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CONVERGENCES $\mathcal{L}^H_S$ FOR THE GROUP OF REAL NUMBERS

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For each subgroup $H$ of the group $R$ of real numbers and each subset $S$ of the quotient group $R/H$ a convergence $\mathcal{L}^H_S$ for the group $R$ is constructed. The relation of the system of convergences $\mathcal{L}^H_S$ to the Čech-Stone compactification of discrete spaces is clarified. Necessary and sufficient conditions are given for $(R, \mathcal{L}^H_S, +)$ to be a complete group with respect to the convergence $\mathcal{L}^H_S$. This gives some views on the structure of the groups $R$ and $R/H$.

The point of our considerations is the group $(R, +)$ of real numbers. We use the fact that $R$ is a linearly ordered point set for which a convergence $\mathcal{L}$ is defined by means of open intervals $(a, b) \subset R$ such that $\lim x_n = x$, $\lim y_n = y$ implies that $\lim(x_n - y_n) = x - y$. In this sense $R$ is a convergence commutative group ([1]). It will be denoted $(R, \mathcal{L}, +)$.

Recall that a convergence $\mathcal{M}$ for a set $M$ is a collection of pairs $((x_n), x)$ where $\langle x_n \rangle$ is a sequence of points $x_n \in M$ and $x \in M$. We assume that the convergence $\mathcal{M}$ satisfies the well known Fréchet axioms of convergence inclusive the axiom of the maximal convergence ($\mathcal{M} = \mathcal{M}^*$). A commutative group $(M, +)$ with a convergence $\mathcal{M}$ will be denoted $(M, \mathcal{M}, +)$. If $((x_n), x) \in \mathcal{M}$, $((y_n), y) \in \mathcal{M}$ implies that $((x_n - y_n), x - y) \in \mathcal{M}$ we have a convergence commutative group $(M, \mathcal{M}, +)$ (abbr. cc-group). In such a group Cauchy sequences are defined to be sequences $\langle x_n \rangle$, $x_n \in M$, such that $((x_n - x_{i_n}), 0) \in \mathcal{M}$ whenever $\langle x_{i_n} \rangle \subset \langle x_n \rangle$. A cc-group $(M, \mathcal{M}, +)$ is complete if each Cauchy sequence $\mathcal{M}$-converges in $M$, more precisely, if $\langle x_n \rangle$, $x_n \in M$, is a Cauchy sequence then there is a point $x \in M$ such that $((x_n), x) \in \mathcal{M}$. 

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Notation. We denote $N$ the set of natural numbers, $N^{-1}$ the set of numbers $n^{-1}, n \in N$, $Q$ the group of rational and $R$ the group of real numbers, $H$ a subgroup of the group $R$ and $S$ a subset of the quotient group $R/H$. A subgroup of $R$ is either discrete or dense. Points $x_1$ and $x_2$ of $R$ are non-equivalent (with respect to $H$) if $(x_1 - x_2) \notin H$. In the section I we consider $R/H$ as a set of points sometimes called indexes. They will be denoted by Greek letters $\xi, \eta, \zeta$.

Let $H$ be a subgroup of the group $R$ and $R/H$ the corresponding quotient group. Elements $\xi \in R/H$ are classes $T_\xi = a_\xi + H$ where $a_\xi$ is a representative of the class $T_\xi$. We identify elements $\xi$ with ordinals $\xi < \omega_H$ where $\omega_H$ is the least ordinal of the power $|R/H|$. We put $a_0 = 0$. Then $T_0 = H$. Notice that $R = UT_\xi, \xi \in R/H, |R/Q| = \exp(\omega), |R/H| = 1$.

Definition D1. Let $H$ be a subgroup of the group $R$. Functions $f : R/H \to N^{-1}$ are called generating functions. Adjoin to $S \subseteq R/H$ a class $F^H_S$ (or simply $F_S$) of generating functions $f$ such that the partial function $f/S$ is a constant function. $f/\emptyset$ is considered as a constant partial function. If $S$ contains only one index $\xi$ we write $F_\xi$ instead of $F_{\{\xi\}}$.

Lemma 1. Let $S_1 \subset S_2 \subset R/H$. Then $F^H_{S_1} \supset F^H_{S_2}$.

Proof. If $f \in F^H_{S_2}$ then $f/S_2$ is constant and $f/S_1$ as well. Hence $f \in F^H_{S_1}$.

Definition D2. Let $H$ be a subgroup of the group $R$. Let $(a,b)$ be an open interval of real numbers. Denote $(a,b)_{\xi} = (a,b) \cap T_\xi, \xi \in R/H$. Let $z$ be a point of $R$ and $S$ a subset of $R/H$. A set $W(z)$ is called a closure neighborhood or, simply, a neighborhood of the point $z$ if there is a generating function $f \in F_S$ such that $W_f(z) \subseteq W(z)$ where

$$W_f(z) = U(z - f(\xi), z + f(\xi))_{\xi}, \quad \xi \in R/H.$$ 

Remark. Let $(a,b) \subset R, z \in (a,b)$. Choose $m_0 \in N$ such that $m_0^{-1} < \min\{z - a, b - z\}$ and put $f(\xi) = m_0^{-1}, \xi \in R/H$. Then $f \in F^H_S$. Hence $W_f(z) \subseteq (a,b)$. Consequently, the open interval $(a,b)$ in $R$ is a closure neighborhood of each point $z \in (a,b)$.

The following are the main properties of closure neighborhoods $W_f(z), f \in F^H_S$.

