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RELATIONS AND TOPOLOGIES

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Recently, ternary relations have been intensively studied by V. Novák and M. Novotný in a series of papers. In [6] these authors dealt with the problem of representation of ternary relations on a set $G$ by binary relations on $G^3$. In [7] M. Novotný represented the former ones by grupoids on the power set of $G$. In this note we investigate the possibility of representing ternary relations on a given set by topologies in Čech’s sense on this set. First of all, however, with any relation of type $\alpha$ ($\alpha$ an ordinal) we associate a topology and study these associated topologies.

1. PRELIMINARIES

Following the convention introduced by J. von Neumann, we identify ordinals with the set of their predecessors, and cardinals with their initial ordinals (see e.g. [1]).

Let $\alpha$ be an ordinal and $G$ a set. By a relation of type $\alpha$ on $G$ we understand any subset $R \subseteq G^\alpha$ (where $G^\alpha$ denotes the set of all mappings of $\alpha$ into $G$). The set $G$ is then called the underlying set of $R$. In other words, a relation of type $\alpha$ on $G$ is a set of sequences of type $\alpha$ consisting of elements of $G$. These relations are introduced and studied in [9]. The ordered pair $(G, R)$ where $G$ is a set and $R$ is a relation of type $\alpha$ on $G$ is said to be a relational system of type $\alpha$. Given two relational systems $(G, R), (H, S)$ of type $\alpha$, a homomorphism of $(G, R)$ into $(H, S)$ is any mapping $f : G \to H$ such that $(x_i \mid i < \alpha) \in R \Rightarrow (f(x_i) \mid i < \alpha) \in S$.

We shall use some fundamental concepts of the category theory—they can be found e.g. in [5]. For an ordinal $\alpha$, by $\text{Rel}_\alpha$ we denote the category of relational systems of type $\alpha$ with homomorphisms as morphisms.

By a topology on a set $G$ we mean a topology in Čech’s sense [3], i.e. a mapping $u : \exp G \to \exp G$ satisfying $u\emptyset = \emptyset$, $X \subseteq G \Rightarrow X \subseteq uX$, and $X \subseteq Y \subseteq G \Rightarrow uX \subseteq uY$. The ordered pair $(G, u)$ is then called a topological space and the elements of $G$ are called points. If $(G, u)$ is a topological space and $x \in G$ a point, then a set $X \subseteq G$ is
said to be a neighborhood of \( x \) if \( x \notin u(G - X) \). A complete system of neighborhoods of \( x \) is any system \( \mathcal{T}(x) \) of neighborhoods of \( x \) such that for each neighborhood \( X \) of \( x \) there exists \( Y \in \mathcal{T}(x) \) with \( Y \subseteq X \). The topology \( u \) is uniquely determined by complete systems of neighborhoods \( \mathcal{T}(x) \) of all \( x \in G \) because for any subset \( Z \subseteq G \) and any point \( x \in G \) we have \( x \in uZ \Leftrightarrow Z \cap X \neq \emptyset \) for every \( X \in \mathcal{T}(x) \).

According to [3], for a topology \( u \) on a set \( G \) we consider the following two axioms:

\[
X \subseteq G \Rightarrow uuX = uX \quad \text{F-axiom},
\]

\[
X, Y \subseteq G \Rightarrow u(X \cup Y) = uX \cup uY \quad \text{A-axiom}.
\]

Next, for any cardinal \( n > 1 \) we introduce another axiom:

\[
X \subseteq G \Rightarrow uX = \bigcup \{uA \mid A \subseteq X, \, \text{card } A < n\} \quad \text{S}_n\text{-axiom}.
\]

