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AN EXAMPLE OF A GROUP CONVERGENCE WITH UNIQUE SEQUENTIAL LIMITS WHICH CANNOT BE ASSOCIATED WITH A HAUSDORFF TOPOLOGY

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As usual, by $C$ we denote the Cantor set equipped with the topology inherited from the real line. We assume that \{0, 1\} is the two-element group equipped with the discrete topology. Throughout the paper we denote by $X$ the set of all continuous functions from $C$ to \{0, 1\}.

We write $x_n \to x(G)$ and say that a sequence $\{x_n\}$ converges to $x$ in $(X, G)$ if $x_n, x \in X$ for $n \in \mathbb{N}$ and for every subsequence $\{u_n\}$ of $\{x_n\}$ there are a subsequence $\{v_n\}$ and an open dense subset $A$ of $C$ such that $v_n(t) \to x(t)$ for $t \in A$.

It is not difficult to prove that $G$ is a FLUSH-convergence, i.e., it satisfies the conditions:

- (F) $x_n \to x$ implies $x_{m_n} \to x$;
- (L) $x_n \to x, y_n \to y$ implies $x_n + y_n \to x + y$;
- (U) if for every subsequence $\{u_n\}$ of a given sequence $\{x_n\}$ there is a subsequence $\{v_n\}$ of $\{u_n\}$ such that $v_n \to x$ for a given $x$, then $x_n \to x$;
- (S) if $x_n = x$ for $n \in \mathbb{N}$, then $x_n \to x$;
- (H) if $x_n \to x$ and $x_n \to y$, then $x = y$.

We claim the following:

**Theorem.** (a) If $V$ is a nonempty subset of $X$ such that $x_n \in V$ for sufficiently large $n$ whenever $x_n \to x(G)$ and $x \in V$, then for every $y \in X$ there is a sequence $\{x_n\}$ of elements $x_n$ in $V$ such that $x_n \to y(G)$.

(b) If $\tau$ is a topology on $X$ which preserves the convergence $G$, i.e., $x_n \to x(G)$ implies $x_n \to x$ in $(X, \tau)$, then nonempty open sets in $(X, \tau)$ are sequentially dense in $X$. 

7
(c) If \( \tau \) is a topology on \( X \) which preserves the convergence \( G \), then the intersection of any two nonempty open sets in \( (X, \tau) \) is nonempty.

(d) \( G \) is a FLUSH-convergence, i.e., \( G \) satisfies the following condition:

- (P) if \( x_{ij} \to x_i \) as \( j \to \infty \) for \( i \in \mathbb{N} \) and for any two subsequences \( \{p_i\} \) and \( \{q_i\} \) of \( \{i\} \) we have \( x_{pi,qi} \to x \) for a given \( x \), then \( x_i \to x \).

Summarizing, we may say that there is no Hausdorff topology which induces the convergence \( G \). An example of a FLUSH-convergence group for which there is no Hausdorff topology inducing the convergence is given in [1]. J. Pochcial notes in [2] that convergences in \( T_3 \)-topological spaces are FLUSH-convergences and convergences in topological groups are FLUSH-convergences.

Observe that (a) implies (b) and (b) implies (c). Hence it suffices to prove (a) and (d).

**Proof** of (a). Let \( a \) be an arbitrary fixed point in \( X \) and let \( U = V - a \). We assert that if \( x \in U \) and \( x_n \to x \) in \( (X,G) \), then \( x_n \in U \) for sufficiently large \( n \).

Indeed, if \( x \in U \) then \( x = v - a \) for some \( v \in V \) and, by (L), \( x_n + a \to v \) in \( (X,G) \). Therefore \( x_n + a \in V \) for sufficiently large \( n \) or, equivalently, \( x_n \in U \) for sufficiently large \( n \). Assume that \( u \in U \) and \( \{w_n\} \) is a sequence of all rational numbers. Let \( \{P_n\} \) be a base at \( w_1 \) of closed-open subsets of \( C \) such that \( P_n \supset P_{n+1} \) for \( n \in \mathbb{N} \). We put

\[ u_n = u \cdot I_{C \setminus P_n} \]

where \( I_{C \setminus P_n} \) is the characteristic function of the set \( C \setminus P_n \). We note that \( u_n \in X \) for \( n \in \mathbb{N} \) and \( u_n(t) \to u(t) \) for \( t \in C \setminus \{w_1\} \). Therefore \( u_n \to u \) in \( (X,G) \). Consequently, there is an index \( n_1 \) such that \( x_1 \in U \) with

\[ x_1 = u_{n_1} = u \cdot I_{C \setminus Q_1} \in U \quad \text{and} \quad Q_1 = P_{n_1}. \]

