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ON $K$-SEQUENCES

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1. We recall that a sequence $\{x_n\}$ in a topological group $X$ is called a $K$-sequence if for every subsequence $\{y_n\}$ of $\{x_n\}$ there are a subsequence $\{t_n\}$ of $\{y_n\}$ and $t \in X$ such that

$$\sum_{n=1}^{\infty} t_n = t$$

(see [1]).

$K$-sequences converge to zero. Sequences converging to zero in a complete metric group are $K$-sequences.

In this note we prove

**Theorem 1.** Assume that $X$ is a topological group, $\{F_k\}$ is a nondecreasing sequence of closed subsets of $X$ such that

$$X = \bigcup_{k=1}^{\infty} F_k$$

and assume that $\{x_n\}$ is a $K$-sequence in $X$. Then there exists an index $k_0$ such that

$$x_n \in F_{k_0} + \left\{ - \sum_{m \in A} x_m : A \subset \{1, \ldots, k_0\} \right\}$$

for every $n \in \mathbb{N}$.

As consequences of Theorem 1 we get the following theorems.

**Theorem 2.** Assume that $f_n$ for $n \in \mathbb{N}$ and $f$ are sequentially continuous non-negative mappings defined on $X$ such that the following conditions hold:
(a) \( f_n \) for \( n \in \mathbb{N} \) are triangle mappings, i.e.
\[
f_n(x + y) \leq f_n(x) + f_n(y) \quad \text{for} \quad x, y \in \mathbb{N};
\]

(b) \( f(0) = 0; \)

(c) \( f_n(x) \rightarrow f(x) \) for every \( x \in X; \)

and assume that \( \{x_n\} \) is a \( K \)-sequence in \( X \).

Then \( f_n(x_n) \rightarrow 0 \) as \( n \rightarrow \infty \).

**Theorem 3.** If \( X \) is a Fréchet topological group such that every sequence converging to zero in \( X \) is a \( K \)-sequence, then \( X \) is of the second category.

We recall that \( X \) is a Fréchet topological group if for every subset \( A \) of \( X \) and for every element \( x \) which belongs to the closure \( \bar{A} \) of \( A \) there is a sequence \( \{x_n\} \) of elements in \( A \) such that \( x_n \rightarrow x \). In the case when \( X \) is a metric group, Theorem 3 was proved in [2]. Theorem 3 in the present form was proved in [3]. The proof of Theorem 3 produced in this paper is simpler than the proof in [3] and suggests a generalization of the theorem.

2. In this section we prove the theorems formulated in Section 1.

**Proof of Theorem 1.** Suppose that Theorem 1 does not hold. Then there are a topological group \( X \), a nondecreasing sequence \( \{F_k\} \) of closed subsets of \( X \), a \( K \)-sequence \( \{x_n\} \) in \( X \) and a subsequence \( \{m_n\} \) of \( \{n\} \) such that

\[
x_{m_{n+1}} \notin F_{m_n} + \left\{ - \sum_{m \in A} x_m : A \subset \{1, \ldots, m_n\} \right\}.
\]

Since \( \{F_k\} \) is a nondecreasing sequence of subsets of \( X \) and subsequences of \( K \)-sequences are \( K \)-sequences, we may assume that \( m_n = n \) for \( n \in \mathbb{N} \) and

\[
x_1 \notin G_1 = \{0\},
\]

\[
x_{n+1} \notin G_{n+1} = F_n + \left\{ - \sum_{m \in A} x_m : A \subset \{1, \ldots, n\} \right\}.
\]