A topology fulfilling a \( \lambda \)-axiom \( (\lambda \in \{F, A, S_n\}) \) is said to be a \( \lambda \)-topology, a topology fulfilling both a \( \lambda \)-axiom and a \( \mu \)-axiom is said to be a \( \lambda \mu \)-topology. The topologies of Bourbaki [2], most frequently understood under topologies in literature, are exactly the \( FA \)-topologies. In [4], \( S_2 \)-topologies are called quasi-discrete closure operations and it is shown that they coincide with reflexive binary relations. Of course, any \( S_2 \)-topology is an \( A \)-topology. Next, any \( S_n \)-topology is an \( S_m \)-topology whenever \( m > n \). Since any topology \( u \) on a set \( G \) is obviously an \( S_n \)-topology for each cardinal \( n \) with \( n > \text{card } G \), there exists a least cardinal \( n \) for which \( u \) is an \( S_n \)-topology. In [3] this cardinal is mentioned as an important invariant of the topology \( u \). Clearly, if \( n \leq \omega_0 \), then any \( AS_n \)-topology is an \( S_2 \)-topology. Finally, let us note that we need the axiom of choice whenever we consider the \( S_n \)-axiom with \( n > \omega_1 \).

**Lemma 1.** Let \( n > 1 \) be a cardinal and let \( u \) be an \( S_n \)-topology on a set \( G \). For any \( x \in G \) put \( \mathcal{T}(x) = \{X \subseteq G \mid \text{for each subset } A \subseteq G \text{ fulfilling card } A < n \text{ and } x \in uA \text{ there exists a non-empty subset } X_A \subseteq A \text{ such that } X = \bigcup \{X_A \mid A \subseteq G, \text{card } A < n, x \in uA\}\} \). Then \( \mathcal{T}(x) \) is a complete system of neighborhoods of \( x \) in \((G, u)\).

**Proof.** As \( u \) is an \( S_n \)-topology, a subset \( X \subseteq G \) is a neighborhood of \( x \) in \((G, u)\) iff \( x \in uA \Rightarrow A \cap X \neq \emptyset \) for each subset \( A \subseteq G \) with \( \text{card } A < n \). Therefore \( \mathcal{T}(x) \) is a system of neighborhoods of \( x \) in \((G, u)\). Let \( Y \subseteq G \) be an arbitrary neighborhood of \( x \) in \((G, u)\). For any subset \( A \subseteq G \) with \( \text{card } A < n \) and \( x \in uA \) put \( X_A = A \cap Y \) \( (\neq \emptyset) \) and \( Z = \bigcup \{X_A \mid A \subseteq G, \text{card } A < n, x \in uA\} \). Then clearly \( Z \subseteq Y \) and \( Z \in \mathcal{T}(x) \). The lemma is proved.

For any two topologies \( u, v \) on a set \( G \) we put \( u \preceq v \) iff \( uX \subseteq vX \) for each \( X \subseteq G \). If \((G, u), (H, v)\) are topological spaces, then a continuous mapping of \((G, u)\)
into \((H, v)\) is any mapping \(f: G \to H\) fulfilling \(f(uX) \subseteq vf(X)\) whenever \(X \subseteq G\). By \(\text{Top}\) we denote the category of topological spaces with continuous mappings as morphisms.

2. Topologies Associated with relations of type \(\alpha\)

Throughout this section, \(\alpha\) denotes an ordinal with \(\alpha > 1\), and \(|\alpha|\) denotes the least cardinal fulfilling \(|\alpha| \geq \alpha\).

Let \(R\) be a relation of type \(\alpha\) on a set \(G\). Then for any subset \(X \subseteq G\) we put

\[
u_R X = X \cup \{x \in G \mid \text{there exist } (x_i \mid i < \alpha) \in R \text{ and an ordinal } i_0, 0 < i_0 < \alpha, \text{ such that } x = x_{i_0} \text{ and } x_i \in X \text{ for all } i < i_0\}.
\]

Clearly, \(\nu_R\) is a topology on \(G\). For relations \(R\) of type 2 the topologies \(\nu_R\) coincide with those dealt with in [8]. In the next section we shall study topologies \(\nu_R\) for relations of the particular type 3.

For any object \((G, R)\) of \(\text{Rel}_\alpha\) we put \(F_\alpha(G, R) = (G, \nu_R)\) and for any morphism \(f\) in \(\text{Rel}_\alpha\) we put \(F_\alpha f = f\).

