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AN EXAMPLE OF A GROUP CONVERGENCE WITH UNIQUE SEQUENTIAL LIMITS WHICH CANNOT BE ASSOCIATED WITH A HAUSDORFF TOPOLOGY

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As usual, by C we denote the Cantor set equipped with the topology inherited from the real line. We assume that $\{0, 1\}$ is the two-element group equipped with the discrete topology. Throughout the paper we denote by X the set of all continuous functions from C to $\{0, 1\}$.

We write $x_n \to x(G)$ and say that a sequence $\{x_n\}$ converges to x in (X, G) if $x_n, x \in X$ for $n \in \mathbb{N}$ and for every subsequence $\{u_n\}$ of $\{x_n\}$ there are a subsequence $\{v_n\}$ and an open dense subset A of C such that

$$v_n(t) \to x(t) \quad \text{for } t \in A.$$

It is not difficult to prove that G is a FLUSH-convergence, i.e., it satisfies the conditions:

(F) $x_n \to x$ implies $x_{m_n} \to x$;

(L) $x_n \to x$, $y_n \to y$ implies $x_n \pm y_n \to x \pm y$;

(U) if for every subsequence $\{u_n\}$ of a given sequence $\{x_n\}$ there is a subsequence $\{v_n\}$ of $\{u_n\}$ such that $v_n \to x$ for a given x, then $x_n \to x$;

(S) if $x_n = x$ for $n \in \mathbb{N}$, then $x_n \to x$;

(H) if $x_n \to x$ and $x_n \to y$, then $x = y$.

We claim the following:

Theorem. (a) *If V is a nonempty subset of X such that $x_n \in V$ for sufficiently large n whenever $x_n \to x(G)$ and $x \in V$, then for every $y \in X$ there is a sequence $\{x_n\}$ of elements x_n in V such that $x_n \to y(G)$.*

(b) *If τ is a topology on X which preserves the covergence G, i.e., $x_n \to x(G)$ implies $x_n \to x$ in (X, τ), then nonempty open sets in (X, τ) are sequentially dense in X.*

(c) *If τ is a topology on X which preserves the convergence G, then the intersection of any two nonepmty open sets in (X, τ) is nonempty.*

(d) *G is a FLUSHP-convergence, i.e., G satisfies the following condition:*

(P) *if $x_{ij} \to x_i$ as $j \to \infty$ for $i \in \mathbb{N}$ and for any two subsequences $\{p_i\}$ and $\{q_i\}$ of $\{i\}$ we have $x_{p_i q_i} \to x$ for a given x, then $x_i \to x$.*

Summarizing, we may say that there is no Hausdorff topology which induces the convergence G. An example of a FLUSH-convergence group for which there is no Hausdorff topology inducing the convergence is given in [1]. J. Pochcial notes in [2] that convergences in T_3-topological spaces are FLUSHP-convergences and convergences in topological groups are FLUSHP-convergences.

Observe that (a) implies (b) and (b) implies (c). Hence it suffices to prove (a) and (d).

Proof of (a). Let a be an arbitrary fixed point in X and let $U = V - a$. We assert that if $x \in U$ and $x_n \to x$ in (X, G), then $x_n \in U$ for sufficiently large n. Indeed, if $x \in U$ then $x = v - a$ for some $v \in V$ and, by (L), $x_n + a \to v$ in (X, G). Therefore $x_n + a \in V$ for sufficiently large n or, equivalently, $x_n \in U$ for sufficiently large n. Assume that $u \in U$ and $\{w_n\}$ is a sequence of all rational numbers. Let $\{P_n\}$ be a base at w_1 of closed-open subsets of C such that $P_n \supset P_{n+1}$ for $n \in \mathbb{N}$. We put

$$u_n = u \cdot I_{C \setminus P_n}$$

where $I_{C \setminus P_n}$ is the characteristic function of the set $C \setminus P_n$. We note that $u_n \in X$ for $n \in \mathbb{N}$ and $u_n(t) \to u(t)$ for $t \in C \setminus \{w_1\}$. Therefore $u_n \to u$ in (X, G). Consequently, there is an index n_1 such that $x_1 \in U$ with

$$x_1 = u_{n_1} = u \cdot I_{C \setminus Q_1} \in U \quad \text{and} \quad Q_1 = P_{n_1}.$$

