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CONVERGENCES $\mathcal{L}_S^H$ FOR THE GROUP OF REAL NUMBERS

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For each subgroup $H$ of the group $R$ of real numbers and each subset $S$ of the quotient group $R/H$ a convergence $\mathcal{L}_S^H$ for the group $R$ is constructed. The relation of the system of convergences $\mathcal{L}_S^H$ to the Čech-Stone compactification of discrete spaces is clarified. Necessary and sufficient conditions are given for $(R, \mathcal{L}_S^H, +)$ to be a complete group with respect to the convergence $\mathcal{L}_S^H$. This gives some views on the structure of the groups $R$ and $R/H$.

The point of our considerations is the group $(R, +)$ of real numbers. We use the fact that $R$ is a linearly ordered point set for which a convergence $\mathcal{L}$ is defined by means of open intervals $(a, b) \subset R$ such that $\lim x_n = x$, $\lim y_n = y$ implies that $\lim(x_n - y_n) = x - y$. In this sense $R$ is a convergence commutative group ([1]). It will be denoted $(R, \mathcal{L}, +)$.

Recall that a convergence $\mathcal{M}$ for a set $M$ is a collection of pairs $((x_n), x)$ where $(x_n)$ is a sequence of points $x_n \in M$ and $x \in M$. We assume that the convergence $\mathcal{M}$ satisfies the well known Fréchet axioms of convergence inclusive the axiom of the maximal convergence $(\mathcal{M} = \mathcal{M}^*)$. A commutative group $(M, +)$ with a convergence $\mathcal{M}$ will be denoted $(M, \mathcal{M}, +)$. If $((x_n), x) \in \mathcal{M}$, $((y_n), y) \in \mathcal{M}$ implies that $((x_n - y_n), x - y) \in \mathcal{M}$ we have a convergence commutative group $(M, \mathcal{M}, +)$ (abbr. cc-group). In such a group Cauchy sequences are defined to be sequences $(x_n), x_n \in M,$ such that $((x_n - x_{i_n}), 0) \in \mathcal{M}$ whenever $(x_{i_n}) \subset (x_n)$. A cc-group $(M, \mathcal{M}, +)$ is complete if each Cauchy sequence $\mathcal{M}$-converges in $M$, more precisely, if $(x_n), x_n \in M,$ is a Cauchy sequence then there is a point $x \in M$ such that $((x_n), x) \in \mathcal{M}$.
Notation. We denote \( N \) the set of natural numbers, \( N^{-1} \) the set of numbers \( n^{-1}, n \in N \), \( Q \) the group of rational and \( R \) the group of real numbers, \( H \) a subgroup of the group \( R \) and \( S \) a subset of the quotient group \( R/H \). A subgroup of \( R \) is either discrete or dense. Points \( x_1 \) and \( x_2 \) of \( R \) are non-equivalent (with respect to \( H \)) if \( (x_1 - x_2) \neq H \). In the section I we consider \( R/H \) as a set of points sometimes called indexes. They will be denoted by Greek letters \( \xi, \eta, \zeta \).

Let \( H \) be a subgroup of the group \( R \) and \( R/H \) the corresponding quotient group. Elements \( \xi \in R/H \) are classes \( T_\xi = a_\xi + H \) where \( a_\xi \) is a representative of the class \( T_\xi \). We identify elements \( \xi \) with ordinals \( \xi < \omega_H \) where \( \omega_H \) is the least ordinal of the power \( |R/H| \). We put \( a_0 = 0 \). Then \( T_0 = H \). Notice that \( R = UT_\xi, \xi \in R/H, |R/Q| = \exp(\omega), |R/H| = 1 \).

Definition D1. Let \( H \) be a subgroup of the group \( R \). Functions \( f: R/H \to N^{-1} \) are called generating functions. Adjoin to \( S \subset R/H \) a class \( F^H_S \) (or simply \( F_S \)) of generating functions \( f \) such that the partial function \( f/S \) is a constant function. \( f/\emptyset \) is considered as a constant partial function. If \( S \) contains only one index \( \xi \) we write \( F_\xi \) instead of \( F_\{\xi\} \).

Lemma 1. Let \( S_1 \subset S_2 \subset R/H \). Then \( F_{S_1} \supset F_{S_2} \).

Proof. If \( f \in F_{S_2} \) then \( f/S_2 \) is constant and \( f/S_1 \) as well. Hence \( f \in F_{S_1} \).

Definition D2. Let \( H \) be a subgroup of the group \( R \). Let \( (a, b) \) be an open interval of real numbers. Denote \( (a, b)_\xi = (a, b) \cap T_\xi, \xi \in R/H \). Let \( z \) be a point of \( R \) and \( S \) a subset of \( R/H \). A set \( W(z) \) is called a closure neighborhood or, simply, a neighborhood of the point \( z \) if there is a generating function \( f \in F_S \) such that \( W_f(z) \subset W(z) \) where

\[
W_f(z) = U(z - f(\xi), z + f(\xi))_{\xi}, \quad \xi \in R/H.
\]

Remark. Let \( (a, b) \subset R, z \in (a, b) \). Choose \( m_0 \in N \) such that \( m_0^{-1} < \min\{z - a, b - z\} \) and put \( f(\xi) = m_0^{-1}, \xi \in R/H \). Then \( f \in F^H_S \). Hence \( W_f(z) \subset (a, b) \). Consequently, the open interval \( (a, b) \) in \( R \) is a closure neighborhood of each point \( z \in (a, b) \).

The following are the main properties of closure neighborhoods \( W_f(z), f \in F^H_S \).