**Theorem 1.** \(F_\alpha\) is a faithful (covariant) functor from \(\text{Rel}_\alpha\) into \(\text{Top}\).

**Proof.** Let \((G, R), (H, S)\) be relational systems of type \(\alpha\) and let \(f: G \to H\) be a homomorphism of \((G, R)\) into \((H, S)\). Let \(X \subseteq G\) be a subset and let \(y \in f(u_RX)\). Then there exists \(x \in u_R X\) with \(y = f(x)\). If \(x \in X\), then \(y \in f(X) \subseteq u_S f(X)\).

Suppose \(x \notin X\). Then there exist \((x_i \mid i < \alpha) \in R\) and an ordinal \(i_0, 0 < i_0 < \alpha\), such that \(x = x_{i_0}\) and \(x_i \in X\) for all \(i < i_0\). Consequently, there exist \((f(x_i) \mid i < \alpha) \in S\) and an ordinal \(i_0, 0 < i_0 < \alpha\), such that \(y = f(x_{i_0})\) and \(f(x_i) \in f(X)\) for all \(i < i_0\). Hence \(y \in u_S f(X)\) and the inclusion \(f(u_RX) \subseteq u_S f(X)\) is valid. Therefore \(f\) is a continuous mapping of the space \((G, u_R)\) into \((H, u_S)\). Thus \(F_\alpha\) is a covariant functor from \(\text{Rel}_\alpha\) into \(\text{Top}\) and it is evident that \(F_\alpha f = f\). \(\square\)

**Proposition 1.** Let \(R\) be a relation of type \(\alpha\) and let \(n = |\alpha|\). Then \(u_R\) is an \(S_n\)-topology.

**Proof.** Let \(G\) be the underlying set of \(R\) and let \(X \subseteq G\) be a subset. Let \(x \in u_R X\) be a point. If \(x \in X\), then \(x \in u_R \{x\} \subseteq \bigcup\{u_RA \mid A \subseteq X, \text{card } A < n\}\).

Suppose \(x \notin X\). Then there exist \((x_i \mid i < \alpha) \in R\) and an ordinal \(i_0, 0 < i_0 < \alpha\), such that \(x = x_{i_0}\) and \(x_i \in X\) for all \(i < i_0\). Hence \(\{x_i \mid i < i_0\} \subseteq X\) and \(\text{card} \{x_i \mid i < i_0\} \leq \text{card } i_0\). But \(\text{card } i_0 < \alpha = n\) whenever \(\alpha\) is a cardinal and \(\text{card } i_0 \leq \text{card } \alpha < n\) whenever \(\alpha\) is not a cardinal. Since \(x \in u_R \{x_i \mid i < i_0\}\), we have \(x \in \bigcup\{u_RA \mid A \subseteq X, \text{card } A < n\}\). Therefore the inclusion \(u_R X \subseteq \bigcup\{u_RA \mid A \subseteq X, \text{card } A < n\}\) is valid and the proof is complete. \(\square\)
By virtue of Proposition 1 one can use Lemma 1 for defining $u_R$ by determining complete systems of neighborhoods of all points of $G$.

For any topology $u$ on a set $G$ we denote by $R_u$ the relation of type $\alpha$ on $G$ defined as follows:

$$(x_i \mid i < \alpha) \in R_u \iff x_{i_0} \in u\{x_i \mid i < i_0\} \text{ for any } i_0 \text{ with } 0 < i_0 < \alpha.$$ 

Obviously, for each relation $S$ of type $\alpha$ we have $S \subseteq R_u$.

**Proposition 2.** For any topology $v$ we have $u_{R_v} \leq v$.

**Proof.** Let $v$ be a topology on a set $G$, let $X \subseteq G$ be a subset and $x \in u_{R_v} X$ a point. If $x \in X$, then $x \in vX$. Suppose $x \notin X$. Then there exist $(x_i \mid i < \alpha) \in R_v$ and an ordinal $i_0$, $0 < i_0 < \alpha$, such that $x = x_{i_0}$ and $x_i \in X$ for all $i < i_0$. Consequently, $x \in v\{x_i \mid i < i_0\} \subseteq vX$. Thus $u_{R_v} X \subseteq vX$ and the assertion is proved.