We note that \( Q_1 \) is a closed-open subset of \( C \) and \( w_1 \in Q_1 \). By induction we find a sequence \( \{x_n\} \) and a sequence \( \{Q_n\} \) of closed-open subsets of \( C \) such that

\[ x_n = u \cdot I_{C \setminus (Q_1 \cup \ldots \cup Q_n)}, \quad x_n \in U \quad \text{and} \quad w_n \in Q_n \]

for \( n \in \mathbb{N} \). We put

\[ A = \bigcup_{n=1}^{\infty} Q_n \]

and note that \( A \) is an open dense subset of \( C \) and \( x_n(t) \to 0 \) for \( t \in A \). This means that \( x_n \to 0 \) in \( (X,G) \) and \( x_n \in U \) for \( n \in \mathbb{N} \). Let \( \{y_n\} \) be a sequence such that \( x_n = y_n - a \). Then \( y_n \in V \) for \( n \in \mathbb{N} \) and, by (L), \( y_n \to a \), which was to be proved. \( \square \)
To complete the proof of our Theorem we should show that $G$ has property (P). To this aim we shall prove a number of lemmas.

**Lemma 1.** The following conditions are equivalent:

(i) $x_n \to x$ in $(X, G)$;

(ii) for every subsequence $\{y_n\}$ of $\{x_n\}$ and for every nonempty open subset $U$ of $C$ there are a subsequence $\{z_n\}$ of $\{y_n\}$ and a nonempty open subset $V$ of $U$ such that $z_n(t) = 0$ for $t \in V$ and $n \in \mathbb{N}$.

**Proof.** Assume that (i) holds, $\{y_n\}$ is a subsequence of $\{x_n\}$ and $U$ is a nonempty subset of $C$. Let $\{u_n\}$ be a subsequence of $\{y_n\}$ and let $A$ be an open dense subset of $C$ such that $u_n(t) \to 0$ for every $t \in A$. We see that $W = U \cap A$ is a nonempty open subset of $U$. We put

$$F_n = \{t \in W: u_m(t) = 0 \text{ for } m \geq n \text{ and } m, n \in \mathbb{N}\}.$$  

Note that $F_n$ are closed subsets of $W$ and $W = \bigcup_{n=1}^{\infty} F_n$. Hence, by the Baire category theorem, there is an index $n_0$ such that $\text{int} F_{n_0} \neq \emptyset$. Assuming $z_n = u_{n+n}$ for $n \in \mathbb{N}$ and $V = \text{int} F_{n_0}$ we see that $z_n(t) = 0$ for every $t \in V$ and $n \in \mathbb{N}$. This shows that (i) implies (ii). To prove that (ii) implies (i) we take a countable base $\{U_n: n \in \mathbb{N}\}$ of open sets in $C$ and a subsequence $\{y_n\}$ of $\{x_n\}$. If (ii) holds, then there are a subsequence $\{z_{1n}\}$ of $\{y_n\}$ and an open subset $V_1$ such that $V_1 \subset U_1$ and $z_1(t) = 0$ for $t \in V_1$ and $n \in \mathbb{N}$. By induction we find a sequence of sequences $\{z_{kn}\}$ and a sequence $\{V_n\}$ of open sets $V_n$ such that $\{z_{k+1,n}\}$ is a subsequence of $\{z_{kn}\}$ for $k \in \mathbb{N}$ and $z_{kn}(t) = 0$ for $t \in V_k$ and $n \in \mathbb{N}$. We put

$$A = \bigcup_{k=1}^{\infty} V_k$$

and

$$v_n = z_{nn}$$

for $n \in \mathbb{N}$. Then $A$ is an open dense subset of $C$, $v_n(t) \to 0$ for $t \in A$ and $\{v_n\}$ is a subsequence of $\{y_n\}$. This shows that $x_n \to 0$ in $(X, G)$ or, equivalently, (ii) implies (i).  