(i) \( z \in W_f(z) \). (If \( z \in T_{\xi_0} \) then \( z \in (z - f(\xi_0), z + f(\xi_0))_{\xi_0} \subset W_f(z) \), by D2).
(ii) If \( W_{f_i}(z), f_i \in F_S, i = 1, 2 \), are neighborhoods of a point \( z \), then \( W_{f_1}(z) \cap W_{f_2}(z) \) is a neighborhood of the point \( z \). (\( W_{f_1}(z) \cap W_{f_2}(z) = W_{f_3}(z) \), where \( f_3(\xi) = \min\{f_1(\xi), f_2(\xi)\}, \xi \in R/H \).)

(iii) If \( z_1 \neq z_2 \) there are \( f_i \in F_S, i = 1, 2 \), such that \( W_{f_1}(z_1) \cap W_{f_2}(z_2) = \emptyset \) (see Remark above).

From (i), (ii), (iii) we deduce that the system of closure neighborhoods \( W_f(z), f \in F_S^H \), of points \( z \in R \) satisfies the axioms of Hausdorff topological spaces except the axiom of open neighborhoods which need not be fulfilled. This is shown in the following

**Lemma 2.** Let \( H \) be a dense subgroup of the group \( R \) and \( S \subset R/H \). Let \( z \in R \). Then there is a complete system of open closure neighborhoods at the point \( z \) if and only if there is a finite \( K \subset R/H \) such that \( S = R/H - K \).

**Proof.** Let \( W_f(z), f \in F_S^H \), be a neighborhood of the point \( z \). Since the partial function \( f/(R/H - K) \) is constant and \( K \) is finite there is a natural number \( p \) such that \( p^{-1} < f(\xi), \xi \in R/H \). Hence \( (z - p^{-1}, z + p^{-1}) \subset W_f(z) \). It follows that the system of intervals \( (z - n^{-1}, z + n^{-1}), n \in N \), is a complete system of open neighborhoods at the point \( z \).

Now, assume that \( R/H - S \) is infinite. Choose distinct \( \xi_n \in (R/H - S) \). Define \( f(\xi) = 1, \xi \neq \xi_n, f(\xi_n) = n^{-1}, n \in N \). Then \( f \in F_S \) and we have a neighborhood \( W_f(z) \). Let \( W_g(z) \subset W_f(z), g \in F_S \). Suppose that (on the contrary) \( W_g(z) \) is open. The neighborhood \( W_g(z) \) is infinite because \( H \) is dense. Choose a point \( t \in W_g(z), t \neq z \). Then there is, by the assumption, a neighborhood \( W_h(t) \subset W_g(z), h \in F_S^H \). Notice that \( h(\xi) \leq g(\xi) \leq f(\xi), \xi \in R/H \). There are \( \varepsilon_n > 0 \) such that

\[
(t - \varepsilon_n, t + \varepsilon_n)_{\xi_n} \subset (t - h(\xi_n), t + h(\xi_n))_{\xi_n} \subset (z - g(\xi_n), z + g(\xi_n))_{\xi_n} \\
\subset (z - f(\xi_n), z + f(\xi_n))_{\xi_n} \subset (z - n^{-1}, z + n^{-1}), \quad n \in N.
\]

Hence \( t \in (z - n^{-1}, z + n^{-1}) \) and so \( t = z \). This is a contradiction. Thus \( W_g(z) \) is not open.

We have seen above that the class \( F_S^H \) generates a complete system of closure neighborhoods \( W_f(z) \) at the point \( z \). By neighborhoods \( W_f(z), f \in F_S^H \), a convergence for the group \( R \) is defined in a well known way. \( \square \)

**Definition D3.** Let \( H \) be a subgroup of the group \( R \), \( S \subset R/H \). Denote \( \mathcal{L}_S^H \) a collection of pairs \( ((x_n), x), x_n \in R, x \in R \), such that if \( W_f(x), f \in F_S^H \), is a neighborhood of the point \( x \) then \( x_n \in W_f(x), n \geq n_0 \). If \( ((x_n), x) \in \mathcal{L}_S^H \) we say that the sequence \( (x_n) \mathcal{L}_S^H \)-converges to the point \( x \) and write \( \mathcal{L}_S^H \lim x_n = x \). The collection \( \mathcal{L}_S^H \) is called a convergence for \( R \). (It will be sometimes denoted \( \mathcal{L}_S \)).

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Fréchet axioms of convergence are clearly satisfied. From (iii) it follows that \( L \Gamma_S \lim x_n = x \), \( L \Delta_S \lim y_n = y \) implies \( x = y \). In view of (i) we have \( L \Delta_S \lim x = x \). If \( \langle (x_n), x \rangle \in L \Gamma_S \), \( \langle x_i \rangle \subset \langle x_n \rangle \), then \( \langle (x_{i_n}), x \rangle \in L \Gamma_S \), by D3. From D3 it instantly follows that \( L \Gamma_S \) is a maximal convergence, i.e. \( L \Delta_S = L \Delta_S^* \).

Denote \( L \) the usual metric convergence for \( R \). We write simply \( \lim x_n = x \) instead of \( L \lim x_n = x \). Note that \( L = L \Gamma_{R/H} \).

**Lemma 3.** Let \( S_1 \subset S_2 \subset R/H \). Then \( L \Gamma_{S_1} \subset L \Gamma_{S_2} \).

**Proof.** Let \( \langle (x_n), z \rangle \in L \Gamma_{S_1} \). Let \( W_f(z), f \in F^H_{S_2} \), be a neighborhood of the point \( z \). Define a generating function \( g(\xi) = f(\xi), \xi = S_1, g(\xi) \leq f(\xi), \xi \in R/H - S_1 \). The partial function \( g/S_1 \) is constant, by Lemma 1, and so \( y \in F^H_{S_1} \).

