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METRIZATION OF UNIFORM LATTICES

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0. INTRODUCTION

In [W2] we have studied uniform lattices as generalization of Boolean rings endowed with an FN -topology and of Riesz spaces endowed with a locally solid linear topology. In these two special cases the uniformity (induced by an FN -topology or a locally solid linear topology) is generated by a system $(d_\alpha)_{\alpha \in A}$ of pseudo-metrics with the property

$$(*) \quad d_\alpha(x \vee z, y \vee z) \leq d_\alpha(x, y), \quad d_\alpha(x \wedge z, y \wedge z) \leq d_\alpha(x, y).$$

More generally, I. Fleischer and T. Traynor [FT] have proved that any uniformity on a lattice induced by a modular function with values in a commutative topological groups is generated by a system (d_α) of pseudo-metrics with the property (*). It is natural question whether that also holds for an arbitrary uniform lattice, i.e. for a uniformity on a lattice such that the lattice operations \vee and \wedge are uniformly continuous. The answer is no in general (see section 2), but yes in the case that the lattice is distributive (see section 1, (1.6)). The setting of section 1 is more general. There we study uniform spaces with one or more operations. In particular, section 1 contains a simple proof of the known fact that the uniformity of a uniform semigroup $(X, +)$ is induced by a system of pseudo-metrics (d_α) such that

$$d_\alpha(x + y, x' + y') \leq d_\alpha(x, x') + d_\alpha(y, y').$$

Hereby uniform semigroup is defined as a semigroup endowed with a uniformity such that the semigroup operation is uniformly continuous.

1. METRIZATION OF UNIFORM SEMIGROUPS AND ALGEBRAS

In the following let (X, u) be a uniform space. We denote by Δ the diagonal $\Delta := \{(x, x) : x \in X\}$.

Proposition 1.1. *Let $+$: $X \times X \rightarrow X$ be an operation on X .*

(a) *$+$ is uniformly continuous iff for every $U \in u$ there is a $V \in u$ such that $V + \Delta \subset U$ and $\Delta + V \subset U$.*

(b) *If $+$ is associative, then $+$ is uniformly continuous iff u has a base of sets U with $U + \Delta \subset U$ and $\Delta + U \subset U$.*

Proof. (a) Since $+$ is uniformly continuous iff for every $U \in u$ there is a $V \in u$ with $V + V \subset U$, one implication (\Rightarrow) is obvious. To prove the other implication (\Leftarrow), let $U \in u$ and $V, W \in u$ with $V \circ V \subset U$ and $\Delta + W, W + \Delta \subset V$. We show that $W + W \subset U$. If $(x, x'), (y, y') \in W$, then $(x + y, x' + y') = (x, x') + (y, y') \in W + \Delta \subset V$ and similarly $(x' + y, x' + y') \in V$, hence $(x + y, x' + y') \in V \circ V \subset U$.

(b) \Leftarrow follows from (a).

\Rightarrow : Let $+$ be associative and $W \in u$. We show that W contains an $U \in u$ with $U + \Delta, \Delta + U \subset U$. Choose $V \in u$ with $V + V + V \subset W, V + V \subset W$ and put

$$U := \{(x, y) \in W : (x, y) + \Delta, \Delta + (x, y), \Delta + (x, y) + \Delta \subset W\}.$$

By definition, $U \subset W$. Since $\Delta \subset V$ and $V + V, V + V + V \subset W$, one gets $V \subset U$, hence $U \in u$. We show that $U + \Delta \subset U$; analogously one obtains $\Delta + U \subset U$. To prove $U + \Delta \subset U$ we have to check that $U + \Delta \subset W$ and $(U + \Delta) + \Delta, \Delta + (U + \Delta), \Delta + (U + \Delta) + \Delta \subset W$. But this holds obviously, since $+$ is associative, $\Delta + \Delta \subset \Delta$ and $U + \Delta, \Delta + U, \Delta + U + \Delta \subset W$ by the definition of U . □

If we write in (1.1) (b) $f(x, y)$ instead of $x + y$, then the inclusions $U + \Delta \subset U$ and $\Delta + U \subset U$ mean that $(f(x, y), f(x', y)) \in U$ and $(f(y, x), f(y, x')) \in U$ hold for any $(x, x') \in U$ and $y \in X$. This formulation is used in the next proposition.