We note that Q_1 is a closed-open subset of C and $w_1 \in Q_1$. By induction we find a sequence $\{x_n\}$ and a sequence $\{Q_n\}$ of closed-open subsets of C such that

$$x_n = u \cdot I_{C \setminus (Q_1 \cup \ldots \cup Q_n)}, \; x_n \in U \quad \text{and} \quad w_n \in Q_n$$

for $n \in \mathbb{N}$. We put

$$A = \bigcup_{n=1}^{\infty} Q_n$$

and note that A is an open dense subset of C and $x_n(t) \to 0$ for $t \in A$. This means that $x_n \to 0$ in (X, G) and $x_n \in U$ for $n \in \mathbb{N}$. Let $\{y_n\}$ be a sequence such that $x_n = y_n - a$. Then $y_n \in V$ for $n \in \mathbb{N}$ and, by (L), $y_n \to a$, which was to be proved. □

To complete the proof of our Theorem we should show that G has property (P). To this aim we shall prove a number of lemmas.

Lemma 1. *The following conditions are equivalent:*

(i) $x_n \to x$ *in* (X, G);

(ii) *for every subsequence* $\{y_n\}$ *of* $\{x_n\}$ *and for every nonempty open subset* U *of* C *there are a subsequence* $\{z_n\}$ *of* $\{y_n\}$ *and a nonempty open subset* V *of* U *such that* $z_n(t) = 0$ *for* $t \in V$ *and* $n \in \mathbb{N}$.

P r o o f. Assume that (i) holds, $\{y_n\}$ is a subsequence of $\{x_n\}$ and U is a nonempty subset of C. Let $\{u_n\}$ be a subsequence of $\{y_n\}$ and let A be an open dense subset of C such that $u_n(t) \to 0$ for every $t \in A$. We see that $W = U \cap A$ is a nonempty open subset of U. We put

$$F_n = \{t \in W : u_m(t) = 0 \text{ for } m \geqslant n \text{ and } m, n \in \mathbb{N}\}.$$

Note that F_n are closed subsets of W and $W = \bigcup_{n=1}^{\infty} F_n$. Hence, by the Baire category theorem, there is an index n_0 such that $\operatorname{int} F_{n_0} \neq 0$. Assuming $z_n = u_{n_0+n}$ for $n \in \mathbb{N}$ and $V = \operatorname{int} F_{n_0}$ we see that $z_n(t) = 0$ for every $t \in V$ and $n \in \mathbb{N}$. This shows that (i) implies (ii). To prove that (ii) implies (i) we take a countable base $\{U_n : n \in \mathbb{N}\}$ of open sets in C and a subsequence $\{y_n\}$ of $\{x_n\}$. If (ii) holds, then there are a subsequence $\{z_{1n}\}$ of $\{y_n\}$ and an open subset V_1 such that $V_1 \subset U_1$ and $z_{1n}(t) = 0$ for $t \in V_1$ and $n \in \mathbb{N}$. By induction we find a sequence of sequences $\{z_{kn}\}$ and a sequence $\{V_n\}$ of open sets V_n such that $\{z_{k+1,n}\}$ is a subsequence of $\{z_{kn}\}$ for $k \in \mathbb{N}$ and $z_{kn}(t) = 0$ for $t \in V_k$ and $n \in \mathbb{N}$. We put

$$A = \bigcup_{k=1}^{\infty} V_k$$

and

$$v_n = z_{nn}$$

for $n \in \mathbb{N}$. Then A is an open dense subset of C, $v_n(t) \to 0$ for $t \in A$ and $\{v_n\}$ is a subsequence of $\{y_n\}$. This shows that $x_n \to 0$ in (X, G) or, equivalently, (ii) implies (i). $\square$