**Theorem 2.** Let $v$ be an $S_n$-topology on a set $G$ where $n = |\alpha|$. Then the following conditions are equivalent:

(i) $v = u_{R_v}$,

(ii) there exists a relation $R$ of type $\alpha$ on $G$ such that $v = u_R$,

(iii) if $\alpha > 2$ and $X \subseteq G$ is a subset with $1 < \text{card } X = m < n$, then for any point $x \in vX - \bigcup\{vA \mid A \subseteq X, \text{card } A < m\}$ there exists a subset $Y \subseteq X$ with $\text{card } Y = m$ such that both $x \in vY$ and there exists a bijection $p: m \rightarrow Y$ with the property $p(j) \in v\{p(i) \mid i < j\}$ whenever $0 < j < m$.

**Proof.** For $\alpha = 2$ the assertion states that $v = u_{R_v}$, which is well known—see [4]. Suppose $\alpha > 2$.

The implication (i) $\Rightarrow$ (ii) is trivial.

Let the condition (ii) be true. Let $X \subseteq G$ be a subset with $1 < \text{card } X = m < n$ and let $x \in vX - \bigcup\{vA \mid A \subseteq X, \text{card } A < m\}$. Then obviously $x \notin X$. Thus $x \in u_R X - X$ and $x \notin u_RA$ for any subset $A \subseteq X$ with $\text{card } A < m$. Consequently, there exist $(x_i \mid i < \alpha) \in R$ and an ordinal $i_0$, $0 < i_0 < \alpha$, such that $x = x_{i_0}$ and $x_i \in X$ for all $i < i_0$. Put $Y = \{x_i \mid i < i_0\}$. Then $x \in vY$ and since $Y \subseteq X$, we have $\text{card } Y = m$ and $x_j \in v\{x_i \mid i < j\}$ whenever $0 < j < i_0$. Put $p(0) = x_0$ and for each ordinal $j$ with $0 < j < m$, having defined $p(i)$ for all $i < j$, put $p(j) = x_k$ where $k < i_0$ is the least ordinal such that $x_k \neq p(i)$ for all $i < j$. Then clearly $p: m \rightarrow Y$ is a bijection with the property $p(j) \in v\{p(i) \mid i < j\}$ whenever $0 < j < m$. The implication (ii) $\Rightarrow$ (iii) is proved.

Let (iii) be true. Let $X \subseteq G$ be a set with $1 < \text{card } X < n$ and let $y \in vX$. If there exists a point $x \in X$ such that $y \in v\{x\}$, then putting $x_0 = x$ and $x_i = y$
for all ordinals $i$ with $0 < i < \alpha$ we get $(x_i \mid i < \alpha) \in R_v$. Hence $y \in u_{R_v}{x} \subseteq u_{R_v}X$. Suppose $y \notin v{x}$ for every $x \in X$. Let $Y \subseteq X$ be a subset with the minimal cardinality such that $y \in vY$. Denote $\text{card} \ Y = m$. Since $y \notin \bigcup\{vA \mid A \subseteq Y, \text{card} \ A < m\}$, there exists a subset $Z \subseteq Y$ with $\text{card} \ Z = m$ such that both $y \in vZ$ and there exists a bijection $p: m \to Z$ with the property $p(j) = v(p(i) \mid i < j)$ whenever $0 < j < m$. For each $i < m$ put $x_i = p(i)$ and for each $i$ with $m \leq i < \alpha$ put $x_i = y$. Then $(x_i \mid i < \alpha) \in R_v$ and consequently $y \in u_{R_v}{p(i) \mid i < m} = u_{R_v}Z \subseteq u_{R_v}X$. We have proved the inclusion $vX \subseteq u_{R_v}X$. Thus, as $v$ and $u_{R_v}$ are $S_{\alpha}$-topologies, we have $v \leq u_{R_v}$. Since the inverse inclusion is valid according to Proposition 2, the equality $v = u_{R_v}$ is true. Therefore (iii) $\Rightarrow$ (i) and the proof is complete.