Proposition 1.2. *Let $f_i : X \times X \rightarrow X$ be operations on X for $i \in I$ and I_0 a finite subset of I ; ($I_0 = \emptyset$ or $I \setminus I_0 = \emptyset$ are admitted). Further, let q be a real number, $q > 1$. Then there is a system D of pseudo-metrics on X , which generates the uniformity u , such that for any $d \in D$ and $x, x', y, y' \in X$*

$$d(f_i(x, y), f_i(x', y')) \leq q(d(x, x') + d(y, y')) \quad \text{for } i \in I_0$$

and

$$d(f_i(x, y), f_i(x', y')) \leq d(x, x') + d(y, y') \text{ for } i \in I \setminus I_0$$

iff f_i are uniformly continuous for $i \in I_0$ and u has a base of sets U such that

$$(f_i(x, y), f_i(x', y)) \in U \text{ and } (f_i(y, x), f_i(y, x')) \in U$$

for any $(x, x') \in U$, $y \in X$ and $i \in I \setminus I_0$. Moreover, if u has a countable base, then one can replace in this equivalence the system D by a single pseudo-metric d .

PROOF. One implication (\Rightarrow) is obvious. Suppose now that f_i is uniformly continuous for $i \in I_0$ and u has a base of sets U such that $(f_i(x, y), f_i(x', y))$, $(f_i(y, x), f_i(y, x')) \in U$ for $(x, x') \in U$, $y \in X$, $i \in I \setminus I_0$.

Let A be the system of all sequence $(U_n)_{n \in \mathbf{N}}$ of symmetric sets of u with the property that for any $n \in \mathbf{N}$ $U_n \circ U_n \circ U_n \subset U_{n-1}$ (with $U_0 := X \times X$) and that for $(x, x') \in U_n$ and $y \in X$ the pairs $(f_i(x, y), f_i(x', y))$ and $(f_i(y, x), f_i(y, x'))$ belong to U_{n-1} for $i \in I_0$ and belong to U_n for $i \in I \setminus I_0$.

For $\alpha = (U_n) \in A$ define g_α by $g_\alpha(x, y) = 2^{-n}$ iff $(x, y) \in U_{n-1} \setminus U_n$ and $g_\alpha(x, y) = 0$ iff (x, y) belongs to each U_n .

Now define $d_\alpha: X \times X \rightarrow [0, 1]$ by

$$d_\alpha(x, y) := \inf \left\{ \sum_{j=0}^n g_\alpha(x_j, x_{j+1}) : n \in \mathbf{N}, x_j \in X, x_0 = x, x_{n+1} = y \right\}.$$

On p. 185 of [K] it is proved that d_α is a pseudo-metric and $U_n \subset \{(x, y) \in X \times X : d_\alpha(x, y) < 2^{-n}\} \subset U_{n-1}$. Therefore $(d_\alpha)_{\alpha \in A}$ generates u , since every $U \in u$ contains a sequence of A .

Let $x, x', y, y' \in X$. Obviously $g_\alpha(f_i(x, y), f_i(x', y)) \leq 2g_\alpha(x, x')$ for $i \in I_0$ and $g_\alpha(f_i(x, y), f_i(x', y)) \leq g_\alpha(x, x')$ for $i \in I \setminus I_0$, hence

$$\begin{aligned} d_\alpha(f_i(x, y), f_i(x', y)) &\leq \\ &\leq \inf \left\{ \sum_{j=0}^n g_\alpha(f_i(x_j, y), f_i(x_{j+1}, y)) : n \in \mathbf{N}, x_j \in X, x_0 = x, x_{n+1} = x' \right\} \\ &\leq \inf \left\{ \sum_{j=0}^n 2g_\alpha(x_j, x_{j+1}) : n \in \mathbf{N}, x_j \in X, x_0 = x, x_{n+1} = x' \right\} = 2d_\alpha(x, x') \\ &\text{for } i \in I_0 \end{aligned}$$

and similarly

$$d_\alpha(f_i(x, y), f_i(x', y)) \leq d(x, x') \text{ for } i \in I \setminus I_0.$$

Analogously one gets

$$d_\alpha(f_i(x', y), f_i(x', y')) \leq 2d_\alpha(y, y') \quad \text{for } i \in I_0$$

and

$$d_\alpha(f_i(x', y), f_i(x', y')) \leq d_\alpha(y, y') \quad \text{for } i \in I \setminus I_0.$$

Finally

$$\begin{aligned} d_\alpha(f_i(x, y), f_i(x', y')) &\leq d_\alpha(f_i(x, y), f_i(x', y)) + d_\alpha(f_i(x', y), f_i(x', y')) \\ &\leq 2(d_\alpha(x, x') + d_\alpha(y, y')) \quad \text{for } i \in I_0 \end{aligned}$$

and analogously

$$d_\alpha(f_i(x, y), f_i(x', y')) \leq d_\alpha(x, x') + d_\alpha(y, y') \quad \text{for } i \in I \setminus I_0.$$

Now choose $n \in \mathbf{N}$ with $2^{1/n} \leq q$. Then the family $D := \{d_\alpha^{1/n} : \alpha \in A\}$ has the desired properties.