Since \( x_n \in W_g(z), n \geq n_0 \), and \( W_g(z) \subset W_f(z) \) we have \( x_n \in W_f(z), n \geq n_0 \). Hence \( \langle (x_n), z \rangle \in L \Gamma_{S_2} \).

The assertion \( L \Gamma_{S_1} \subset L \Gamma_{S_2} \) implies \( S_1 \subset S_2 \) is not correct. Let \( H \) be a subgroup of \( R, R \neq H \). Choose indexes \( \xi \neq \xi_2 \) and put \( S_1 = \{ \xi \}, S_2 = \{ \xi_2 \} \). Then \( L \Gamma_{S_1} = L \Gamma_{S_2} \), but \( S_1 \notin S_2 \). This example shows that the map \( \varphi(S) = L \Gamma_S, S \subset R/H, H \neq R \), is not one-to-one even when it preserves the order relation \( \subset \), by Lemma 3. Next we investigate the structure of the system of classes \( \varphi^{-1}(L \Gamma_S), S \subset R/H \).

Let \( H \) be a subgroup of the group \( R, S \subset R/H \). Denote \( R_S = U T_\xi, \xi \in S \). Notice that \( R_S \subset R, R_\emptyset = \emptyset, R_{\{0\}} = H, R_{R/R} = R \).

**Lemma 4.** Let \( H \) be a subgroup of the group \( R, S \subset R/H \). Then \( L \Gamma_S - \lim z_n = z \) if and only if \( \lim z_n = z \) and there is a finite \( K \subset R/H \) such that \( z_n \in R_{S\cup K}, n \in N \).

**Proof.** Let \( L \Gamma_S - \lim z_n = z \). Then \( \lim z_n = z \) because \( L \Gamma_S \subset L \), by Lemma 3. Suppose that (on the contrary) there is a subsequence \( \langle z_{i_n} \rangle \subset \langle z_n \rangle \), \( z_{i_n} \neq z \), and distinct indexes \( \eta_n \in (R/H - S) \) such that \( z_{i_n} \in T_{\eta_n} \). Put \( f(\xi) = 1, \xi \neq \eta_n \), and choose \( f(\eta_n) \in N^{-1} \) such that \( z_{i_n} \notin (z - f(\eta_n), f(\eta_n)) \), \( n \in N \). This is possible because \( z_{i_n} \neq z \). Then \( f \in F_S \) and we have a neighborhood \( W_f(z) \) of \( z \) which contains no point \( z_i \). Hence \( \langle z_{i_n} \rangle \) does not \( L \Gamma_S \)-converge to \( z \). This is in contradiction with the assumption \( \langle (z_n), z \rangle \in L \Gamma_S \).

Now, let \( \lim z_n = z \), \( z_n \in R_{S\cup K} \). We use the property \( L \Gamma_S = L \Gamma_S^* \) to prove that \( L \Gamma_S - \lim z_n = z \). Let \( \langle z_{i_n} \rangle \) be a subsequence of \( \langle z_n \rangle \). Either there is a subsequence \( \langle t_n \rangle \subset \langle z_{i_n} \rangle \) of non-equivalent points \( t_n \in R_S \) and then \( L \Gamma_S - \lim t_n = z \) or it is not so, and there is an index \( \xi_0 \in (S \cup K) \) and a subsequence \( \langle u_n \rangle \subset \langle z_{i_n} \rangle, u_n \in T_{\xi_0} \).

Hence \( L \Gamma_S - \lim u_n = z \). It follows that \( \langle (z_n), z \rangle \in L \Gamma_S \).

**Lemma 5.** Let \( H \) be a subgroup of the group \( R \). Let \( S_1 \subset R/H, i = 1, 2 \). Let \( S_1 \div S_2 \) be a finite set. Then \( L \Gamma_{S_1} = L \Gamma_{S_2} \).
Proof. Let \((z^n, z) \in \mathcal{L}_{S_1} \) and \(W_f(z), f \in \mathcal{F}_{S_2}\), be a neighborhood of the point \(z \in R\). We are to prove that \(z^n \in W_f(z), n \geq n_0\). Notice that \(S_1 \cup S_2 = (S_1 \setminus S_2) \cup (S_1 \cap S_2)\). The partial function \(f/S_1 \cap S_2\) is constant, by Lemma 1, and \(S_1 \setminus S_2\) is a finite set. Therefore the number \(d = \min\{f(\xi)\}, \xi \in S_1 \cup S_2\), belongs to the set \(N^{-1}\). Put \(g(\xi) = d, \xi \in S_1 \cup S_2\), and \(g(\xi) \in N^{-1}\), \(\xi \in (R/H - (S_1 \cup S_2))\). Then \(g \in \mathcal{F}_{S_2}\) and so \(z^n \in W_g(z), n \geq n_0\). Hence \(z^n \in W_f(z), n \geq n_0\) and therefore \(\mathcal{L}_{S_1} \subseteq \mathcal{L}_{S_2}\).

Analogously we prove that \(\mathcal{L}_{S_2} \subseteq \mathcal{L}_{S_1}\). \(\Box\)

Lemma 6. Let \(H\) be a subgroup of the group \(R\). Let \(S_i \subseteq R/H, i = 1, 2\). Let \(\mathcal{L}^H_{S_i} \subseteq \mathcal{L}^H_{S_2}\). Then \(S_1 - S_2\) is a finite set.