We introduce auxiliary convergences on $X$. We write $x_n \to x(T_0)$ or $x_n \to x$ in $(X, T_0)$ iff $x_n, x \in X$ for $n \in \mathbb{N}$ and there is a dense subset $A$ of $C$ such that $x_n(t) \to x(t)$ for $t \in A$. We write $x_n \to x(T)$ or $x_n \to x$ in $(X, T)$ iff for every subsequence $\{u_n\}$ of $\{x_n\}$ there is a subsequence $\{v_n\}$ of $\{u_n\}$ such that $v_n \to x(T_0)$. Obviously, $x_n \to x(G)$ implies $x_n \to x(T)$ but not conversely.
Lemma 2. \((X,T)\) is a FUS-convergence space with the following properties:

\((L_0)\) If \(x_n \to x\) in \((X,T)\) and \(y \in X\), then \(x_n + y \to x + y\) in \((X,T)\). If \(x_n \to x\) in \((X,T)\), then \(-x_n \to -x\) in \((X,T)\).

\((H_0)\) If \(x_n = x\) and \(x_n \to y\) in \((X,T)\), then \(x = y\).

Proof. Properties FUS of \(T\) are obvious. Properties \((L_0)\) and \((H_0)\) follow from the fact that if \(x\) and \(y\) are continuous functions and \(x(t) = y(t)\) for \(t\) belonging to a dense subset of \(C\), then \(x = y\).

Lemma 3. For every sequence \(\{x_n\}\) in \(X\) the following conditions are equivalent:

(i) \(x_n \to 0\) in \((X,T)\);

(ii) for every subsequence \(\{y_n\}\) of \(\{x_n\}\) the set

\[A = \{t \in C: y_n(t) = 0 \text{ for infinitely many } n \in \mathbb{N}\}\]

is dense in \(C\);

(iii) for every subsequence \(\{y_n\}\) of \(\{x_n\}\) and for every open set \(U \subset C\) there is \(t \in U\) such that \(y_n(t) = 0\) for infinitely many \(n \in \mathbb{N}\).

Proof. Obviously, (i) implies (ii) and (ii) implies (iii). To prove that (iii) implies (i) we take a countable base \(\{U_n: n \in \mathbb{N}\}\) of open sets in \(C\) and a subsequence \(\{y_n\}\) of \(\{x_n\}\). If (iii) holds, then there is an element \(t_1\) of \(U_1\) and a subsequence \(\{z_{1n}\}\) of \(\{y_n\}\) such that \(z_{1n}(t_1) \to 0\). By induction we select a sequence of sequences \(\{z_{kn}\}\) and a sequence \(\{t_k\}\) such that, for every \(k \in \mathbb{N}\), \(\{z_{k+1,n}\}\) is a subsequence of \(\{z_{kn}\}\), \(t_k \in U_k\) and \(z_{kn}(t_k) \to 0\) as \(n \to \infty\). Denoting \(z_k = z_{kk}\) for \(k \in \mathbb{N}\) and \(A = \{t_k: k \in \mathbb{N}\}\) we see that \(A\) is a dense subset of \(C\) and \(z_n(t) \to 0\) for \(t \in A\). This shows that (iii) implies (i).

Lemma 4. If no subsequence of \(\{x_n\}\) converges to zero in \((X,T)\), then for every subsequence \(\{u_n\}\) of \(\{x_n\}\) there are a subsequence \(\{v_n\}\) of \(\{u_n\}\) and a nonempty open set \(V\) in \(C\) such that \(v_n(t) = 1\) for \(t \in V\) and \(n \in \mathbb{N}\).

Proof. We claim that, under the conditions of the lemma, for every subsequence \(\{u_n\}\) of \(\{x_n\}\) there are a subsequence \(\{z_n\}\) of \(\{u_n\}\) and an open set \(U\) in \(C\) such that, for every \(t \in U\), \(z_n(t) = 0\) for sufficiently large \(n\). Otherwise, by Lemma 3 (iii), there would exist a subsequence \(\{u_n\}\) of \(\{x_n\}\) such that \(u_n \to 0\) in \((X,T)\). We put

\[F_n = \{t \in U: z_m(t) = 1 \text{ for } m \geq n\}\]

and note that \(F_n\) are closed subsets of \(C\) and

\[U = \bigcup_{n=1}^{\infty} F_n.\]
By the Baire theorem there is an index \( n_0 \) such that \( \text{int} \, F_{n_0} \neq \emptyset \). Denoting \( V = \text{int} \, F_{n_0} \) and \( v_n = z_{n_0+n} \) for \( n \in \mathbb{N} \) we see that \( v_n(t) = 0 \) for every \( t \in V \) and \( n \in \mathbb{N} \), which was to be proved.