If u has a countable base, A contains one sequence $\gamma = (U_n)$, which is a base of u . In this case we can take $D = \{d_\gamma^{1/n}\}$. □

Some remarks to (1.2) are given in (2.1) and (2.2).

The next two corollaries immediately follow from (1.1) (b) and (1.2) (applied with $|I| = 1$, $I_0 = \emptyset$ or $|I| = 2$, $|I_0| = 1$, respectively).

Corollary 1.3. *If $(X, u, +)$ is a uniform semigroup, then u is generated by a system D of pseudo-metrics on X such that $d(x + y, x' + y') \leq d(x, x') + d(y, y')$ for all $x, x', y, y' \in X$ and $d \in D$.*

Note that in the commutative case in (1.3) the condition “ $d(x + y, x' + y') \leq d(x, x') + d(y, y')$ for all $x, x', y, y' \in X$ ” is equivalent to the condition “ $d(x + z, y + z) \leq d(x, y)$ for all $x, y, z \in X$ ”.

(1.3) was first given in [W1, Hilfssatz (1.1)]. The proof, the idea of which was given in [W1, p. 414], was elaborated in detail in [FM, p. 3-7] and [P, p. 8-11] and is quite long. In the proof given here, however, we can at once apply with the help of (1.1)(b) the metrization lemma [K, p. 185], which leads to an essentially simpler proof.

Corollary 1.4. *Let (X, u, \vee, \wedge) be a uniform lattice and $q > 1$. Then u is generated by a system D of pseudo-metrics on X such that $d(x \vee z, y \vee z) \leq d(x, y)$ and $d(x \wedge z, y \wedge z) \leq q \cdot d(x, y)$ for all $x, y, z \in X$ and $d \in D$.*

In general one cannot replace in (1.4) q by 1 (see (2.3)), but that is possible in the distributive case; more general holds:

Theorem 1.5. *Assume that $+, \cdot: X \times X \rightarrow X$ are two uniformly continuous associative operations on (X, u) , which satisfy the distributive laws*

$$(x + y) \cdot z = (x \cdot z) + (y \cdot z)$$

and

$$z \cdot (x + y) = (z \cdot x) + (z \cdot y) \text{ for all } x, y, z \in X.$$

Then u is generated by a system D of pseudo-metrics on X such that

$$\begin{aligned} d(x + y, x' + y') &\leq d(x, x') + d(y, y'), \\ d(x \cdot y, x' \cdot y') &\leq d(x, x') + d(y, y') \end{aligned}$$

for all $x, x', y, y' \in X$ and $d \in D$.

Proof. Let $W \in u$. By (1.2) it is enough to prove that W contains a $U \in u$ with $U + \Delta, \Delta + U, U \cdot \Delta, \Delta \cdot U \subset U$.

By (1.1) (b), there is a $V \in u$ such that $V + \Delta, \Delta + V \subset V \subset W$. Put

$$U := \{(x, y) \in V : (x, y) \cdot \Delta, \Delta \cdot (x, y), \Delta \cdot (x, y) \cdot \Delta \subset V\}.$$

Of course $U \subset W$. As in the proof of (1.1) (b) one gets that $U \in u$ and $U \cdot \Delta, \Delta \cdot U \subset U \subset V$. Now we prove that $U + \Delta \subset U$; analogously one gets $\Delta + U \subset U$. To prove $U + \Delta \subset U$ we have to check that $U + \Delta \subset V$ and $(U + \Delta) \cdot \Delta, \Delta \cdot (U + \Delta), \Delta \cdot (U + \Delta) \cdot \Delta \subset V$. First we have $U + \Delta \subset V + \Delta \subset V$. Further $(U + \Delta) \cdot \Delta \subset U \cdot \Delta + \Delta \cdot \Delta \subset U + \Delta \subset V + \Delta \subset V$, analogously $\Delta \cdot (U + \Delta) \subset U + \Delta \subset V$. Finally $[\Delta \cdot (U + \Delta)] \cdot \Delta \subset (U + \Delta) \cdot \Delta \subset V$. \square