We introduce auxiliary convergences on X. We write $x_n \to x(T_0)$ or $x_n \to x$ in (X, T_0) iff $x_n, x \in X$ for $n \in \mathbb{N}$ and there is a dense subset A of C such that $x_n(t) \to x(t)$ for $t \in A$. We write $x_n \to x(T)$ or $x_n \to x$ in (X, T) iff for every subsequence $\{u_n\}$ of $\{x_n\}$ there is a subsequence $\{v_n\}$ of $\{u_n\}$ such that $v_n \to x(T_0)$. Obviously, $x_n \to x(G)$ implies $x_n \to x(T)$ but not conversely.

Lemma 2. *(X,T) is a FUS-convergence space with the following properties:*

(L_0) *If $x_n \to x$ in (X,T) and $y \in X$, then $x_n + y \to x + y$ in (X,T). If $x_n \to x$ in (X,T), then $-x_n \to -x$ in (X,T).*

(H_0) *If $x_n = x$ and $x_n \to y$ in (X,T), then $x = y$.*

P r o o f. Properties FUS of T are obvious. Properties (L_0) and (H_0) follow from the fact that if x and y are continuous functions and $x(t) = y(t)$ for t belonging to a dense subset of C, then $x = y$. □

Lemma 3. *For every sequence $\{x_n\}$ in X the following conditions are equivalent:*

(i) *$x_n \to 0$ in (X,T);*

(ii) *for every subsequence $\{y_n\}$ of $\{x_n\}$ the set*

$$A = \{t \in C\colon y_n(t) = 0 \text{ for infinitely many } n \in \mathbb{N}\}$$

is dense in C;

(iii) *for every subsequence $\{y_n\}$ of $\{x_n\}$ and for every open set $U \subset C$ there is $t \in U$ such that $y_n(t) = 0$ for infinitely many $n \in \mathbb{N}$.*

P r o o f. Obviously, (i) implies (ii) and (ii) implies (iii). To prove that (iii) implies (i) we take a countable base $\{U_n : n \in \mathbb{N}\}$ of open sets in C and a subsequence $\{y_n\}$ of $\{x_n\}$. If (iii) holds, then there is an element t_1 of U_1 and a subsequence $\{z_{1n}\}$ of $\{y_n\}$ such that $z_{1n}(t_1) \to 0$. By induction we select a sequence of sequences $\{z_{kn}\}$ and a sequence $\{t_k\}$ such that, for every $k \in \mathbb{N}$, $\{z_{k+1,n}\}$ is a subsequence of $\{z_{kn}\}$, $t_k \in U_k$ and $z_{kn}(t_k) \to 0$ as $n \to \infty$. Denoting $z_k = z_{kk}$ for $k \in \mathbb{N}$ and $A = \{t_k : k \in \mathbb{N}\}$ we see that A is a dense subset of C and $z_n(t) \to 0$ for $t \in A$. This shows that (iii) implies (i). □

Lemma 4. *If no subsequence of $\{x_n\}$ converges to zero in (X,T), then for every subsequence $\{u_n\}$ of $\{x_n\}$ there are a subsequence $\{v_n\}$ of $\{u_n\}$ and a nonempty open set V in C such that $v_n(t) = 1$ for $t \in V$ and $n \in \mathbb{N}$.*

P r o o f. We claim that, under the conditions of the lemma, for every subsequence $\{u_n\}$ of $\{x_n\}$ there are a subsequence $\{z_n\}$ of $\{u_n\}$ and an open set U in C such that, for every $t \in U$, $z_n(t) = 0$ for sufficiently large n. Otherwise, by Lemma 3 (iii), there would exist a subsequence $\{u_n\}$ of $\{x_n\}$ such that $u_n \to 0$ in (X,T). We put

$$F_n = \{t \in U : z_m(t) = 1 \text{ for } m \geqslant n\}$$

and note that F_n are closed subsets of C and

$$U = \bigcup_{n=1}^{\infty} F_n.$$

By the Baire theorem there is an index n_0 such that $\operatorname{int} F_{n_0} \neq 0$. Denoting $V = \operatorname{int} F_{n_0}$ and $v_n = z_{n_0+n}$ for $n \in \mathbb{N}$ we see that $v_n(t) = 0$ for every $t \in V$ and $n \in \mathbb{N}$, which was to be proved. □