**Lemma 5.** Assume that \( \{x_n\} \) is a sequence in \( X \) such that \( x_n \to 0(T) \) and the only limit of every subsequence of \( \{x_n\} \) is zero. Then \( x_n \to 0(G) \).

**Proof.** Let \( U \) be a nonempty open subset of \( C \). We may assume that \( U \) is an open-closed set. Let \( \chi \) be the characteristic function of \( U \), let \( \{u_n\} \) be a subsequence of \( \{x_n\} \) and let \( \{v_n\} \) be a subsequence of \( \{u_n\} \) such that \( v_n \to 0 \) in \((X, T_0)\). Assume that for a subsequence \( \{i_n\} \) of \( \{v_n\} \) we have \( i_n \to x \) in \((X, T)\) and \( x \neq 0 \) which is impossible. Therefore, no subsequence of \( \{v_n - x\} \) converges to zero in \((X, T)\). Hence, by Lemma 4, there exist an open set \( V \) and a subsequence \( \{w_n\} \) of \( \{v_n - x\} \) such that \( w_n(t) = 1 \) for every \( t \in V \) and \( n \in \mathbb{N} \). We claim that \( V \subset U \). Otherwise, \( V \setminus U \) would be a nonempty open subset of \( C \) and, consequently, there would be an element \( t \in V \setminus U \) such that \( w_n(t) = 0 \) for sufficiently large \( n \) and \( x(t) = 0 \). On the other hand, \( w_n(t) + x(t) = 1 \). Hence \( w_n(t) = 1 \) for sufficiently large \( n \), which is impossible since \( w_n(t) = 0 \) for sufficiently large \( n \). This contradiction shows that \( V \subset U \). Therefore, \( w_n(t) = 0 \) for \( t \in V \) and \( n \in \mathbb{N} \). In this way we have proved that, under the conditions of Lemma 4, condition (ii) of Lemma 1 is satisfied or, equivalently, \( x_n \to 0 \) in \((X, G)\), which completes the proof of Lemma 5.

From Lemma 5 we get

**Corollary 1.** We have \( x_n \to x \) in \((X, G)\) iff \( x_n \to x \) in \((X, T)\) and there is no subsequence of \( \{x_n\} \) which converges in \( \{X, T\} \) to an element different from \( x \).

**Lemma 6.** The convergence \((X, T)\) satisfies the following diagonal type condition:

(\( \Phi \)) \text{ If } x_{ij} \in X \text{ for } i, j \in \mathbb{N}, \text{ } x_{ij} \to x_i \text{ in } (X, T) \text{ as } j \to \infty \text{ for } i \in \mathbb{N} \text{ and } x_i \to 0 \text{ in } (X, T), \text{ then there are subsequences } \{m_i\} \text{ and } \{n_i\} \text{ of } \{i\} \text{ such that } x_{m_i, n_i} \to 0 \text{ in } (X, T). \]

**Proof.** We may and will assume that \( x_{ij} \to x_i \) in \((X, T_0)\) as \( j \to \infty \) for \( i \in \mathbb{N} \), and \( x_i \to 0 \) in \((X, T_0)\). Otherwise, applying the diagonal procedure, we would take such a submatrix. Let \( V_1, V_2, \ldots \) be a base for the topology in \( C \). Note that if \( y_n \to y \) in \((X, T_0)\), \( V \) is an open set in \( C \) and \( y^{-1}(\{0\}) \cap V \neq \emptyset \), then there are an element \( t \in y^{-1}(\{0\}) \cap V \) and an index \( n_0 \) such that \( y_n(t) = 0 \) for \( n \geq n_0 \). Consequently, \( y_n^{-1}(\{0\}) \cap V \neq \emptyset \) for \( n \geq n_0 \). This remark implies that there is a subsequence \( \{m_i\} \) if \( \{i\} \) such that \( x_{m_i}^{-1}(\{0\}) \cap V_k \neq \emptyset \) for \( i \in \mathbb{N} \) and \( k = 1, \ldots, i \). By the same remark
there exists a subsequence \( \{n_i\} \) of \( \{i\} \) such that

\[
x_{m_i,n_i,1}^{-1}(\{0\}) \cap x_{m_i,n_i,0}^{-1}(\{0\}) \cap V_k \neq \emptyset.
\]