Theorem 2 immediately results in

**Corollary 1.** Let $S, T$ be relations of type $\alpha$ on a given set. Then $u_S \neq u_T$ iff $R_{u_S} \neq R_{u_T}$.

The following proposition is obvious:

**Proposition 3.** Let $R$ be a relation of type $\alpha$. Then $u_R$ is an $S_2$-topology iff for every $(x_i \mid i < \alpha) \in R$ the following condition is true:

for any ordinal $i_1$, $1 < i_1 < \alpha$, with the property that $x_i \neq x_{i_1}$ for all $i < i_1$ there exist $(y_j \mid j < \alpha) \in R$ and ordinals $i_0$, $j_0$, $i_0 < i_1$, $0 < j_0 < \alpha$, such that $y_{j_0} = x_{i_1}$ and $y_j = x_{i_0}$ for all $j < j_0$.

**Proposition 4.** Let $R$ be a relation of type $\alpha$ and let $\alpha \leq \omega_0$. If $u_R$ is an $F$-topology, then $u_R$ is an $S_2$-topology.

**Proof.** Let $G$ be the underlying set of $R$ and let $X \subseteq G$ be a subset with card $X < \alpha$. We are to prove that $u_{R_X}X \subseteq \bigcup_{x \in X} u_R{x}$. To this end, let $y \in u_{R_X}X$ be a point. If $y \in X$, then $y \in u_R{y} \subseteq \bigcup_{x \in X} u_R{x}$. Let $y \notin X$. Then there exist $(x_i \mid i < \alpha) \in R$ and an ordinal $i_0$, $0 < i_0 < \alpha$, such that $y = x_{i_0}$ and $x_i \in X$ for all $i < i_0$. Consequently, $y \in u_R{x_0, \ldots, x_{i_0-1}}$ and $(x_0, \ldots, x_j) \subseteq u_R{x_0, \ldots, x_{j-1}}$ whenever $1 \leq j \leq i_0 - 1$. Hence, we have $y \in u_R{x_0, \ldots, x_{i_0-1}} \subseteq u_Ru_R{x_0, \ldots, x_{i_0-2}} = u_R{x_0, \ldots, x_{i_0-2}} \subseteq u_Ru_R{x_0, \ldots, x_{i_0-3}} = u_R{x_0, \ldots, x_{i_0-3}} \subseteq \ldots = u_R{x_0}$. Thus $y \in \bigcup_{x \in X} u_R{x}$ and the inclusion $u_{R_X}X \subseteq \bigcup_{x \in X} u_R{x}$ is proved.  

\[ \square \]
3. Topologies associated with ternary relations

In this section we further investigate topologies associated with relations of type $\alpha$ for the special case $\alpha = 3$, i.e. for ternary relations. Clearly, for any ternary relation $R$ on a set $G$ the $S_3$-topology $u_R$ on $G$ is given by

$$X \subseteq G \Rightarrow u_RX = X \cup \{z \in G \mid \exists x \in X, y \in G: (x, z, y) \in R \lor \exists x, y \in X: (x, y, z) \in R\}.$$ 

In [6] and [7] some properties of ternary relations are introduced and subsequently used such as symmetry, asymmetry, antisymmetry, transitivity and cyclicity. However, among them only the cyclicity is useful for the study of topologies associated with ternary relations. For our purpose we introduce several other properties of ternary relations:

A ternary relation $R$ on a set $G$ will be called

- **irreflexive** if $(x, y, z) \in R \Rightarrow y \neq x \neq z$,
- **irreversible** if $(x, y, z) \in R \Rightarrow (y, x, p) \notin R$ for any $p \in G$,
- **fleebly regular** if $(x, y, p) \in R, (y, z, q) \in R \Rightarrow (x, y, z) \in R$,
- **regular** if $R$ is fleebly regular and $(x, y, p) \in R, (x, z, q) \in R \Rightarrow (x, y, z) \in R$,
- **fleebly translative** if $(x, y, z) \in R, (y, p, q) \in R \Rightarrow \exists r \in G: (x, p, r) \in R$,
- **translative** if $R$ is fleebly translative and $(x, y, z) \in R \Rightarrow \exists r \in G: (x, z, r) \in R$ or $(y, z, r) \in R$,
- **cyclic** if $(x, y, z) \in R \Rightarrow (y, z, x) \in R$.

**Theorem 3.** Let $(G, R), (H, S)$ be ternary relational systems and $f: G \rightarrow H$ a continuous injective mapping of the space $(G, u_R)$ into $(H, u_S)$. If $R$ is irreflexive and $S$ is both irreflexive and regular, then $f$ is a homomorphism of $(G, R)$ into $(H, S)$.

**Proof.** Let $(x, y, z) \in R$. Then $y \in u_R\{x\}$ and $z \in u_R\{x, y\}$. Hence $f(y) \in u_S\{f(x)\}$ and $f(z) \in u_S\{f(x), f(y)\}$. As $R$ is irreflexive and $f$ is injective, we have $f(y) \neq f(x) \neq f(z)$. Thus, by virtue of the irreflexivity of $S$, it follows from $f(y) \in u_S\{f(x)\}$ that there exists $p \in H$ with $(f(x), f(y), p) \in S$. If $f(y) = f(z)$, then the regularity of $S$ implies $(f(x), f(y), f(z)) \in S$. Suppose $f(y) \neq f(z)$. As $S$ is irreflexive, it follows from $f(z) \in u_S\{f(x), f(y)\}$ that $(f(x), f(y), f(z)) \in S$ or $(f(y), f(x), f(z)) \in S$ or there exists $q \in H$ such that $(f(x), f(z), q) \in S$ or there exists $r \in H$ such that $(f(y), f(z), r) \in S$. If $(f(y), f(x), f(z)) \in S$, then the regularity of $S$ yields $(f(x), f(y), f(z)) \in S$, which is a contradiction with the irreflexivity of $S$. In both cases $(f(x), f(z), q) \in S$ and $(f(y), f(z), r) \in S$ the regularity of $S$ results in $(f(x), f(y), f(z)) \in S$. Thus, $(f(x), f(y), f(z)) \in S$ always holds and the statement is proved. \qed
Corollary 2. Let \( R, S \) be ternary relations on a given set. If \( R \) is irreflexive and \( S \) is both irreflexive and regular, then the equivalence \( R \subseteq S \Leftrightarrow u_R \leq u_S \) is valid.

Proof. Let \( G \) be a set and let \( \alpha \) be an ordinal. Then clearly \( \subseteq \) and \( \leq \) are orderings of the fibres of \( G \) in \( \text{Rel}_\alpha \) and \( \text{Top} \), respectively, i.e. for any two relations \( R, S \) of type \( \alpha \) (topologies \( u, v \)) on \( G \) we have \( R \subseteq S \) (\( u \leq v \)) iff the identity mapping of \( G \) is a homomorphism (continuous mapping) of \( (G, R) \) into \( (G, S) \) (of \( (G, u) \) into \( (G, v) \)). By this fact, the corollary follows from Theorems 1 and 3. \( \square \)

Corollary 3. Let \( R, S \) be irreflexive and regular ternary relations on a given set. Then \( R \neq S \) implies \( u_R \neq u_S \).