Since \( G_n \) for \( n \in \mathbb{N} \) are closed subsets of \( X \), there are continuous pseudonorms \( p_n \) on \( X \) and numbers \( \varepsilon_n > 0 \) such that

\[
\inf \{p_n(x_n - z) : z \in G_n\} > \varepsilon_n
\]

for \( n \in \mathbb{N} \). As \( p_1(x_n) \rightarrow 0 \), there is an index \( r_1 \) such that \( p_1(x_{r_1}) < 2^{-2} \varepsilon_1 \). As \( p_2(x_n) \rightarrow 0 \), there is an index \( r_2 \) such that

\[
p_1(x_{r_2}) < 2^{-3} \varepsilon_1 \quad \text{and} \quad p_2(x_{r_2}) < 2^{-4} \varepsilon_2.
\]
By induction, we select a subsequence \( \{r_n\} \) or \( \{n\} \) such that

\[
p_n(x_{r_m}) < 2^{-n-m} \varepsilon_n
\]

for \( n \leq m \) and \( m, n \in \mathbb{N} \). Since \( \{x_{r_n}\} \) is a subsequence of the \( K \)-sequence \( \{x_n\} \), there are a subsequence \( \{s_n\} \) of \( \{r_n\} \) and \( x \in X \) such that

\[
\sum_{n=1}^{\infty} x_{s_n} = x.
\]

Let \( n_0 \) be an index such that \( x \in F_{n_0-1} \). We put

\[
z = x - \sum_{n<n_0} x_{s_n}.
\]

Then

\[
z \in G_{n_0} \quad \text{and} \quad x_{s_{n_0}} - z = \sum_{n=n_0+1}^{\infty} x_{s_n}
\]

for \( n \in \mathbb{N} \). Hence, by (2), we get

\[
p_{s_{n_0}}(x_{s_{n_0}} - z) \leq \varepsilon_{s_{n_0}},
\]

which contradicts (1). This contradiction completes the proof. \( \square \)

Remark 1. Under the assumptions of Theorem 1 there is an index \( k_0 \) such that \( x_n \in F_{k_0} - F_{k_0} \), and there are subsequence \( \{y_n\} \) of \( \{x_n\} \), an index \( k_0 \), a set \( A \subset \{1, \ldots, k_0\} \) and a sequence \( \{z_n\} \) in \( F_{k_0} \) such that

\[
y_n = - \sum_{m \in A} x_m + z_n
\]

for \( n \in \mathbb{N} \). If, moreover, \( F_k \) for \( k \in \mathbb{N} \) are subgroups of \( X \), then there is an index \( k_0 \) such that \( x_n \in F_{k_0} \) for \( n \in \mathbb{N} \).

Proof of Theorem 2. Suppose that Theorem 2 does not hold. Then there are number \( \varepsilon > 0 \) and a subsequence \( \{m_n\} \) of \( \{n\} \) such that

\[
f_{m_n}(x_{m_n}) > \varepsilon
\]

for \( n \in \mathbb{N} \). Since \( f \) is continuous, \( f(0) = 0 \) and \( x_n \to 0 \), there is a subsequence \( \{p_n\} \) of \( \{m_n\} \) such that

\[
\sum_{n=1}^{\infty} [f(x_{p_n}) + f(-x_{p_n})] < \varepsilon/3.
\]
We put

\[(5) \quad F_k = \{x \in X : |f_{p_n}(x) - f(x)| \leq \varepsilon/4 \text{ for } n \geq k\} \]

We note that $F_k$ for $k \in \mathbb{N}$ are closed subsets of $X$,

\[X = \bigcup_{k=1}^{\infty} F_k\]

and $\{x_{p_n}\}$ is a $K$-sequence. Hence, by Theorem 1, there is an index $k_0$ such that

\[x_{p_n} \in F_{k_0} + \left\{- \sum_{k \in A} x_{p_n} : A \subseteq \{1, \ldots, k_0\}\right\}\]

for $n \in \mathbb{N}$. According to Remark 1, there is a subsequence $\{q_n\}$ of $\{p_n\}$, a set $A \subseteq \{1, \ldots, k_0\}$ and a sequence $\{y_n\}$ in $F_{k_0}$ such that

\[(6) \quad x_{q_n} = - \sum_{m \in A} x_{p_m} + y_n\]

for $n \in \mathbb{N}$. It follows from (a) that

\[f_{q_n}(x_{q_n}) \leq f_{q_n}\left(- \sum_{m \in A} x_{p_m}\right) + |f_{q_n}(y_n) - f(y_n)| + f(y_n)\]