(i) $z \in W_f(z)$. (If $z \in T_{\xi_0}$ then $z \in (z - f(\xi_0), z + f(\xi_0))_{\xi_0} \subseteq W_f(z)$, by D2).
(ii) If \( W_{f_i}(z), f_i \in F_S, i = 1, 2, \) are neighborhoods of a point \( z, \) then \( W_{f_1}(z) \cap W_{f_2}(z) \) is a neighborhood of the point \( z. \) \((W_{f_1}(z) \cap W_{f_2}(z) = W_{f_3}(z), \) where \( f_3(\xi) = \min\{f_1(\xi), f_2(\xi)\}, \xi \in R/H).\)

(iii) If \( z_1 \neq z_2 \) there are \( f_i \in F_S, i = 1, 2, \) such that \( W_{f_1}(z_1) \cap W_{f_2}(z_2) = \emptyset \) (see Remark above).

From (i), (ii), (iii) we deduce that the system of closure neighborhoods \( W_f(z), f \in F_S^H, \) of points \( z \in R \) satisfies the axioms of Hausdorff topological spaces except the axiom of open neighborhoods which need not be fulfilled. This is shown in the following

**Lemma 2.** Let \( H \) be a dense subgroup of the group \( R \) and \( S \subset R/H. \) Let \( z \in R. \) Then there is a complete system of open closure neighborhoods at the point \( z \) if and only if there is a finite \( K \subset R/H \) such that \( S = R/H - K. \)

**Proof.** Let \( W_f(z), f \in F_S^H, \) be a neighborhood of the point \( z. \) Since the partial function \( f/(R/H - K) \) is constant and \( K \) is finite there is a natural number \( p \) such that \( p^{-1} < f(\xi), \xi \in R/H. \) Hence \( (z - p^{-1}, z + p^{-1}) \subset W_f(z). \) It follows that the system of intervals \( (z - n^{-1}, z + n^{-1}), n \in N, \) is a complete system of open neighborhoods at the point \( z. \)

Now, assume that \( R/H - S \) is infinite. Choose distinct \( \xi_n \in (R/H - S). \) Define \( f(\xi) = 1, \xi \neq \xi_n, f(\xi_n) = n^{-1}, n \in N. \) Then \( f \in F_S \) and we have a neighborhood \( W_f(z). \) Let \( W_g(z) \subset W_f(z), g \in F_S. \) Suppose that (on the contrary) \( W_g(z) \) is open. The neighborhood \( W_g(z) \) is infinite because \( H \) is dense. Choose a point \( t \in W_g(z), t \neq z. \) Then there is, by the assumption, a neighborhood \( W_h(t) \subset W_g(z), h \in F_S^H. \) Notice that \( h(\xi) \leq g(\xi) \leq f(\xi), \xi \in R/H. \) There are \( \varepsilon_n > 0 \) such that

\[
(t - \varepsilon_n, t + \varepsilon_n)_{\xi_n} \subset (t - h(\xi_n), t + h(\xi_n))_{\xi_n} \subset (z - g(\xi_n), z + g(\xi_n))_{\xi_n} \\
\subset (z - f(\xi_n), z + f(\xi_n))_{\xi_n} \subset (z - n^{-1}, z + n^{-1}), \quad n \in N.
\]

Hence \( t \in (z - n^{-1}, z + n^{-1}) \) and so \( t = z. \) This is a contradiction. Thus \( W_g(z) \) is not open.

We have seen above that the class \( F_S^H \) generates a complete system of closure neighborhoods \( W_f(z) \) at the point \( z. \) By neighborhoods \( W_f(z), f \in F_S^H, \) a convergence for the group \( R \) is defined in a well known way. \( \square \)

**Definition D3.** Let \( H \) be a subgroup of the group \( R, S \subset R/H. \) Denote \( L^H_S \) a collection of pairs \( ((x_n), x), x_n \in R, x \in R, \) such that if \( W_f(x), f \in F_S^H, \) is a neighborhood of the point \( x, \) then \( x_n \in W_f(x), n \geq n_0. \) If \( ((x_n), x) \in L^H_S \) we say that the sequence \( \langle x_n \rangle \) \( L^H_S \)-converges to the point \( x \) and write \( L^H_S - \lim x_n = x. \) The collection \( L^H_S \) is called a convergence for \( R. \) (It will be sometimes denoted \( L_S. \))
Fréchet axioms of convergence are clearly satisfied. From (iii) it follows that \( \mathcal{L}^H_S - \lim x_n = x, \mathcal{L}^H_S - \lim y_n = y \) implies \( x = y \). In view of (i) we have \( \mathcal{L}^H_S - \lim x = x \). If \( (\langle x_n \rangle, x) \in \mathcal{L}^H_S, \langle x_i \rangle \subset \langle x_n \rangle \), then \( (\langle x_{i_n} \rangle, x) \in \mathcal{L}^H_S \), by D3. From D3 it instantly follows that \( \mathcal{L}^H_S \) is a maximal convergence, i.e. \( \mathcal{L}^H_S = \mathcal{L}^H_S^* \).

Denote \( \mathcal{L} \) the usual metric convergence for \( R \). We write simply \( \lim x_n = x \) instead of \( \mathcal{L} - \lim x_n = x \). Note that \( \mathcal{L} = \mathcal{L}^H_{R/H} \).