Proof. First prove the following statement: If \(S_0\) is an infinite subset of \(R/H\) then there is a sequence of non-equivalent points \(x_n \in T_\xi, \xi \in S_0\), and a point \(z \in R\) such that \(\mathcal{L}^H_{S_0} - \lim x_n = z\). Distinguish two cases. 1) \(H\) is dense. Let \(\langle \xi_n \rangle\) be one-to-one sequence of indexes \(\xi \in S_0\). Choose a point \(z \in R\). Since \(H\) is dense there is a sequence \(\langle x_n \rangle\) of non-equivalent points \(x_n \in T_\xi, \lim x_n = z\). Hence \(\mathcal{L}^H_{S_0} - \lim x_n = z, \text{by Lemma 4}\). 2) \(H\) is discrete. Denote \(d\) the least positive number of \(H\). Choose numbers \(b_\xi \in T_\xi\) such that \(0 \leq b_\xi < d, \xi \in R/H\). Since \(S_0\) is infinite there is a one-to-one sequence \(\langle \xi_n \rangle\), \(\xi_n \in S_0\), and a point \(z \in R, 0 \leq z \leq d\), such that \(\lim b_{\xi_n} = z\). Denote \(b_{\xi_n} = x_n\). Then \(\langle x_n \rangle\) is a sequence of non-equivalent points \(x_n\) with \(\mathcal{L}^H_{S_0} - \lim x_n = z, \text{by Lemma 4}\).

Suppose that \(S_1 - S_2\) is infinite and denote \(S_0 = S_1 - S_2\). Then \(S_0 \subseteq S_1\) and \((\langle x_n \rangle, z) \in \mathcal{L}^H_{S_1}\), by Lemma 3 where \(\langle x_n \rangle\) is the sequence constructed above. On the other hand, \((\langle x_n \rangle, z) \notin \mathcal{L}^H_{S_2}\), by Lemma 4. This is a contradiction. \(\Box\)

Proposition 1. Let \(H\) be a subgroup of the group \(R, S_i \subseteq R/H, i = 1, 2\). Then \(\mathcal{L}^H_{S_1} = \mathcal{L}^H_{S_2}\) if and only if \(S_1 - S_2\) is a finite set.

Proof follows instantly from Lemmas 5 and 6.

From Proposition 1 it follows that there is a connection between convergences \(\mathcal{L}^H_{S_i}\) and some subsets of the Čech-Stone compactification of a discrete topological space. Consider \(R/H\) as a discrete topological space of isolated points \(\xi\) and denote \(\beta^*S = \beta S - R/H, \text{where} \ \beta\) is a topological operator in the Čech-Stone compactification \(\beta(R/H)\). It is well known that \(\beta^*S_1 = \beta^*S_2\) if and only if \(S_1 - S_2\) is finite. Hence \(\mathcal{L}^H_{S_1} = \mathcal{L}^H_{S_2}\) if and only if \(\beta^*S_1 = \beta^*S_2\), by Proposition 1.

Let \(H\) be a subgroup of the group \(R\). We denote, as above, functions \(\varphi(S) = \mathcal{L}^H_{S}\), \(S \subseteq R/H\). We have shown that \(\varphi\) is not one-to-one except in the case when \(H = R\). From Proposition 1 it follows that \(S_1\) and \(S_2\) are equivalent (i.e. \(S_2 \in \varphi^{-1}(\mathcal{L}_{S_1})\)) iff \(S_1 - S_2\) is finite. Now, define a quasi-order \(<\) as follows: \(S_1 < S_2\) if there is a finite \(K \subseteq R/H\) such that \(S_1 \subseteq S_2 \cup K\).
Lemma 7. Let $H$ be a subgroup of the group $R$. Then $S_1 \prec S_2$ if and only if $\mathcal{L}^H_{S_1} \subset \mathcal{L}^H_{S_2}$.

Proof. Let $S_1 \prec S_2$. Then $S_1 \subset S_2 \cup K$. It follows $\mathcal{L}_{S_1} \subset \mathcal{L}_{S_2}$, by Lemma 3 and Proposition 1. Now, let $\mathcal{L}_{S_1} \subset \mathcal{L}_{S_2}$. According to Lemma 6 the set $S_1 - S_2$ is finite. Since $S_1 \subset S_2 \cup (S_1 - S_2)$ we have $S_1 \prec S_2$. □

Proposition 2. Let $H$ be a subgroup of the group $R$. There is a similar map (with respect to the inclusion $\subset$), on the system $\mathcal{L}^H$ of convergences $\mathcal{L}^H_S$, $S \subset R/H$, onto the system of clopen sets $\beta^*(S)$ of the space $\beta^*(R/H)$.