For every subsequence \( \{r_i\} \) of \( \{i\} \) we put

\[
A = \bigcap_{m=1}^{\infty} \bigcup_{i=m}^{p_i} \bigcup_{j=1}^{q_i} x_{p_i,q_i,0}^{-1}(\{0\}) \cap V_j,
\]

where \( p_i = m_{r_i} \) and \( q_i = n_{r_i} \) for \( i \in \mathbb{N} \). First note that \( A \) is the intersection of a countable family of dense and open subset of \( C \). Therefore, by the Baire Category Theorem, \( A \) is a dense subset of \( C \). Moreover, notice that if \( t \in A \), then \( x_{p_i,q_i,0}(t) = \emptyset \) for infinitely many \( i \in \mathbb{N} \). Hence, by Lemma 2(b), \( x_{m,n} \to 0 \) in \((X,T)\), which was to be proved. \( \square \)

Assume that \( Y \) is an abelian group equipped with a convergence \( W \). By \( W_* \) we denote the convergence in \( Y \) such that

\[
x_n \to x(W_*) \iff z_n \to 0(W) \text{ implies } x_n + z_n \to x(W).
\]

We see that \( x_n \to x(W_*) \) implies \( x_n \to x(W) \).

**Lemma 7.** Assume that \( W \) is a FL\(_0\)USII\(_0\)-convergence in \( Y \). Then

(i) \( W_* \) is a FLUSII-convergence in \( Y \);

(ii) if \( x_n \to x(W_*) \), then the only limit of every subsequence of \( \{x_n\} \) is \( x \), i.e., if \( x_n \to 0(W_*) \) and \( \{y_n\} \) is a subsequence of \( \{x_n\} \) such that \( y_n \to y(W) \), then \( y = x \);

(iii) if \( W \) has property \((\Phi)\), then \( W_* \) has property \((P)\).

**Proof of (i).** Assume that \( x_n \to x(W_*) \), \( \{x_{m,n}\} \) is a subsequence of \( \{x_n\} \) and \( z_n \to 0(W) \). We put \( u_{m,n} = z_n \) for \( n \in \mathbb{N} \) and \( u_k = 0 \) if \( k \in \mathbb{N} \) and \( k \neq m_n \) for \( n \in \mathbb{N} \). By \((H_0)\), \((U)\) and \((F)\), \( u_n \to 0(W) \). Hence \( x_n + u_n \to 0(W) \). By \((F)\), \( x_{m,n} + z_n \to 0(W) \) which proves \((F)\). To prove \((L)\) we note that \( x_n \to x(W_*) \) iff \( x_n - x \to 0(W_*) \). Indeed, assume that \( x_n \to x(W_*) \) and \( z_n \to 0(W) \). Then \( x_n + z_n \to x(W) \). Hence by \((L_0)\) we have \( x_n - x + z_n \to 0(W) \) or, equivalently, \( x_n - x \to 0(W_*) \). Assume now that \( x_n - x \to 0(W_*) \) and \( z_n \to 0(W) \). Then \( x_n - x + z_n \to 0(W) \). Hence, by \((L_0)\), \( x_n + z_n \to x(W) \) or, equivalently, \( x_n \to x(W_*) \). Now assume that \( x_n \to x(W_*) \) and \( y_n \to y(W_*) \) and \( z_n \to 0(W) \). Then \( x_n - x \to 0(W_*) \) and \( y_n - y + z_n \to 0(W) \). Hence we get

\[
(x_n - x) + (y_n - y) + z_n \to 0(W)
\]

or, equivalently, \( x_n + y_n - x - y \to 0(W_*) \) and \( x_n + y_n \to x + y(W_*) \). This proves \((L)\). Assume that \( x \in Y \), \( \{x_n\} \) is a sequence in \( Y \), and for every subsequence \( \{u_n\} \) of
there is a subsequence \( \{v_n\} \) of \( \{u_n\} \) such that \( v_n \to x(W_*) \). Moreover assume that \( z_n \to 0(W) \). Then, by (F), \( x_n + z_n \to x(W) \) or, equivalently, \( x_n \to x(W_*) \). This proves (U). Properties (S) and (II) follow from (II_0) and (L_0).