**Lemma 3.** Let \( S_1 \subset S_2 \subset R/H \). Then \( \mathcal{L}^H_{S_1} \subset \mathcal{L}^H_{S_2} \).

**Proof.** Let \( ((x_n), z) \in \mathcal{L}_{S_1} \). Let \( W_f(z), f \in F^H_{S_2} \), be a neighborhood of the point \( z \). Define a generating function \( g(\xi) = f(\xi), \xi = S_1, g(\xi) \in f(\xi), \xi \in R/H - S_1 \). The partial function \( g/S_1 \) is constant, by Lemma 1, and so \( y \in F^H_{S_1} \).

Since \( x_n \in W_f(z), n \geq n_0 \), and \( W_f(z) \subset W_f(z) \) we have \( x_n \in W_f(z), n \geq n_0 \). Hence \( ((x_n), z) \in \mathcal{L}_{S_2} \).

The assertion \( \mathcal{L}^H_{S_1} \subset \mathcal{L}_{S_2} \) implies \( S_1 \subset S_2 \) is not correct. Let \( H \) be a subgroup of \( R \), \( R \neq H \). Choose indexes \( \xi \neq \xi_2 \) and put \( S_1 = \{\xi_1\}, S_2 = \{\xi_2\} \). Then \( \mathcal{L}_{S_1} = \mathcal{L}_{S_2} \), but \( S_1 \notin S_2 \). This example shows that the map \( \varphi(S) = \mathcal{L}^H_S, S \subset R/H, H \neq R \), is not one-to-one even when it preserves the order relation \( \subset \), by Lemma 3. Next we investigate the structure of the system of classes \( \varphi^{-1}(\mathcal{L}^H_S), S \subset R/H \).

Let \( H \) be a subgroup of the group \( R, S \subset R/H \). Denote \( R_S = UT_\xi, \xi \in S \). Notice that \( R_S \subset R, R_0 = \emptyset, R_{\{0\}} = H, R_{R/R} = R \).

**Lemma 4.** Let \( H \) be a subgroup of the group \( R, S \subset R/H \). Then \( \mathcal{L}^H_S - \lim z_n = z \) if and only if \( \lim z_n = z \) and there is a finite \( K \subset R/H \) such that \( z_n \in R_{S \cup K}, n \in N \).

**Proof.** Let \( \mathcal{L}^H_S - \lim z_n = z \). Then \( \lim z_n = z \) because \( \mathcal{L}^H_S \subset \mathcal{L} \), by Lemma 3. Suppose that (on the contrary) there is a subsequence \( (\langle z_{i_n} \rangle) \subset (\langle z_n \rangle), z_{i_n} \neq z \), and distinct indexes \( \eta_n \in (R/H - S) \) such that \( z_{i_n} \in T_{\eta_n} \). Put \( f(\xi) = 1, \xi \neq \eta_n \), and choose \( f(\eta_n) \in N^{-1} \) such that \( z_{i_n} \notin (z - f(\eta_n), f(\eta_n)) \), \( n \in N \). This is possible because \( z_{i_n} \neq z \). Then \( f \in F_S \) and we have a neighborhood \( W_f(z) \) of \( z \) which contains no point \( z \). Hence \( (z_{i_n}) \) does not \( \mathcal{L}^H_S \)-converge to \( z \). This is in contradiction with the assumption \( ((z_n), z) \in \mathcal{L}^H_S \).

Now, let \( \lim z_n = z, z_n \in R_{S \cup K} \). We use the property \( \mathcal{L}_S = \mathcal{L}_S^* \) to prove that \( \mathcal{L}_S - \lim z_n = z \). Let \( \langle z_{i_n} \rangle \subset (z_{i_n}) \) be a subsequence of \( (z_{i_n}) \). Either there is a subsequence \( \langle t_n \rangle \subset \langle z_{i_n} \rangle \) of non-equivalent points \( t_n \in R_S \) and then \( \mathcal{L}_S - \lim t_n = z \) or it is not so, and there is an index \( \xi_0 \in (S \cup K) \) and a subsequence \( \langle u_n \rangle \subset \langle z_{i_n} \rangle, u_n \in T_{\xi_0} \). Hence \( \mathcal{L}_S - \lim u_n = z \). It follows that \( ((z_n), z) \in \mathcal{L}_S \).

**Lemma 5.** Let \( H \) be a subgroup of the group \( R \). Let \( S_i \subset R/H, i = 1, 2 \). Let \( S_1 \cap S_2 \) be a finite set. Then \( \mathcal{L}^H_{S_1} = \mathcal{L}^H_{S_2} \).
Proof. Let \((z_n, z) \in \mathcal{L}_{S_1}\) and \(W_f(z), f \in F_{S_2}\), be a neighborhood of the point \(z \in R\). We are to prove that \(z_n \in W_f(z), n \geq n_0\). Notice that \(S_1 \cup S_2 = (S_1 \setminus S_2) \cup (S_1 \cap S_2)\). The partial function \(f/S_1 \cap S_2\) is constant, by Lemma 1, and \(S_1 \setminus S_2\) is a finite set. Therefore the number \(d = \min\{f(\xi)\}, \xi \in S_1 \cup S_2\), belongs to the set \(N^{-1}\). Put \(g(\xi) = d, \xi \in S_1 \cup S_2\), and \(g(\xi) \leq f(\xi), g(\xi) \in N^{-1}, \xi \in (R/H - (S_1 \cup S_2))\). Then \(g \in F_{S_2}\) and so \(z_n \in W_g(z), n \geq n_0\). Hence \(z_n \in W_f(z), n \geq n_0\) and therefore \(\mathcal{L}_{S_1} \subset \mathcal{L}_{S_2}\).