Proof. Denote $\psi(\mathcal{L}^H_S) = \beta^*S$, $S \subset R/H$. Let $S_1$, $S_2$ be subsets of $R/H$, $\mathcal{L}_{S_1} \subset \mathcal{L}_{S_2}$. Then $S_1 \prec S_2$, by Lemma 7. Therefore, according to the definition of the quasi-order $\prec$ it follows that $\beta^*S_1 \subset \beta^*S_2$. It remains to prove the following implication: If $A$ is a clopen subset of $\beta(R/H)$ then there is $S \subset R/H$ such that $\beta^*S = A$. This is true because there is a clopen set $B$ in $\beta(R/H)$ such that $A = B \cap \beta^*(R/H)$ and so there is $S \subset R/H$ such that $A = \beta^*(S)$. □

Remark. Notice that $\mathcal{R}_{\alpha_1} \subset \mathcal{R}_{\alpha_2}$ implies $\mathcal{R}_{\alpha_1} \cdot \mathcal{R}_{\alpha_2} = \mathcal{R}_{\alpha_2}$. Let $S \subset R/Q$. Denote $F = \{K \subset R/Q, K$ finite$, \}, X_S = \{S \div K : K \in F\}, Y = \{S ; S \subset R/Q\}, Z = Y/F, \mathcal{R}_{\alpha_1} = |X_S|, \mathcal{R}_{\alpha_2} = |Z|$. Clearly $\mathcal{R}_{\alpha_1} = \exp(\omega), \mathcal{R}_{\alpha_2} = \exp(\exp(\omega))$. Then $|L^Q| = |X_S| \cdot |Z| = \exp(\omega) \cdot \exp(\exp(\omega)) = \exp(\exp(\omega))$. Thus the number of convergences $L^Q_S$, $S \subset R/Q$, is $\exp(\exp(\omega))$.

Let $H$ be a subgroup of the group $R$, $S \subset R/H$. We have seen that a closure topology for $R$ is defined by means of the class $F^H_S$ of generating functions. The corresponding closure operator will be denoted $w^H_S$ (or simply $w_S$). Hence $w_S A = \{x \in R : A \cap W_f(x) \neq \emptyset, f \in F^H_S\}$. Another closure topology for $R$ is defined by means of the convergence $\mathcal{L}^H_S$. Denote $\lambda^H_S$ (or $\lambda_S$) the corresponding closure operator: $\lambda^H_S A = \{x \in R ; x = \mathcal{L}^H_S - \lim x_n, x_n \in A, n \in N\}$. Hence we have closure spaces $(R, w^H_S)$ and $(R, \lambda^H_S)$.

Now, we are interested in the question what is the relation between closures $\lambda^H_S$ and $w^H_S$. It is well known that there are closure spaces $(P, u)$ and adjoint convergence spaces $(P, \lambda_u)$ such that $u \neq \lambda_u$. It is not the case if $P = R$, $u = w^H_S$. We show that $w^H_S = \lambda^H_S$. It is evident that $\lambda_S A \subset w_S A$, $A \subset R$. Suppose that there is $z \in R$ and $A \subset R$ such that $z \in (w_S A - \lambda_S A)$. Then $z \notin A$ and there is no sequence of points $x_n \in A$ such that $\mathcal{L}^H_S - \lim x_n = z$. In view of Lemma 4 there is a generating function $f \in F_S$ such that $A \cap (z - f(\xi), z + f(\xi)) = \emptyset, \xi \in R/H$. Hence $A \cap W_f(z) = \emptyset$. This is a contradiction. Consequently, $w_S = \lambda_S$.

Notice that $\omega_1$-iterated closure $\lambda^{\omega_1}_S = w^{\omega_1}_S$ are topologies for $R$. 

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In this section we investigate some convergence and group properties of the structures \((R, \mathcal{L}_S^H, +)\). For this purpose we consider indexes \(\xi\) of the set \(R/H\) as elements of the group \((R/H, +)\). If \(\xi_1, \xi_2\) are elements of \(R/H\) then \(\xi_1 + \xi_2 = \xi_3\) where \(\xi_3\) is uniquely determined by the addition \(T_{\xi_1} + T_{\xi_2} = T_{\xi_3}\) in the group \((R/H, +)\). The inverse element to the element \(\xi \in R/H\) is the element \(\eta \in R/H\) such that \(T_{\eta} = -T_{\xi}\). It will be denoted \(-\xi\).

Now, we are going to examine conditions under which \((R, \mathcal{L}_S^H, +)\) is a \(cc\)-group. First we give an example to show that \((R, \mathcal{L}_S^H, +)\) need not be a \(cc\)-group even when \(R_S\) is a subgroup of the group \(R\).

\textbf{Example.} Let \(H = Q\) and let \(R_S\) be the group of algebraic numbers. Put \(x_n = n^{-1}/\sqrt{2}, y_n = \pi + n^{-1}\). Then \(\lim x_n = 0\), \(\lim y_n = \pi\) and \(\mathcal{L}_S^H - \lim x_n = 0\), \(\mathcal{L}_S^H - \lim y_n = \pi\), by Lemma 4. On the other hand, \((x_n + y_n)\) is a sequence of non-equivalent transcendent numbers which, by the same lemma, does not \(\mathcal{L}_S^H\)-converge to the point \(\pi\).

\textbf{Definition D4.} Let \((M, \mathcal{M}, +)\) be a commutative group with a convergence \(\mathcal{M}\) for \(M\). We say that \((M, \mathcal{M}, +)\) satisfies condition \((-)\) provided that the following implication holds

\[\text{If } (\langle x_n \rangle, x) \in \mathcal{M} \text{ then } (\langle -x_n \rangle, -x) \in \mathcal{M}.\]

\((M, \mathcal{M}, +)\) satisfies condition \((+)\) provided that

\[\text{If } (\langle x_n \rangle, x) \in \mathcal{M}, (\langle y_n \rangle, y) \in \mathcal{M} \text{ then } (\langle x_n + y_n \rangle, x + y) \in \mathcal{M}.\]

It is clear that \((M, \mathcal{M}, +)\) is a \(cc\)-group if and only if both the conditions \((-)\) and \((+)\) are satisfied.