Proof. Let \( G \) be the underlying set of both \( R \) and \( S \) and let \( \text{id}_G \) denote the identity mapping of \( G \). Assume \( u_R = u_S \). Then \( \text{id}_G \) is a continuous mapping of \( (G, u_R) \) into \( (G, u_S) \) as well as of \( (G, u_\ast) \) into \( (G, u_R) \). Thus, by Theorem 3, \( \text{id}_G \) is a homomorphism of \( (G, R) \) into \( (G, S) \) as well as of \( (G, S) \) into \( (G, R) \). Therefore \( R \subseteq S \) and \( S \subseteq R \), i.e. \( R = S \). This proves the statement. \( \square \)

We denote by \( \text{Rel}_3 \) the full subcategory of \( \text{Rel}_3 \) whose objects are precisely the ternary relational systems \( (G, R) \) for which \( R \) is both irreflexive and regular. As a consequence of Theorem 1 and Corollary 3 we get

Corollary 4. The functor \( F_3 \) (restricted onto \( \text{Rel}_3 \)) is an embedding of the category \( \text{Rel}_3 \) into \( \text{Top} \).

Next, for any set \( G \) we denote by \( \mathcal{R}(G) \) and \( \mathcal{U}(G) \) the fibres of \( G \) in \( \text{Rel}_3 \) and \( \text{Top} \), respectively, i.e. \( \mathcal{R}(G) \) is the set of all irreflexive and regular ternary relations on \( G \) ordered by the set inclusion and \( \mathcal{U}(G) \) is the set of all topologies on \( G \) ordered by \( \subseteq \). Then Corollaries 2 and 3 result in

Corollary 5. For any set \( G \) the correspondence \( R \mapsto u_R \) defines an embedding of the ordered set \( \mathcal{R}(G) \) into \( \mathcal{U}(G) \).

Remark. By Theorem 2, for any \( S_3 \)-topology \( u \) on a set \( G \) there exists a ternary relation \( R \) on \( G \) such that \( u = u_R \) if and only if the following condition is fulfilled:

for any two points \( x, y \in G \) the implication \( u\{x, y\} \neq u\{x\} \cup u\{y\} \Rightarrow x \in u\{y\} \) or \( y \in u\{x\} \) is valid.

Thus, if we denote by \( \text{Top}_3 \) the full subcategory of \( \text{Top} \) whose objects precisely the topological spaces \( (G, u) \) for which \( u \) is an \( S_3 \)-topology fulfilling the condition mentioned above, then Corollary 4 is still valid when \( \text{Top} \) is replaced by \( \text{Top}_3 \). Similarly, in Corollary 5 we can consider the fibre \( \mathcal{U}(G) \) of \( G \) in \( \text{Top}_3 \) instead of in \( \text{Top} \).
Example 1. Let \( g \) be an asymmetric binary relation on a set \( G \). Let us define
\[
R_e \subseteq G^3
\]
as follows:
\[
(x, y, z) \in R_e \iff xgy \text{ and either } xgz \text{ or } ygz.
\]
Then \( R_e \) is an irreflexive and regular (and irreversible) ternary relation on \( G \). In particular, if \( < \) is a strict order (i.e. an asymmetric and transitive binary relation) on \( G \), then \((x, y, z) \in R_< \) iff \( x < y \) and \( x < z \), i.e. iff \( x \) is a lower bound of the set \( \{y, z\} \).

Obviously, if \( R \) is a cyclic ternary relation, then \( u_R \) is an \( S_2 \)-topology. Unfortunately, if a non-empty cyclic ternary relation is regular, then it is not irreflexive (because any regular ternary relation \( R \) fulfills \((x, y, z) \in R \iff (x, y, y) \in R \)). Therefore Corollary 3 can not be applied to cyclic ternary relations. Nevertheless, we have

**Proposition 5.** Let \( R, S \) be irreflexive, feebly regular and cyclic ternary relations on a given set. Then \( R \neq S \) implies \( u_R \neq u_S \).