Analogously we prove that \(\mathcal{L}_{S_2} \subset \mathcal{L}_{S_1}\). □

Lemma 6. Let \(H\) be a subgroup of the group \(R\). Let \(S_i \subset R/H, i = 1,2\). Let \(\mathcal{L}^H_{S_i}\) \(\subset \mathcal{L}^H_{S_2}\). Then \(S_1 - S_2\) is a finite set.

Proof. First prove the following statement: If \(S_0\) is an infinite subset of \(R/H\) then there is a sequence of non-equivalent points \(x_n \in T_<, \xi_n \in S_0,\) and a point \(z \in R\) such that \(\mathcal{L}^H_{S_0} - \lim x_n = z\). Distinguish two cases. 1) \(H\) is dense. Let \(\langle \xi_n \rangle\) be one-to-one sequence of indexes \(\xi_n \in S_0\). Choose a point \(z \in R\). Since \(H\) is dense there is a sequence \(\langle x_n \rangle\) of non-equivalent points \(x_n \in T_<\) with \(\lim x_n = z\). Hence \(\mathcal{L}^H_{S_0} - \lim x_n = z\), by Lemma 4. 2) \(H\) is discrete. Denote \(d\) the least positive number of \(H\). Choose numbers \(b_\xi \in T_<\) such that \(0 \leq b_\xi < d, \xi \in R/H\). Since \(S_0\) is infinite there is a one-to-one sequence \(\langle \xi_n \rangle\), \(\xi_n \in S_0\), and a point \(z \in R, 0 \leq z \leq d,\) such that \(\lim b_{\xi_n} = z\). Denote \(b_{\xi_n} = x_n\). Then \(\langle x_n \rangle\) is a sequence of non-equivalent points \(x_n\) with \(\mathcal{L}^H_{S_0} - \lim x_n = z\), by Lemma 4.

Suppose that \(S_1 - S_2\) is infinite and denote \(S_0 = S_1 - S_2\). Then \(S_0 \subset S_1\) and \(\langle x_n \rangle, z) \in \mathcal{L}^H_{S_1}\), by Lemma 3 where \(\langle x_n \rangle\) is the sequence constructed above. On the other hand, \((\langle x_n \rangle, z) \notin \mathcal{L}^H_{S_2}\), by Lemma 4. This is a contradiction. □

Proposition 1. Let \(H\) be a subgroup of the group \(R, S_i \subset R/H, i = 1,2\). Then \(\mathcal{L}^H_{S_1} = \mathcal{L}^H_{S_2}\) if and only if \(S_1 - S_2\) is a finite set.

Proof follows instantly from Lemmas 5 and 6.

From Proposition 1 it follows that there is a connection between convergences \(\mathcal{L}^H_{S_i}\) and some subsets of the Čech-Stone compactification of a discrete topological space. Consider \(R/H\) as a discrete topological space of isolated points \(\xi\) and denote \(\beta^*S = \beta S - R/H,\) where \(\beta\) is a topological operator in the Čech-Stone compactification \(\beta(R/H)\). It is well known that \(\beta^*S_1 = \beta^*S_2\) if and only if \(S_1 - S_2\) is finite. Hence \(\mathcal{L}^H_{S_1} = \mathcal{L}^H_{S_2}\) if and only if \(\beta^*S_1 = \beta^*S_2\), by Proposition 1.

Let \(H\) be a subgroup of the group \(R\). We denote, as above, functions \(\varphi(S) = \mathcal{L}^H_S, S \subset R/H\). We have shown that \(\varphi\) is not one-to-one except in the case when \(H = R\). From Proposition 1 it follows that \(S_1\) and \(S_2\) are equivalent (i.e. \(S_2 \in \varphi^{-1}(\mathcal{L}_{S_1})\)) iff \(S_1 - S_2\) is finite. Now, define a quasi-order \(<\) as follows: \(S_1 < S_2\) if there is a finite \(K \subset R/H\) such that \(S_1 \subset S_2 \cup K\).
Lemma 7. Let $H$ be a subgroup of the group $R$. Then $S_1 < S_2$ if and only if $\mathcal{L}^H_{S_1} \subset \mathcal{L}^H_{S_2}$.

Proof. Let $S_1 < S_2$. Then $S_1 \subset S_2 \cup K$. It follows $\mathcal{L}_{S_1} \subset \mathcal{L}_{S_2}$, by Lemma 3 and Proposition 1. Now, let $\mathcal{L}_{S_1} \subset \mathcal{L}_{S_2}$. According to Lemma 6 the set $S_1 - S_2$ is finite. Since $S_1 \subset S_2 \cup (S_1 - S_2)$ we have $S_1 < S_2$. □

Proposition 2. Let $H$ be a subgroup of the group $R$. There is a similar map (with respect to the inclusion $\subset$), on the system $\mathcal{L}^H_S$, $S \subset R/H$, onto the system of clopen sets $\beta^*(S)$ of the space $\beta^*(R/H)$.