\textbf{Definition D5.} Let \(H\) be a subgroup of the group \(R, S \subset R/H\). We denote \(S^-\) the set of elements \(\eta \in R/H\) such that \(T_{\eta} = -T_{\xi}\), \(\xi \in S\).

\textbf{Lemma 8.} \(|S^-| = |S^-|, |S^- - S^-| = |S^- - S|, (S_1 \cup S_2)^- = S_1^- \cup S_2^-, (S_1 \cap S_2)^- = S_1^- \cap S_2^-\), \(x \in R_S\) if and only if \(-x \in R_S^-\).

Proof follows instantly from D5 and from the equivalence \(\xi \in (S - S^-)\) if and only if \(-\xi \in (S^- - S)\).

The properties \((-)\) and \((+)\) can be formulated by means of \(\check{\text{C}}\)ech-Stone operator \(\beta^*\). In the proofs we use the equivalence
(i) $\beta^*S_1 \subset \beta^*S_2$ and only if $S_1 \subset S_2 \cup K$ where $K$ is finite. From (i) it follows
(ii) $\beta^*S_1 = \beta^*S_2$ if and only if $S_1 \div S_2$ is finite.

**Lemma 9.** Let $H$ be a subgroup of the group $R, S \subset R/H$. Then $(R, \mathcal{L}_S^H, +)$ satisfies $(-)$ if and only if $\beta^*S = \beta^*(S^-)$.

**Proof.** Let $\beta^*S = \beta^*(S^-)$. The set $S \div S^-$ is finite, by (ii). Let $\mathcal{L}_S^H - \lim x_n = x$. In view of Lemma 4, there is a finite $K \subset R/H$ such that $x_n \in R_{S \cup K}$ and $\lim x_n = x$. Hence $\lim(-x_n) = -x$ and $-x_n \in R_{S-\cup K}$, by Lemma 8. Notice that $S^- \cup K^- = (S^- \cap S) \cup (S^- - S) \cup K^- \subset S \cup K_1$ where $K_1 = (S^- - S) \cup K^-$. It follows that $K_1$ is a finite subset of $R/H$ and $R_{S-\cup K} \subset R_{S \cup K_1}$. Thus $\mathcal{L}_S^H - \lim(-x_n) = -x$, by Lemma 4.

Let $\beta^*S \neq \beta^*(S^-)$. Then $S \div S^-$ is infinite and both the sets $S - S^-$ and $S^- - S$ are infinite, by Lemma 8. In view of statement (see the proof of Lemma 6) there is a sequence of non-equivalent points $x_n \in R_{S-\cup S^+}$ and a point $z \in R$ such that $\mathcal{L}_{S-\cup S^+} - \lim x_n = z$. Notice that $\langle -x_n \rangle$ is a sequence of non-equivalent points $-x_n \in R_{S-\cup S^+}$. Consequently, $-x_n \notin R_S$. From Lemma 4 it follows that the sequence $\langle -x_n \rangle$ does not $\mathcal{L}_S^H$-converge to $-z$.

**Lemma 10.** Let $H$ be a subgroup of the group $R, S \subset R/H$. $(R, \mathcal{L}_S^H, +)$ satisfies $(+)$ if and only if $\beta^*((S \cup L_1) + (S \cup L_2)) \subset \beta^*S$ whenever $L_1, L_2$ are finite subsets of $R/H$.

**Proof.** Let $\beta^*((S \cup L_1) + (S \cup L_2)) \subset \beta^*S$. Let $\mathcal{L}_S^H - \lim x_n = x, \mathcal{L}_S^H - \lim y_n = y$. There are finite subsets $K_1, K_2$ of $R/H$ such that $x_n \in R_{S \cup K_1}, \lim x_n = x$, and $y_n \in R_{S \cup K_2}, \lim y_n = y$. Hence $\lim(x_n + y_n) = x + y$. Since $\beta^*((S \cup K_1) + (S \cup K_2)) \subset \beta^*S$ there is, according to (i) above, a finite $K \subset R/H$ such that $((S \cup K_1) + (S \cup K_2)) \subset S \cup K$. Consequently, $R_{((S \cup K_1) + (S \cup K_2))} \subset R_{S \cup K}$ and so $(x_n + y_n) \in R_{S \cup K}, \lim(x_n + y_n) = x + y$. We have $\mathcal{L}_S^H - \lim(x_n + y_n) = x + y$, by Lemma 4.

Suppose that there are finite subsets $K_1, K_2$ of $R/H$ such that $\beta^*((S \cup K_1) + (S \cup K_2)) \notin \beta^*S$. According to (i) we deduce $((S \cup K_1) + (S \cup K_2)) \notin S \cup K$ for every finite subset $K$ of $R/H$. It follows that there is an infinite set of elements $\zeta'_n = \xi'_n + n, \zeta'_n \in (S \cup K_1), n \in (S \cup K_2)$ such that if $K$ is finite then there is $n_K \in N$ such that $\zeta'_n \notin (S \cup K), n \geq n_K$. Since the sequence $\langle \zeta'_n \rangle$ is one-to-one there is a sequence $\langle \xi_n \rangle \subset \langle \zeta'_n \rangle, \xi_n = \xi_n + n$ such that either $\langle \xi_n \rangle, \langle \eta_n \rangle$ are one-to-one or one of them, say $\langle \eta_n \rangle$, is one-to-one whereas the other is a constant one, i.e. $n = n, n \in N$. In the first case there is (in view of the statement in the proof of Lemma 6) a subsequence $\langle \xi_{n_k} \rangle \subset \langle \xi_n \rangle$ points $x \in R$ and $y \in R$, sequences $\langle x_n \rangle, x_n \in T_{\xi_{n_k}}, \langle y_n \rangle, y_n \in T_{\eta_{n_k}}$, such that $\mathcal{L}_S^H - \lim x_n = x$ and $\mathcal{L}_S^H - \lim y_n = y$. In the second case
we choose \( y \in T_\eta \) and put \( y_n = y, \ n \in N \). Then \( \mathcal{L}^H_S - \lim x_n = x \) and \( \mathcal{L}^H_S - \lim y_n = y \). In both cases we have a sequence \( (x_n + y_n) \) of non-equivalent points \((x_n + y_n) \in T_\xi, \) which does not \( \mathcal{L}^H_S \)-converge to the point \( x + y \) because there is no finite \( K \subset R/H \) such that \( \xi_n \in (S \cup K), \ n \geq n_0 \). □