Since $y_n \in F_k$ and for sufficiently large $n$ we have $q_n > k_0$, in view of (5) we get

\[|f_{q_n}(y_n) - f(y_n)| < \varepsilon/3\]

for sufficiently large $n$. Note that, by (6), (a), (c) and (4), we can write

\[f(y_n) \leq f(x_{q_n}) + \sum_{m \in A} f(x_{p_m}) < \varepsilon/3\]

Since $A$ is a finite set, we infer from (c) and (2) that

\[f_{q_n}\left(- \sum_{m \in A} x_{p_m}\right) < \varepsilon/3\]

for sufficiently large $n$. From the above estimates we get $f_{q_n}(x_{q_n}) < \varepsilon$ for sufficiently large $n$, which contradicts (3). This contradiction prove the theorem. $\square$
We precede the proof of Theorem 3 with two lemmas.

**Lemma 1.** If $X$ is a Fréchet topological group, $x_{ij} \in X$ for $i, j \in \mathbb{N}$ and $x_{ij} \to 0$ as $j \to \infty$ for $i \in \mathbb{N}$, then there are two subsequences $\{p_i\}, \{q_i\}$ of $\{i\}$ such that $x_{p_i, q_i} \to 0$.

**Proof.** We may assume that, under the assumptions of Lemma 1, there is a sequence $\{x_n\}$ in $X$ such that $x_n \neq 0$ for every $n \in \mathbb{N}$ and $x_n \to 0$. Otherwise the lemma is trivially true. We see that $x_{ij} + x_i \to x_i$ as $j \to \infty$ for $i \in \mathbb{N}$ and $x_i \neq 0$. Therefore, there is a subsequence $\{m_i\}$ of $\{i\}$ such that $x_{ij} \neq 0$ for $j \geq m_i$ and $i \in \mathbb{N}$. Assume that

$$A = \{x_{ij} : j \geq m_i, i, j \in \mathbb{N}\}.$$ 

Then $0 \notin A$ but $0 \in \text{cl} A$. Since $X$ is a Fréchet topological group, there are two sequences $\{r_i\}$ and $\{s_i\}$ of positive integers such that $m_i \leq s_i$ for $i \in \mathbb{N}$ and $x_{r_i, s_i} \to 0$. We assert that $r_i \to \infty$. Otherwise there would exist a constant subsequence $\{v_i\}$ of $\{r_i\}$ such that $v_i = v$ for $i \in \mathbb{N}$ and $x_{v_i, s_i} \to 0$ but $x_{v_i, s_i} \to x_v$ and $x_v \neq 0$. Consequently, $r_i \to \infty$ and $s_i \to \infty$. Thus there is a subsequence $\{k_i\}$ of $\{i\}$ such that $\{r_{k_i}\}$ and $\{s_{k_i}\}$ are subsequences of $\{i\}$. Assuming $p_i = r_{k_i}$ and $q_i = s_{k_i}$ for $i \in \mathbb{N}$ we get the lemma. □

**Lemma 2.** If $X$ is a Fréchet topological group and $\{A_n\}$ is a nonincreasing sequence of dense subsets $A_n$ of $X$, then there is a sequence $\{x_n\}$ such that $x_n \in A_n$ for $n \in \mathbb{N}$ and $x_n \to 0$.