Proof. Denote $\psi(\mathcal{L}^H_S) = \beta^*S$, $S \subset R/H$. Let $S_1$, $S_2$ be subsets of $R/H$, $\mathcal{L}_{S_1} \subset \mathcal{L}_{S_2}$. Then $S_1 < S_2$, by Lemma 7. Therefore, according to the definition of the quasi-order $<$ it follows that $\beta^*S_1 \subset \beta^*S_2$. It remains to prove the following implication: If $A$ is a clopen subset of $\beta^*(R/H)$ then there is $S \subset R/H$ such that $\beta^*S = A$. This is true because there is a clopen set $B$ in $\beta(R/H)$ such that $A = B \cap \beta^*(R/H)$ and so there is $S \subset R/H$ such that $A = \beta^*(S)$. □

Remark. Notice that $\mathcal{R}_1 \leq \mathcal{R}_2$ implies $\mathcal{R}_1 \cdot \mathcal{R}_2 = \mathcal{R}_2$. Let $S \subset R/Q$. Denote $F = \{K; K \subset R/Q, K \text{ finite}\}$, $X_S = \{S \setminus K: K \in F\}$, $Y = \{S; S \subset R/Q\}$, $Z = Y/F$, $\mathcal{R}_1 = |X_S|$, $\mathcal{R}_2 = |Z|$. Clearly $\mathcal{R}_1 = \exp(\omega)$, $\mathcal{R}_2 = \exp(\exp(\omega))$. Then $|\mathcal{L}_Q^S| = |X_S| \cdot |Z| = \exp(\omega) \cdot \exp(\exp(\omega)) \exp(\omega))$. Thus the number of convergences $\mathcal{L}^Q_S$, $S \subset R/Q$, is $\exp(\exp(\omega))$.

Let $H$ be a subgroup of the group $R$, $S \subset R/H$. We have seen that a closure topology for $R$ is defined by means of the class $F^H_s$ of generating functions. The corresponding closure operator will be denoted $w^H_S$ (or simply $w_S$). Hence $w_SA = \{x \in R; A \cap W_f(x) \neq \emptyset, f \in F^H_s\}$. Another closure topology for $R$ is defined by means of the convergence $\mathcal{L}^H_s$. Denote $\lambda^H_s$ (or $\lambda_s$) the corresponding closure operator: $\lambda^H_s A = \{x \in R; x = \mathcal{L}^H_s \lim x_n, x_n \in A, n \in N\}$. Hence we have closure spaces $(R, w^H_S)$ and $(R, \lambda^H_s)$.

Now, we are interested in the question what is the relation between closures $\lambda^H_s$ and $w^H_S$. It is well known that there are closure spaces $(P, u)$ and adjoint convergence spaces $(P, \lambda_u)$ such that $u \neq \lambda_u$. It is not the case if $P = R$, $u = w^H_S$. We show that $w^H_S = \lambda^H_s$. It is evident that $\lambda_S A \subset w_S A$, $A \subset R$. Suppose that there is $z \in R$ and $A \subset R$ such that $z \in (w_SA - \lambda_S A)$. Then $z \notin A$ and there is no sequence of points $x_n \in A$ such that $\mathcal{L}^H_s \lim x_n = z$. In view of Lemma 4 there is a generating function $f \in F_S$ such that $A \cap (z - f(\xi), z + f(\xi)) = \emptyset, \xi \in R/H$. Hence $A \cap W_f(z) = \emptyset$. This is a contradiction. Consequently, $w_SA = \lambda_S$.

Notice that $\omega_1$-iterated closure $\lambda^{\omega_1}_s = w^{\omega_1}_S$ are topologies for $R$. 20
In this section we investigate some convergence and group properties of the structures \((R, \mathcal{L}^H, +)\). For this purpose we consider indexes \(\xi\) of the set \(R/H\) as elements of the group \((R/H, +)\). If \(\xi_1, \xi_2\) are elements of \(R/H\) then \(\xi_1 + \xi_2 = \xi_3\) where \(\xi_3\) is uniquely determined by the addition \(T_{\xi_1} + T_{\xi_2} = T_{\xi_3}\) in the group \((R/H, +)\). The inverse element to the element \(\xi \in R/H\) is the element \(\eta \in R/H\) such that \(T_\eta = -T_\xi\). It will be denoted \(-\xi\).

Now, we are going to examine conditions under which \((R, \mathcal{L}^H, +)\) is a cc-group. First we give an example to show that \((R, \mathcal{L}^H, +)\) need not be a cc-group even when \(R_S\) is a subgroup of the group \(R\).

**Example.** Let \(H = \mathbb{Q}\) and let \(R_S\) be the group of algebraic numbers. Put 
\[x_n = n^{-1} \sqrt{2}, \ y_n = \pi + n^{-1} \]
Then \(\lim x_n = 0, \lim y_n = \pi\) and \(\mathcal{L}^H_S - \lim x_n = 0, \mathcal{L}^H_S - \lim y_n = \pi\), by Lemma 4. On the other hand, \((x_n + y_n)\) is a sequence of non-equivalent transcendent numbers which, by the same lemma, does not \(\mathcal{L}^H_S\)-converge to the point \(\pi\).

**Definition D4.** Let \((M, \mathcal{M}, +)\) be a commutative group with a convergence \(\mathcal{M}\) for \(M\). We say that \((M, \mathcal{M}, +)\) satisfies condition \((-)\) provided that the following implication holds
\[
(-) \quad \text{If} \ ((x_n), x) \in \mathcal{M} \text{ then } ((-x_n), -x) \in \mathcal{M}.
\]
\((M, \mathcal{M}, +)\) satisfies condition \((+)\), provided that
\[
(+) \quad \text{If} \ ((x_n), x) \in \mathcal{M}, ((y_n), y) \in \mathcal{M} \text{ then } ((x_n + y_n), x + y) \in \mathcal{M}.
\]
It is clear that \((M, \mathcal{M}, +)\) is a cc-group if and only if both the conditions \((-)\) and \((+)\) are satisfied.