**Lemma 11.** Let \( H \) be a subgroup of the group \( R \) and \( S \) a finite subset of \( R/H \). Then \((R, \mathcal{L}^H_S, +)\) is a cc-group.

**Proof.** \( S \) and \( S^- \) are finite sets. Hence \( \beta^*(S) = \emptyset, \beta^*(S^-) = \emptyset \). The condition \((-)\) is satisfied, by Lemma 9. Now, let \( L_1, L_2 \) be finite. Then \((\mathcal{L}^H_S, +) = \emptyset \), \( \beta^*(S) = \emptyset \). Hence \((+)\) is satisfied, by Lemma 10. □

Next we use lemmas 9 and 10 to answer the question: Given a subgroup \( H \subset R \) does there exist more than two cc-groups \((R, \mathcal{L}^H_S, +)\)?

**Lemma 12.** Let \( S \) be an infinite and \( K \) a finite subset of \( R/H \). Let \( \{\xi_n\} \) be a one-to-one sequence of elements \( \xi_n \in S \cup K \). Then there is \( n_0 \) such that \( \xi_n \in S, \ n \geq n_0 \).

**Proof.** Since \( \{\xi_n\} \) is one-to-one the finite set \( K \) contains at most a finite number of elements \( \xi_n \). □

**Lemma 13.** Let \( H \) be a subgroup of the group \( R \). Let \( S \) be an infinite subset of \( R/H \). Let \((R, \mathcal{L}^H_S, +)\) be a cc-group. Let \( \{\xi_n\} \) be a one-to-one sequence of elements \( \xi_n \in S \). Let \( \eta \in R/H \). Then there is \( n_0 \) such that \( (\xi_n + \eta) \in S, \ n \geq n_0 \).

**Proof.** Put \( L_1 = \emptyset, L_2 = \{\eta\} \). Then \( \xi_n \in (S \cup L_1), \ \eta \in (S \cup L_2) \). Since \((R, \mathcal{L}^H_S, +)\) is a cc-group the condition \((+)\) is satisfied. We can apply Lemma 10. There is a finite \( K \subset R/H \) such that \((\xi_n + \eta) \in S \cup K, \ n \in N \). Therefore \((\xi_n + \eta) \in S, n \geq n_0 \), by Lemma 12. □

**Lemma 14.** Let \( H \) be a subgroup of the group \( R \). Let \( S \) and \( R/H - S \) be infinite subsets of \( R/H \). Then \((R, \mathcal{L}^H_S, +)\) fails to be a cc-group.

**Proof.** Suppose that, on the contrary, \((R, \mathcal{L}^H_S, +)\) is a cc-group. Denote \( S' = R/H - S \). Let \( \{\xi_n\}, \xi_n \in S, \{\eta_n\}, \eta_n \in S' \), be one-to-one sequences. According to Lemma 12 there is \( n_1 \) such that \((\xi_n + \eta_1) \in S, n \geq n_1 \). Put \( m_1 = n_1 \) and \( \xi_1 = \xi_{m_1} + 1 \). Suppose that we have chosen natural numbers \( m_1 < m_2 < \ldots < m_p \) and non-equivalent elements \( \xi_i \in S, i \leq p, \) where \( \xi_i = \xi_{m_i} + \eta_i, i \leq p \). Notice, that \((\xi_n + \eta_{p+1})\) is a one-to-one sequence such that \((\xi_n + \eta_{p+1}) \in S, n \geq n_0 \), by Lemma 13. It follows that there is a natural number \( m_{p+1} > m_p + n_0 \) such that
$(\xi_{m_{p+1}} + \eta_{p+1}) \neq \zeta_i$, $i \leq p$. Put $\zeta_{p+1} = \xi_{m_{p+1}} + \eta_{p+1}$. Hence we have an increasing sequence $m_1 < m_2 < \ldots < m_{p+1}$ and a one-to-one sequence $\zeta_1, \zeta_2, \ldots, \zeta_{p+1}$ of elements of $S$. We have constructed, by means of mathematical induction, a one-to-one sequence of elements $\zeta_i \in S$, $\zeta_i = \xi_{m_i} + \eta_i$, $i \in N$. Elements $\xi_{m_i}$ belong to the set $S$ and elements $-\xi_{m_i}$ to the set $S^- = S \cap S^- \cup (S^- - S)$. $(R, \mathcal{L}_S^H, +)$ satisfies $(-)$ and so $S^- - S$ is a finite set, by Lemma 9. Put $L_1 = \emptyset$, $L_2 = S^- - S$. Then $\zeta_i \in S \cup L_1$ and $-\xi_{m_i} \in S \cup L_2$. According to Lemma 10 there is a finite $K \subset R/H$ such that $(\zeta_i - \xi_{m_i}) \in S \cup K$, i.e. $\eta_i \in S \cup K$. The sequence $\langle \eta_i \rangle$ is one-to-one. According to Lemma 12 there is $i_0$ such that $\eta_i \in S$, $i \geq i_0$. On the other hand, $\eta_i \in S^r$, $i \in N$. Thus we got a contradictory result.