Lemma 5. *Assume that $\{x_n\}$ is a sequence in X such that $x_n \to 0(T)$ and the only limit of every subsequence of $\{x_n\}$ is zero. Then $x_n \to 0(G)$.*

P r o o f. Let U be a nonempty open subset of C. We may assume that U is an open-closed set. Let x be the characteristic fuction of U, let $\{u_n\}$ be a subsequence of $\{x_n\}$ and let $\{v_n\}$ be a subsequence of $\{u_n\}$ such that $v_n \to 0$ in (X, T_0). Assume that for a subsequence $\{w_n - x\}$ of $\{v_n - x\}$ we have $w_n - x \to 0$ in (X, T). Then, by (L_0), $w_n \to x$ in (X, T) and $x \neq 0$ which is impossible. Therefore, no subsequence of $\{v_n - x\}$ converges to zero in (X, T). Hence, by Lemma 4, there exist an open set V and a subsequence $\{w_n - x\}$ of $\{v_n - x\}$ such that $w_n(t) - x(t) = 1$ for every $t \in V$ and $n \in \mathbb{N}$. We claim that $V \subset U$. Otherwise, $V \setminus U$ would be a nonempty open subset of C and, consequently, there would be an element $t \in V \setminus U$ such that $w_n(t) = 0$ for sufficiently large n and $x(t) = 0$. On the other hand, $w_n(t) + x(t) = 1$. Hence $w_n(t) = 1$ for sufficiently large n, which is impossible since $w_n(t) = 0$ for sufficiently large n. This contradiction shows that $V \subset U$. Therefore, $w_n(t) = 0$ for $t \in V$ and $n \in \mathbb{N}$. In this way we have proved that, under the conditions of Lemma 4, condition (ii) of Lemma 1 is satisfied or, equivalently, $x_n \to 0$ in (X, G), which completes the proof of Lemma 5. □

From Lemma 5 we get

Corollary 1. *We have $x_n \to x$ in (X, G) iff $x_n \to x$ in (X, T) and there is no subsequence of $\{x_n\}$ which converges in $\{X, T\}$ to an element different from x.*

Lemma 6. *The convergence (X, T) satisfies the following diagonal type condition:*

(Φ) If $x_{ij} \in X$ for $i, j \in \mathbb{N}$, $x_{ij} \to x_i$ in (X, T) as $j \to \infty$ for $i \in \mathbb{N}$ and $x_i \to 0$ in (X, T), then there are subsequences $\{m_i\}$ and $\{n_i\}$ of $\{i\}$ such that $x_{m_i n_i} \to 0$ in (X, T).

P r o o f. We may and will assume that $x_{ij} \to x_i$ in (X, T_0) as $j \to \infty$ for $i \in \mathbb{N}$, and $x_i \to 0$ in (X, T_0). Otherwise, applying the diagonal procedure, we would take such a submatrix. Let $V_1, V_2, \ldots$ be a base for the topology in C. Note that if $y_n \to y$ in (X, T_0), V is an open set in C and $y^{-1}(\{0\}) \cap V \neq \emptyset$, then there are an element $t \in y^{-1}(\{0\}) \cap V$ and an index n_0 such that $y_n(t) = 0$ for $n \geqslant n_0$. Consequently, $y_n^{-1}(\{0\}) \cap V \neq \emptyset$ for $n \geqslant n_0$. This remark implies that there is a subsequence $\{m_i\}$ if $\{i\}$ such that $x_{m_i}^{-1}(\{0\}) \cap V_k \neq \emptyset$ for $i \in \mathbb{N}$ and $k = 1, \ldots, i$. By the same remark there exists a subsequence $\{n_i\}$ of $\{i\}$ such that