**Proof.** Let \( G \) be the underlying set of both \( R \) and \( S \). Assume \( R \neq S \). Without loss of generality we can suppose that \( R \nsubseteq S \). Then there exists \((x, y, z) \in R \) such that \((x, y, z) \notin S \). Suppose \( u_R = u_S \). Since \( y \in u_R\{x\} \) and \( z \in u_R\{y\} \), we have \( y \in u_S\{x\} \) and \( z \in u_S\{y\} \). The irreflexivity and cyclicity of \( R \) imply \( x \neq y \neq z \). Therefore there exist \( p, q \in G \) with \((x, y, p) \in S \) and \((y, z, q) \in S \). Thus, by the feebly regularity of \( S \), we have \((x, y, z) \in S \). But this is a contradiction. Hence \( u_R \neq u_S \).

Example 2. Let \( G = \{x, y, z, p, q\} \) and let \( R \subseteq G^3 \) be defined as follows:
\[
R = \{(x, y, z), (x, y, p), (y, p, q), (y, z, q)\}.
\]
Denote by \( R^c \) the cyclic hull of \( R \), i.e. the least (w.r.t. the set inclusion) cyclic ternary relation on \( G \) fulfilling \( R \subseteq R^c \). Then \( R^c \) is an irreflexive, feebly regular and cyclic ternary relation on \( G \).

**Theorem 4.** Let \( R \) be a ternary relation. If \( R \) is translative, then \( u_R \) is an \( F \)-topology.

**Proof.** Let \( G \) be the underlying set of \( R \) and let \( R \) be translatative. As \( u_R \) is evidently and \( S_2 \)-topology, \( u_R \) is an \( F \)-topology iff \( u_R u_R \{x\} \subseteq u_R \{x\} \) holds for any \( x \in G \). Let \( x \in G \), \( y \in u_R u_R \{x\} \) and suppose \( y \notin u_R \{x\} \). Then there exists \( p \in u_R \{x\} \) with \( y \in u_R \{p\} \). Because \( x \neq p \neq y \), we have \((x, x, p) \in R \) or there exists \( q \in G \) such that \((x, p, q) \in R \), and either \((p, p, y) \in R \) holds or there exists \( r \in G \) such that \((p, y, r) \in R \). By virtue of the translativity of \( R \) there always exists \( s \in G \) with \((x, y, s) \in R \). Consequently, \( y \in u_R \{x\} \), which is a contradiction. Therefore \( u_R u_R \{x\} \subseteq u_R \{x\} \) and the proof is complete.

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Example 3. Let $D$ be a preorder on a set $G$. Let us define a ternary relation $R$ on $G$ as follows:

$$(x, y, z) \in R \iff x \leq y, y \leq z.$$ 

Then $R$ is translative.

Obviously, a cyclic ternary relation is translative iff it is feebly translative. Thus, Theorem 4 results in

Corollary 6. Let $R$ be a cyclic ternary relation. If $R$ is feebly translative, then $u_R$ is an $F$-topology.

Example 4. Let $G = \{x, y, z, p\}$ and let $R = \{(x, y, z), (x, z, p), (x, p, y), (y, p, z), (x, x, x), (y, y, y), (z, z, z), (p, p, p)\}$. Then $R^c$ (see Example 2) is a feebly translative cyclic ternary relation on $G$.

Proposition 6. Let $R$ be an irreflexive and irreversible ternary relation. If $u_R$ is an $F$-topology, then $R$ is translative.

Proof. Let $G$ be the underlying set of $R$ and let $u_R$ be an $F$-topology. Let $(x, y, z) \in R$, $(y, p, q) \in R$. Then $p \in u_R\{y\}$ and $y \in u_R\{x\}$, hence $p \in u_Ru_R\{x\} = u_R\{x\}$. Since $R$ is irreversible, we have $p \neq x$. Thus, by the irreflexivity of $R$, there exists $r \in G$ with $(x, p, r) \in R$. By Proposition 3, $u_R$ is an $S_2$-topology. Therefore it follows from $(x, y, z) \in R$ that $z \in u_R\{x\}$ or $z \in u_R\{y\}$. If $y = z$, then $(x, z, z) \in R$. Suppose $y \neq z$. Because $x \neq z$ and $R$ is irreflexive, there exists $r \in G$ with $(x, z, z) \in R$ or $(y, z, z) \in R$. Hence $R$ is translative.

References


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