There is a close connection between cc-groups $(R, \mathcal{L}_S^H, +)$ and complete groups with respect to the convergence $\mathcal{L}_S^H$. This is shown in the following lemma. □

**Lemma 15.** $(R, \mathcal{L}_S^H, +)$ is a cc-group if and only if it is a complete group.

**Proof.** Let $(R, \mathcal{L}_S^H, +)$ be a cc-group. By Lemma 14 there is a finite $K \subset R/H$ such that $S = K$ or $S = R/H - K$. If $S = R/H - K$ then $(R, \mathcal{L}_S^H, +)$ is a complete because $\mathcal{L}_S^H = \emptyset$. Now, suppose that $S$ is a finite set. Let $\langle c_n \rangle$, $c_n \in R$, be a Cauchy sequence of points $c_n$ in $(R, \mathcal{L}_S^H, +)$. Distinguish two cases:

1) There is a finite subset $K_0$ such that $c_n \in R_{K_0}$. The sequence $\langle c_n \rangle$ is a Cauchy sequence with respect to $\mathcal{L}$, because $\mathcal{L}_S^H \subset \mathcal{L}$, by Lemma 3. Hence there is a point $x \in R$ such that $\lim c_n = x$. We have $\mathcal{L}_S^H - \lim c_n = x$, according to Lemma 4.

2) There is a subsequence $\langle b_n \rangle \subset \langle c_n \rangle$ of non-equivalent points $b_n \in R$. We construct, analogously as in [1], a subsequence $\langle b_{i_n} \rangle \subset \langle b_n \rangle$ such that $\langle b_{i_n} - b_{i_n} \rangle$ does not $\mathcal{L}_S^H$-converge to 0. Put $i_1 = 1$. Suppose that we have chosen points $b_{i_1}, b_{i_2}, \ldots, b_{i_k}$, $i_1 < i_2 < \ldots < i_k$, such that no two numbers $t_m = b_m - b_{m-1}$, $m \leq k$, are equivalent. We prove that there is a point $b_{i_{k+1}}, i_{k+1} > i_k$, in the sequence $\langle b_n \rangle$ such that no two numbers $t_m, 1 \leq m \leq k + 1$ are equivalent. Let $q > k$. Suppose (indirect proof) that there is no point $b_s$, $i_k < s < i_k + q$ in the sequence $\langle b_n \rangle$ such that any two numbers $b_{k+1} - b_s$ and $t_m$, $m \leq k$, are non-equivalent. Denote $u_s = b_{k+1} - b_s$. Let $f$: $\{i_k < s \leq i_k + q\} \rightarrow \{1, 2, \ldots, k\}$ be a (one-valued) function such that $u_s$ and $t_f(s)$ are equivalent numbers. Since $q > k$ there are $s_1 > i_k$ and $s_2 \leq i_k + q, s_1 < s_2$, such that $f(s_1) = f(s_2)$. Consequently, the numbers $u_{s_1}, t_{f(s_1)}$ are equivalent and also numbers $u_{s_2}, t_{f(s_2)}$ are equivalent. It follows that $(b_{f(s_1)} - b_{s_1}) \in H$, $(b_{f(s_2)} - b_{s_2}) \in H$. Hence $(b_{s_1} - b_{s_2}) \in H$ and so $b_{s_1}, b_{s_2}$ are equivalent points. This is a contradiction because $b_n$ are non-equivalent points. We conclude that there is $s_0 \in \{i_k + 1, i_k + 2, \ldots, i_k + q\}$ such that points $b_{s_0}, b_{i_m}$, $m \leq k$, are non-equivalent. Hence, it suffices to put $i_{k+1} = s_0$.

In such a way we have constructed a sequence $\langle b_n - b_{i_n} \rangle$ of non-equivalent points. Since $S$ is finite it follows from Lemma 4 that the sequence $\langle b_n - b_{i_n} \rangle$ does not $\mathcal{L}_S^H$-
converge to 0. Therefore \((c_n)\) is not a Cauchy sequence with respect to \(\mathcal{L}^H_S\). The case 2) cannot occur.

Let \((R, \mathcal{L}^H_S, +)\) be a complete group with respect to the convergence \(\mathcal{L}^H_S\). Then it is a cc-group, by the definition on p. 25.

Lemmas 11 and 14 give us a complete information about structures \((R, \mathcal{L}^H_S, +)\) which are cc-groups. If \(H = R\) then \((R, \mathcal{L}, +)\) is the unique cc-group. If \(H \neq R\) then there are exactly two different cc-groups, i.e. \((R, \mathcal{L}^H_0, +)\) and \((R, \mathcal{L}, +)\).

Closing remarks. If \(Q\) is a subgroup of \(H\) and \(H\) a subgroup of \(R\), \(H \neq R\), then there are two different completions \((R, \mathcal{L}^H_S, +)\) of \(Q\), namely, \((R, \mathcal{L}^H_0, +)\) and \((R, \mathcal{L}, +)\). It follows that there is more than one completions of \(Q\). There would be interesting to know what is the number of completion of the group of rational numbers \(Q\).

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Addendum after the proofs. P. Simon and R. Frič proved, independently from each other, that the number of completions of the group \(Q\) is \(\exp(\exp(\omega))\) [2].

References


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