**Proof.** Under the assumptions of Lemma 2, for every $i \in \mathbb{N}$ there is a sequence $\{x_{ij}\}$ such that $x_{ij} \in A_i$ for $j \in \mathbb{N}$ and $x_{ij} \to 0$ as $j \to \infty$ for $i \in \mathbb{N}$. By Lemma 1, there are two subsequences $\{p_i\}$ and $\{q_i\}$ of $\{i\}$ such that $x_{p_i, q_i} \to 0$. Moreover, we have $x_{p_i, q_i} \in A_{p_i} \subset A_i$ for $i \in \mathbb{N}$. Putting $x_i = x_{p_i, q_i}$ for $i \in \mathbb{N}$ we get the assertion. □

**Proof of Theorem 3.** Suppose that $X$ is a Fréchet topological group in which null sequences are $K$-sequences and $X$ is not of the second category. Then there are closed subsets $F_k$ of $X$ such that $\text{int} F_k = \emptyset$ for $k \in \mathbb{N}$ and

$$X = \bigcup_{k=1}^{\infty} F_k.$$ 

To get a contradiction we construct a matrix $\{x_{ij}\}$ such that

$$(i) \quad x_{ij} \to 0 \text{ as } j \to \infty \quad \text{for} \quad i \in \mathbb{N}.$$
and

\[(ii) \quad x_{ij} \in \left[ F_j + \left\{ \sum_{(m,n) \in A} x_{mn} : A \supset \{(k,l) : 1 \leq k \leq i, 1 \leq l \leq j\} \right\} \right]'

for \(i = 2, 3, \ldots\) and \(j \in \mathbb{N}\). Let \(\{x_{ij}\}\) be a sequence in \(X\) such that \(x_{ij} \to 0\). Suppose that the first \((n - 1)\) rows of the matrix have been constructed in such a way that (i) and (ii) hold. Assume that

\[F_{nj} = F_j + \left\{ \sum_{(m,n) \in A} x_{mn} : A \supset \{(k,l) : 1 \leq k \leq n, 1 \leq l \leq j\} \right\}

for \(j \in \mathbb{N}\). Then \(F_{nj}\) for \(n, j \in \mathbb{N}\) are closed subsets of \(X\), \(\text{int} F_{nj} = \emptyset\) and \(F_{nj} \subset F_{n,j+1}\). Consequently, the components \(F'_{nj}\) are open dense subsets of \(X\) and \(F'_{nj} \supset F'_{n,j+1}\) for \(j \in \mathbb{N}\). Thus, by Lemma 2, there a sequence \(\{x_{nj}\}\) such that

\[x_{nj} \in F'_{nj} \text{ for } j \in \mathbb{N} \text{ and } x_{nj} \to 0 \text{ as } j \to \infty.

Consequently, (i) and (ii) hold for \(i = n\). Hence, by induction, the existence of a matrix \(\{x_{ij}\}\) such that (i) and (ii) hold follows. By Lemma 1, there are subsequences \(\{p_i\}\) and \(\{q_i\}\) of \(\{i\}\) such that \(x_{p_i,q_i} \to 0\).

It follows from (ii) that

\[x_{p_i,q_i} \notin F_{q_i} + \left\{ \sum_{k \in A} x_{pkqk} : A \supset \{1, \ldots, i\} \right\}'

for \(i \in \mathbb{N}\). On the other hand, \(\{x_{p_i,q_i}\}\) is a \(K\)-sequence. Hence, by Theorem 1, there exists an index \(i_0\) such that

\[x_{p_i,q_i} \in F_{i_0} + \left\{ \sum_{k \in A} x_{pkqk} : A \supset \{1, \ldots, i_0\} \right\}

for \(i \in \mathbb{N}\). This obvious contradiction completes the proof of Theorem 3. \(\square\)

Remark 2. Observe that we can modify the proof of Theorem 3 in such a way that the elements of \(\{x_{ij}\}\) are in a given dense subset \(G\) of \(X\). Therefore the assertion of Theorem 3 is valid whenever there exists a dense subset \(G\) of a Fréchet topological group \(X\) such that null sequences in \(G\) are \(K\)-sequences in \(X\).

References


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