Proof of (ii). Assume that \( x_n \to x(W_*) \), \( x_{m_n} \to y(W) \) and \( \{z_n\} \) is a sequence such that \( z_{m_n} = y - x_{m_n} \) for \( n \in \mathbb{N} \) and \( z_k = 0 \) for \( k \in \mathbb{N} \) and \( k \neq m_n \) for \( n \in \mathbb{N} \). From (L_0), (H_0), (F) and (U) it follows that \( z_n \to 0(W) \). Thus \( x_n + z_n \to x(W) \) and \( x_{m_n} + z_{m_n} = y \) for \( n \in \mathbb{N} \). Hence, by (F) and (H_0), \( y = x \), which proves (ii).

Proof of (iii). Assume that \( x_{ij} \in Y \) for \( i, j \in \mathbb{N} \), \( x_{ij} \to x_i(W_*) \) as \( j \to \infty \) for \( i \in \mathbb{N} \) and for any subsequences \( \{m_i\} \), \( \{n_i\} \) of \( \{i\} \) we have

\[
x_{m_{n_i}} \to 0(W_*)
\]

To show that \( x_i \to 0(W_*) \) we take an arbitrary sequence \( \{z_i\} \) such that \( z_i \to 0(W) \), and choose a subsequence \( \{p_i\} \) of \( \{i\} \). Then, by the definition of \( W_* \) and properties (F) and (L) for \( W \), we can write

\[
x_{p_i} - x_{p_ip_j} - z_{p_j} \to z_{p_i}(W)
\]
as \( j \to \infty \) for \( i \in \mathbb{N} \) and \( z_{p_j} \to 0(W) \). Now, if the convergence \( W \) has property(\( \Phi \)), there exist two subsequences \( \{r_i\} \) and \( \{s_i\} \) such that

\[
(x_{k_i} + z_{k_i}) - x_{k_i,l_i} \to 0(W)
\]

and

\[
x_{k_i,l_i} \to 0(W_*)
\]
with \( k_i = p_{r_i} \) and \( l_i = p_{s_i} \) for \( i \in \mathbb{N} \). This together with the definition of \( W \) implies

\[
x_{k_i} + z_{k_i} \to 0(W).
\]

In this way we have shown that every subsequence of \( \{x_i + z_i\} \) has a subsequence which converges to zero in \( (X,W) \) or, equivalently, \( x_i + z_i \to 0(W) \). Consequently, \( x_i \to 0(W_*) \), which proves (iii).

Now we can prove statement (d).

Proof of (d). By Lemmas 2 and 6, \( T \) is a FLQUSH_\( \Phi \)-convergence in \( X \). Therefore, by Lemma 7, \( T_* \) is a FLUSHP-convergence in \( X \). We claim that \( G = T_* \). Indeed, assume that \( x_n \to x \) in \( (X,G) \), \( z_n \to 0 \) in \( (X,T) \) and \( \{p_n\} \) is a subsequence of \( \{n\} \). Let \( \{r_n\} \) be a subsequence of \( \{p_n\} \) and let \( A \) be an open dense subset of \( C \) such that \( x_{r_n}(t) \to x \) for \( t \in A \). Let \( \{q_n\} \) be a subsequence of \( \{r_n\} \) and let \( B \) be a
dense subset of $C$ such that $z_{q_n}(t) \to 0$ for $t \in B$. Then $A \cap B$ is a dense subset of $C$ and $x_{q_n}(t) + z_{q_n}(t) \to x(t)$ for $t \in A \cap B$. Consequently, $x_n + z_n \to x(T)$. This shows that $x_n \to x(T)$, i.e., $G \subseteq T_*$. Assume now that $x_n \to x(T)$ and $\{y_n\}$ is a subsequence of $\{x_n\}$ such that $y_n \to y(T)$. Then, by Lemma 7 (ii), $y = x$. Hence, by Corollary 1, $x_n \to x(G)$ which shows that $G \supseteq T_*$. Finally, $G = T_*$. Since $T_*$ is a FLUSHI-convergence on $X$, $G$ is a FLUSHI-convergence in $X$ and this proves (d). 

References


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