$$x_{m_i n_i}^{-1}(\{0\}) \cap x_{m_i}^{-1}(\{0\}) \cap V_k \neq \emptyset.$$

For every subsequence $\{r_i\}$ of $\{i\}$ we put

$$A = \bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} \bigcup_{j=1}^{p_i} x_{p_i q_i}^{-1}(\{0\}) \cap x_{p_i}^{-1}(\{0\}) \cap V_j,$$

where $p_i = m_{r_i}$ and $q_i = n_{r_i}$ for $i \in \mathbb{N}$. First note that A is the intersection of a countable family of dense and open subset of C. Therefore, by the Baire Category Theorem, A is a dense subset of C. Moreover, notice that if $t \in A$, then $x_{p_i q_i}(t) = \emptyset$ for infinitely many $i \in \mathbb{N}$. Hence, by Lemma 2(b), $x_{m_i n_i} \to 0$ in (X, T), which was to be proved. □

Assume that Y is an abelian group equipped with a convergence W. By W_* we denote the convergence in Y such that

$$x_n \to x(W_*) \quad \text{iff} \quad z_n \to 0(W) \text{ implies } x_n + z_n \to x(W).$$

We see that $x_n \to x(W_*)$ implies $x_n \to x(W)$.

Lemma 7. *Assume that W is a* $\mathrm{FL_0USH_0}$*-convergence in Y. Then*

(i) *W_* is a* FLUSH*-convergence in Y;*

(ii) *if $x_n \to x(W_*)$, then the only limit of every subsequence of $\{x_n\}$ is x, i.e., if $x_n \to 0(W_*)$ and $\{y_n\}$ is a subsequence of $\{x_n\}$ such that $y_n \to y(W)$, then $y = x$;*

(iii) *if W has property* (Φ)*, then W_* has property* (P).

Proof of (i). Assume that $x_n \to x(W_*)$, $\{x_{m_n}\}$ is a subsequence of $\{x_n\}$ and $z_n \to 0(W)$. We put $u_{m_n} = z_n$ for $n \in \mathbb{N}$ and $u_k = 0$ if $k \in \mathbb{N}$ and $k \neq m_n$ for $n \in \mathbb{N}$. By $(\mathrm{H_0})$, (U) and (F), $u_n \to 0(W)$. Hence $x_n + u_n \to 0(W)$. By (F), $x_{m_n} + z_n \to 0(W)$ which proves (F). To prove (L) we note that $x_n \to x(W_*)$ iff $x_n - x \to 0(W_*)$. Indeed, assume that $x_n \to x(W_*)$ and $z_n \to 0(W)$. Then $x_n + z_n \to x(W)$. Hence by $(\mathrm{L_0})$ we have $x_n - x + z_n \to 0(W)$ or, equivalently, $x_n - x \to 0(W_*)$. Assume now that $x_n - x \to 0(W_*)$ and $z_n \to 0(W)$. Then $x_n - x + z_n \to 0(W)$. Hence, by $(\mathrm{L_0})$, $x_n + z_n \to x(W)$ or, equivalently, $x_n \to x(W_*)$. Now assume that $x_n \to x(W_*)$ and $y_n \to y(W_*)$ and $z_n \to 0(W)$. Then $x_n - x \to 0(W_*)$ and $y_n - y + z_n \to 0(W)$. Hence we get

$$(x_n - x) + (y_n - y) + z_n \to 0(W)$$

or, equivalently, $x_n + y_n - x - y \to 0(W_*)$ and $x_n + y_n \to x + y(W_*)$. This proves (L). Assume that $x \in Y$, $\{x_n\}$ is a sequence in Y, and for every subsequence $\{u_n\}$ of

$\{x_n\}$ there is a subsequence $\{v_n\}$ of $\{u_n\}$ such that $v_n \to x(W_*)$. Moreover assume that $z_n \to 0(W)$. Then, by (F), $x_n + z_n \to x(W)$ or, equivalently, $x_n \to x(W_*)$. This proves (U). Properties (S) and (H) follow from (H_0) and (L_0). □