**Definition D5.** Let \(H\) be a subgroup of the group \(R, S \subset R/H\). We denote \(S^-\) the set of elements \(\eta \in R/H\) such that \(T_\eta = -T_\xi, \xi \in S\).

**Lemma 8.** \(|S^-| = |S^-|, |S - S^-| = |S^- - S|, (S_1 \cup S_2)^- = S_1^- \cup S_2^-\), \((S_1 \cap S_2)^- = S_1^- \cap S_2^-\), \(x \in R_S\) if and only if \(-x \in R_{S^-}\).

Proof follows instantly from D5 and from the equivalence \(\xi \in (S - S^-)\) if and only if \(-\xi \in (S^- - S)\).

The properties \((-)\) and \((+)\) can be formulated by means of Čech-Stone operator \(\beta^*\). In the proofs we use the equivalence
Lemma 9. Let $H$ be a subgroup of the group $R$, $S \subset R/H$. Then $(R, \mathcal{L}_S^H, +)$ satisfies (–) if and only if $S^- = \beta^*(S^-)$.

Proof. Let $S^- = \beta^*(S^-)$. The set $S^-S$ is finite, by (ii). Let $\mathcal{L}_S^H - \lim x_n = x$. In view of Lemma 4, there is a finite $K \subset R/H$ such that $x_n \in R_{S^-U K}$ and $\lim x_n = x$. Hence $\lim(-x_n) = -x$ and $-x_n \in R_{S^-U K}$, by Lemma 8. Notice that $S^- U K = (S^- \cap S) U (S^- - S) U K \subset S U K$ where $K = (S^- - S) U K$. It follows that $K$ is a finite subset of $R/H$ and $R_{S^-U K} \subset R_S U K$. Thus $\mathcal{L}_S^H - \lim (-x_n) = -x$, by Lemma 4.

Let $\beta^* S \neq \beta^*(S^-)$. Then $S^-S$ is infinite and both the sets $S^-S$ and $S^-S$ are infinite, by Lemma 8. In view of statement (see the proof of Lemma 6) there is a sequence of non-equivalent points $x_n \in R_{S^-S}$ and a point $z \in R$ such that $\mathcal{L}_S^H - \lim x_n = z$. Notice that $(-x_n)$ is a sequence of non-equivalent points $-x_n \in R_{S^-S}$. Consequently, $-x_n \notin R_S$. From Lemma 4 it follows that the sequence $(-x_n)$ does not $\mathcal{L}_S^H$-converge to $-z$.

Lemma 10. Let $H$ be a subgroup of the group $R$, $S \subset R/H$. $(R, \mathcal{L}_S^H, +)$ satisfies (+) if and only if $\beta^*((S U L_1) + (S U L_2)) \subset \beta^* S$ whenever $L_1, L_2$ are finite subsets of $R/H$.

Proof. Let $\beta^*((S U L_1) + (S U L_2)) \subset \beta^* S$. Let $\mathcal{L}_S^H - \lim x_n = x, \mathcal{L}_S^H - \lim y_n = y$. There are finite subsets $K_1, K_2$ of $R/H$ such that $x_n \in R_{S U K_1}$, $\lim x_n = x$, and $y_n \in R_{S U K_2}$, $\lim y_n = y$. Hence $\lim(x_n + y_n) = x + y$. Since $\beta^*((S U K_1) + (S U K_2)) \subset \beta^* S$ there is, according to (i) above, a finite $K \subset R/H$ such that $((S U K_1) + (S U K_2)) \subset S U K$. Consequently, $R_{((S U K_1) + (S U K_2))} \subset R_{S U K}$ and so $(x_n + y_n) \in R_{S U K}$, $\lim(x_n + y_n) = x + y$. We have $\mathcal{L}_S^H - \lim(x_n + y_n) = x + y$, by Lemma 4.

Suppose that there are finite subsets $K_1, K_2$ of $R/H$ such that $\beta^*((S U K_1) + (S U K_2)) \notin \beta^* S$. According to (i) we deduce $((S U K_1) + (S U K_2)) \notin S U K$ for every finite subset $K$ of $R/H$. It follows that there is an infinite set of elements $\xi'_n = \xi'_n + \eta'_n, \xi'_n \in (S U K_1), \eta'_n \in (S U K_2)$ such that if $K$ is finite then there is $n_K \in N$ such that $\xi'_n \notin (S U K), n \geq n_K$. Since the sequence $\langle \xi'_n \rangle$ is one-to-one there is a sequence $\langle \xi_n \rangle \subset \langle \xi'_n \rangle$, $\xi_n = \xi'_n + \eta_n$ such that either $\langle \xi_n \rangle, \langle \eta_n \rangle$ are one-to-one or one of them, say $\langle \xi_n \rangle$, is one-to-one whereas the other is a constant one, i.e. $\eta_n = \eta, n \in N$. In the first case there is (in view of the statement in the proof of Lemma 6) a subsequence $\langle \xi_{n_k} \rangle \subset \langle \xi_n \rangle$, points $x \in R$ and $y \in R$, sequences $\langle x_n \rangle, x_n \in T_{\xi_n}$, and $\langle y_n \rangle, y_n \in T_{\eta_n}$, such that $\mathcal{L}_S^H - \lim x_n = x$ and $\mathcal{L}_S^H - \lim y_n = y$. In the second case
we choose \( y \in T \) and put \( y_n = y, n \in \mathbb{N} \). Then \( \mathfrak{L}_S^H - \lim x_n = x \) and \( \mathfrak{L}_S^H - \lim y_n = y \). In both cases we have a sequence \( (x_n + y_n) \) of non-equivalent points \( (x_n + y_n) \in T \), which does not \( \mathfrak{L}_S^H \)-converge to the point \( x + y \) because there is no finite \( K \subset R/H \) such that \( \zeta_n \in (S \cup K), n \geq n_0 \). \( \square \)

**Lemma 11.** Let \( H \) be a subgroup of the group \( R \) and \( S \) a finite subset of \( R/H \). Then \( (R, \mathfrak{L}_S^H, +) \) is a cc-group.

