_3} \ldots < F_{e_n}$. We also suppose that $e_{i+1} e_i = e_{i+1} e_i = e_{i+1}$ for $i = 1, 2, \ldots, n-1$.

Then $F_{e_i} \cdot F_{e_j} = e_j$ if $e_i < e_j$ for all (i, j), $i \neq j$
$= e_i$ if $e_j < e_i$ for all (i, j), $i \neq j$.

In Theorem 3, we see that $e_i \prec e_{i+1}$, $F_{e_i} < F_{e_{i+1}}$, and $e_i e_{i+1} = e_{i+1} = e_{i+1} e_i$ implies $F_{e_i} F_{e_{i+1}} = F_{e_{i+1}} = F_{e_{i+1}} F_{e_i}$.

Now we calculate $F_{e_i} F_{e_k}$ with $i < k$:

$$\begin{aligned} F_{e_i} F_{e_k} \geqslant F_{e_i \cdot e_k} &= F_{e_i} \cdot (e_k)^{k-i+1} \\ &\geqslant F_{e_i} \cdot e_i e_{i+1} \ldots e_k \\ &= e_i e_{i+1} \ldots e_k = e_k. \end{aligned}$$

But $F_{e_i} F_{e_k} \leqslant e_k \cdot F e_k = e_k$.

So $F_{e_i} F_{e_k} = e_k$, and similarly $F_{e_k} F_{e_i} = e_k$ if $i < k$. So, we have

Theorem 5. *Let S be the union of n weakly negative lattice ordered nilsemigroups F_{e_i}; S becomes a weakly negative ordered periodic semigroup with the properties $F_{e_1} < F_{e_2} < \ldots < F_{e_n}$, $e_i e_{i+1} = e_{i+1} e_i = e_{i+1}$ for $i = 1, 2, \ldots, n-1$, if and only if $F_{e_i} F_{e_j} = e_j$ for $i < j$ and $F_{e_i} \cdot F_{e_j} = e_i$ for $j < i$.*

References

[1] *G. Birkhoff*: Lattice theory, AMS, Providence, 1967.
[2] *Clifford A. H. and G. B. Preston*: The algebraic theory of semigroups. vol I, AMS, Providence, 1961.
[3] *Merlier Th.*: Sur les bandes réticulées, Semigroup Forum *22* (1981), 191–198.
[4] *Merlier Th.*: Sur les demi-groupes réticulés et les o.demi.groupes, Semigroup Forum *2* (1971), 64–70.
[5] *Merlier Th.*: Some properties of lattice ordered Rees Matrix semigroups, Czech. Math. Jour. *38 (113)* no. 4 (1988), 573–577.

[6] *Merlier Th.*: Totally orderable semigroups and locally finite semigroups, Colloq. Math. Soci. Janos Bolyai. 20 Algebraic theory of semigroups SZEGED, 1976.

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