P r o o f of (ii). Assume that $x_n \to x(W_*)$, $x_{m_n} \to y(W)$ and $\{z_n\}$ is a sequence such that $z_{m_n} = y - x_{m_n}$ for $n \in \mathbb{N}$ and $z_k = 0$ for $k \in \mathbb{N}$ and $k \neq m_n$ for $n \in \mathbb{N}$. From (L_0), (H_0), (F) and (U) it follows that $z_n \to 0(W)$. Thus $x_n + z_n \to x(W)$ and $x_{m_n} + z_{m_n} = y$ for $n \in \mathbb{N}$. Hence, by (F) and (H_0), $y = x$, which proves (ii). □

P r o o f of (iii). Assume that $x_{ij} \in Y$ for $i, j \in \mathbb{N}$, $x_{ij} \to x_i(W_*)$ as $j \to \infty$ for $i \in \mathbb{N}$ and for any subsequences $\{m_i\}$, $\{n_i\}$ of $\{i\}$ we have

$$x_{m_i n_i} \to 0(W_*).$$

To show that $x_i \to 0(W_*)$ we take an arbitrary sequence $\{z_i\}$ such that $z_i \to 0(W)$, and choose a subsequence $\{p_i\}$ of $\{i\}$. Then, by the definition of W_* and properties (F) and (L) for W, we can write

$$x_{p_i} - x_{p_i p_j} + z_{p_i} \to z_{p_i}(W)$$

as $j \to \infty$ for $i \in \mathbb{N}$ and $z_{p_i} \to 0(W)$. Now, if the convergence W has property(Φ), there exist two subsequences $\{r_i\}$ and $\{s_i\}$ such that

$$(x_{k_i} + z_{k_i}) - x_{k_i l_i} \to 0(W)$$

and

$$x_{k_i l_i} \to 0(W_*)$$

with $k_i = p_{r_i}$ and $l_i = p_{s_i}$ for $i \in \mathbb{N}$. This together with the definition of W implies

$$x_{k_i} + z_{k_i} \to 0(W).$$

In this way we have shown that every subsequence of $\{x_i + z_i\}$ has a subsequence which converges to zero in (X, W) or, equivalently, $x_i + z_i \to 0(W)$. Consequently, $x_i \to 0(W_*)$, which proves (iii). □

Now we can prove statement (d).

P r o o f of (d). By Lemmas 2 and 6, T is a $FL_0USH_0\Phi$-convergence in X. Therefore, by Lemma 7, T_* is a FLUSHP-convergence in X. We claim that $G = T_*$. Indeed, assume that $x_n \to x$ in (X, G), $z_n \to 0$ in (X, T) and $\{p_n\}$ is a subsequence of $\{n\}$. Let $\{r_n\}$ be a subsequence of $\{p_n\}$ and let A be an open dense subset of C such that $x_{r_n}(t) \to x$ for $t \in A$. Let $\{q_n\}$ be a subsequence of $\{r_n\}$ and let B be a dense subset of C such that $z_{q_n}(t) \to 0$ for $t \in B$. Then $A \cap B$ is a dense subset of C and $x_{q_n}(t) + z_{q_n}(t) \to x(t)$ for $t \in A \cap B$. Consequently, $x_n + z_n \to x(T)$. This shows that $x_n \to x(T_*)$, i.e., $G \subset T_*$. Assume now that $x_n \to x(T_*)$ and $\{y_n\}$ is a subsequence of $\{x_n\}$ such that $y_n \to y(T)$. Then, by Lemma 7 (ii), $y = x$. Hence, by Corollary 1, $x_n \to x(G)$ which shows that $G \supset T_*$. Finally, $G = T*$. Since T_* is a FLUSHP-convergence on X, G is a FLUSHP-convergence in X and this proves (d). $\square$

References

[1] *J. Novák*: On convergence groups, Czechoslovak Math. J. *20(95)* (1970), 357–374.

[2] *J. Pochcial*: On functional convergences, Rend. Ist. Univ. di Trieste *XVII* (1985), 47–54.

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