**Proof.** \( S \) and \( S^- \) are finite sets. Hence \( \beta^*(S) = \emptyset, \beta^*(S^-) = \emptyset \). The condition (–) is satisfied, by Lemma 9. Now, let \( L_1, L_2 \) be finite. Then \( ((S \cup L_1) + (S \cup L_2)) \) is a finite subset of \( R/H \) and so \( \beta^*((S \cup L_1) + (S \cup L_2)) = \emptyset, \beta^*(S) = \emptyset \). Hence (+) is satisfied, by Lemma 10. \( \square \)

Next we use lemmas 9 and 10 to answer the question: Given a subgroup \( H \subset R \) does there exist more than two cc-groups \( (R, \mathfrak{L}_S^H, +)? \)

**Lemma 12.** Let \( S \) be an infinite and \( K \) a finite subset of \( R/H \). Let \( \langle \xi_n \rangle \) be a one-to-one sequence of elements \( \xi_n \in S \cup K \). Then there is \( n_0 \) such that \( \xi_n \in S, n \geq n_0 \).

**Proof.** Since \( \langle \xi_n \rangle \) is one-to-one the finite set \( K \) contains at most a finite number of elements \( \xi_n \). \( \square \)

**Lemma 13.** Let \( H \) be a subgroup of the group \( R \). Let \( S \) be an infinite subset of \( R/H \). Let \( (R, \mathfrak{L}_S^H, +) \) be a cc-group. Let \( \langle \xi_n \rangle \) be a one-to-one sequence of elements \( \xi_n \in S \). Let \( \eta \in R/H \). Then there is \( n_0 \) such that \( \langle \xi_n + \eta \rangle \in S, n \geq n_0 \).

**Proof.** Put \( L_1 = \emptyset, L_2 = \{ \eta \} \). Then \( \xi_n \in (S \cup L_1), \eta \in (S \cup L_2) \). Since \( (R, \mathfrak{L}_S^H, +) \) is a cc-group the condition (+) is satisfied. We can apply Lemma 10. There is a finite \( K \subset R/H \) such that \( \langle \xi_n + \eta \rangle \in S \cup K, n \in \mathbb{N} \). Therefore \( \langle \xi_n + \eta \rangle \in S, n \geq n_0 \), by Lemma 12. \( \square \)

**Lemma 14.** Let \( H \) be a subgroup of the group \( R \). Let \( S \) and \( R/H - S \) be infinite subsets of \( R/H \). Then \( (R, \mathfrak{L}_S^H, +) \) fails to be a cc-group.

**Proof.** Suppose that, on the contrary, \( (R, \mathfrak{L}_S^H, +) \) is a cc-group. Denote \( S' = R/H - S \). Let \( \langle \xi_n \rangle, \xi_n \in S, \langle \eta_n \rangle, \eta_n \in S', \) be one-to-one sequences. According to Lemma 12 there is \( n_1 \) such that \( \langle \xi_n + \eta_1 \rangle \in S, n \geq n_1 \). Put \( m_1 = n_1 \) and \( \zeta_1 = \xi_{m_1} + \eta_1 \). Suppose that we have chosen natural numbers \( m_1 < m_2 < \ldots < m_p \) and non-equivalent elements \( \zeta_i \in S, i \leq p \), where \( \zeta_i = \xi_{m_i} + \eta_i, i \leq p \). Notice, that \( \langle \xi_n + \eta_{p+1} \rangle \) is a one-to-one sequence such that \( \langle \xi_n + \eta_{p+1} \rangle \in S, n \geq n_0 \), by Lemma 13. It follows that there is a natural number \( m_{p+1} > m_p + n_0 \) such that
$(\xi_{m+1} + \eta_{p+1}) \neq \zeta_i, i \leq p$. Put $\zeta_{p+1} = \xi_{m+1} + \eta_{p+1}$. Hence we have an increasing sequence $m_1 < m_2 < \ldots < m_{p+1}$ and a one-to-one sequence $\zeta_1, \zeta_2, \ldots, \zeta_{p+1}$ of elements of $S$. We have constructed, by means of mathematical induction, a one-to-one sequence of elements $\zeta_i \in S, \zeta_i = \xi_{m_i} + \eta_i, i \in N$. Elements $\xi_{m_i}$ belong to the set $S$ and elements $-\xi_{m_i}$ to the set $S^* = S \cap S^* \cup (S^* - S)$. $(R, \mathcal{L}_S^H, +)$ satisfies (−) and so $S^* - S$ is a finite set, by Lemma 9. Put $L_1 = \emptyset, L_2 = S^* - S$. Then $\zeta_i \in S \cup L_1$ and $-\xi_{m_i} \in S \cup L_2$. According to Lemma 10 there is a finite $K \subset R/H$ such that $(\zeta_i, -\xi_{m_i}) \in S \cup K$, i.e. $\eta_i \in S \cup K$. The sequence $\langle \eta_i \rangle$ is one-to-one. According to Lemma 12 there is $i_0$ such that $\eta_i \in S, i \geq i_0$. On the other hand, $\eta_i \in S', i \in N$. Thus we get a contradictory result.

There is a close connection between cc-groups $(R, \mathcal{L}_S^H, +)$ and complete groups with respect to the convergence $\mathcal{L}_S^H$. This is shown in the following lemma.

**Lemma 15.** $(R, \mathcal{L}_S^H, +)$ is a cc-group if and only if it is a complete group.

**Proof.** Let $(R, \mathcal{L}_S^H, +)$ be a cc-group. By Lemma 14 there is a finite $K \subset R/H$ such that $S = K$ or $S = R/H - K$. If $S = R/H - K$ then $(R, \mathcal{L}_S^H, +)$ is a complete because $\mathcal{L}_S^H = \mathcal{L}$. Now, suppose that $S$ is a finite set. Let $\langle c_n \rangle, c_n \in R$, be a Cauchy sequence of points $c_n$ in $(R, \mathcal{L}_S^H, +)$. Distinguish two cases

1) There is a finite subset $K_0$ such that $c_n \in R_{K_0}$. The sequence $\langle c_n \rangle$ is a Cauchy sequence with respect to $\mathcal{L}$, because $\mathcal{L}_S^H \subset \mathcal{L}$, by Lemma 3. Hence there is a point $x \in R$ such that $\lim c_n = x$. We have $\mathcal{L}_S^H - \lim c_n = x$, according to Lemma 4.

2) There is a subsequence $\langle b_n \rangle \subset \langle c_n \rangle$ of non-equivalent points $b_n \in R$. We construct, analogously as in [1], a subsequence $\langle b_{i_n} \rangle \subset \langle b_n \rangle$ such that $\langle b_n - b_{i_n} \rangle$ does not $\mathcal{L}_S^H$-converge to 0. Put $i_1 = 1$. Suppose that we have chosen points $b_{i_1}, b_{i_2}, \ldots, b_{i_k}, i_1 < i_2 < \ldots < i_k$, such that no two numbers $t_m = b_m - b_{i_m}$, $m \leq k$, are equivalent. We prove that there is a point $b_{i_{k+1}}$, $i_{k+1} > i_k$, in the sequence $\langle b_n \rangle$ such that no two numbers $t_m, 1 \leq m \leq k + 1$ are equivalent. Let $q > k$. Suppose (indirect proof) that there is no point $b_s, i_k < s \leq i_k + q$ in the sequence $\langle b_n \rangle$ such that any two numbers $b_{k+1} - b_s$ and $t_m, m \leq k$, are non-equivalent. Denote $u_s = b_{k+1} - b_s$. Let $f: \{i_k < s \leq i_k + q\} \to \{1, 2, \ldots, k\}$ be a (one-valued) function such that $u_s$ and $t_{f(s)}$ are equivalent numbers. Since $q > k$ there are $s_1 > i_k$ and $s_2 \leq i_k + q, s_1 < s_2$, such that $f(s_1) = f(s_2)$. Consequently, the numbers $u_{s_1}, u_{s_2}, f(s_{s_1})$ are equivalent and also numbers $u_{s_2}, f(s_{s_2})$ are equivalent. It follows that $(b_{f(s_1)} - b_{s_1}) \in H, (b_{f(s_2)} - b_{s_2}) \in H$. Hence $(b_{s_1} - b_{s_2}) \in H$ and so $b_{s_1}, b_{s_2}$ are equivalent points. This is a contradiction because $b_n$ are non-equivalent points. We conclude that there is $s_0 \in \{i_k + 1, i_k + 2, \ldots, i_k + q\}$ such that points $b_{s_0}, b_{i_m}, m \leq k$, are non-equivalent. Hence, it suffices to put $i_{k+1} = s_0$.

In such a way we have constructed a sequence $\langle b_n - b_{i_n} \rangle$ of non-equivalent points. Since $S$ is finite it follows from Lemma 4 that the sequence $\langle b_n - b_{i_n} \rangle$ does not $\mathcal{L}_S^H$-
converge to 0. Therefore \( (c_n) \) is not a Cauchy sequence with respect to \( \mathcal{L}_S^H \). The case 2) cannot occur.

Let \( (R, \mathcal{L}_S^H, +) \) be a complete group with respect to the convergence \( \mathcal{L}_S^H \). Then it is a cc-group, by the definition on p. 25.

Lemmas 11 and 14 give us a complete information about structures \( (R, \mathcal{L}_S^H, +) \) which are cc-groups. If \( H = R \) then \( (R, \mathcal{L}, +) \) is the unique cc-group. If \( H \neq R \) then there are exactly two different cc-groups, i.e. \( (R, \mathcal{L}_R^H, +) \) and \( (R, \mathcal{L}, +) \).

Closing remarks. If \( Q \) is a subgroup of \( H \) and \( H \) a subgroup of \( R \), then there are two different completions \( (R, \mathcal{L}_S^H, +) \) of \( Q \), namely, \( (R, \mathcal{L}_R^H, +) \) and \( (R, \mathcal{L}, +) \). It follows that there is more than one completions of \( Q \). There would be interesting to know what is the number of completion of the group of rational numbers \( Q \).

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Addendum after the proofs. P. Simon and R. Frič proved, independently from each other, that the number of completions of the group \( Q \) is \( \exp(\exp(\omega)) \) [2].

References


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