Corollary 1.6. *If (X, u, \vee, \wedge) is a distributive uniform lattice, then u is generated by a system D of pseudo-metrics on X such that $d(x \vee z, y \vee z) \leq d(x, y)$ and $d(x \wedge z, y \wedge z) \leq d(x, y)$ for all $x, y, z \in X$ and $d \in D$.*

I. Fleischer and T. Traynor [FT] have proved that the uniformity on a lattice induced by a modular function with values in a quasinormed group is generated by a pseudo-metric d such that

- (i) $d(x \vee z, y \vee z) \leq d(x, y), d(x \wedge z, y \wedge z) \leq d(x, y)$,
- (ii) $d(u, v) \leq d(x, y)$ if $x \leq u \leq v \leq y$,
- (iii) $d(x \wedge y, x) = d(y, x \vee y)$

(iv) $d(x, y) = d(x \wedge y, x \vee y)$.

In brief, we examine these properties in a more general setting.

Proposition 1.7. *Let d be a pseudo-metric on a lattice X such that for all $x, y, z \in X$ hold*

$$d(x \vee z, y \vee z) \leq d(x, y) \quad \text{and} \quad d(x \wedge z, y \wedge z) \leq d(x, y).$$

Then

(a) $x \leq u \leq v \leq y$ implies $d(u, v) \leq d(x, y)$,

(b) $d(x \wedge y, x) = d(y, x \vee y) \leq d(x, y)$,

(c) $\frac{1}{2}d(x, y) \leq d(x \wedge y, x \vee y) \leq 2d(x, y)$ for all $x, y, u, v \in X$.

Proof. (a) $d(u, v) = d(u \wedge v, y \wedge v) \leq d(u, y) = d(x \vee u, y \vee u) \leq d(x, y)$.

(b) $d(x \wedge y, x) = d(y \wedge x, (x \vee y) \wedge x) \leq d(y, x \vee y)$, dually $d(x \vee y, y) = d(x \vee y, (x \wedge y) \vee y) \leq d(x, x \wedge y)$. Hence $d(x \wedge y, x) = d(x \vee y, y) = d(x \vee y, y \vee y) \leq d(x, y)$.

(c) $d(x, y) \leq d(x, x \wedge y) + d(x \wedge y, y) \leq 2d(x \wedge y, x \vee y)$ by (a). $d(x \wedge y, x \vee y) \leq d(x \wedge y, x) + d(x, x \vee y) \leq 2d(x, y)$ by (b).

The inequalities in (1.7) (c) are sharp: Define on the free lattice $\{0, a, b, 1\}$ with generators a, b a metric by $d(0, a) = d(0, b) = d(1, a) = d(1, b) = 1$ and $d(a, b) = 2 \cdot d(0, 1) = 2$ or $d(0, 1) = 2 \cdot d(a, b) = 2$; in the first case we have $\frac{1}{2}d(a, b) = d(a \wedge b, a \vee b)$, in the second case $d(a \wedge b, a \vee b) = 2 \cdot d(a, b)$.

Given (X, d) as in (1.7). I don't know whether there exists another pseudo-metric on X with the properties (i) to (iv), which generates the same uniformity as d . If we define $d_1(x, y) := d(x \wedge y, x) + d(x \wedge y, y)$, then d_1 is a pseudo-metric with $d \leq d_1 \leq 2d$, with the properties (i) to (iii) and at least $d_1(x, y) \geq d_1(x \wedge y, x \vee y)$ for all $x, y \in X$; but in the example given before ($X = \{0, a, b, 1\}$, d with $d(0, 1) = 1$) we have $d_1(a, b) > d_1(a \wedge b, a \vee b)$. It would be near at hand to take as distance function

$$d_2(x, y) := d(x \wedge y, x \vee y)$$

or

$$d_3(x, y) := \sup\{d(u, v) : x \wedge y \leq u, v \leq x \vee y\}.$$

But neither d_2 nor d_3 satisfies, in general, the triangular inequality, as shows the following example: Define on the lattice $X = \{x_1, \dots, x_9\}$ of figure 1 a metric by $d(x_i, x_{i+1}) = 1 (i = 1, \dots, 8)$, $d(x_i, x_9) = 1 (i = 2, 3, 4, 6, 7)$, $d(x_1, x_8) = d(x_2, x_4) = d(x_6, x_8) = 1$, $d(x_1, x_5) = 3$ and $d(x_i, x_j) = 2$ for all other pairs with $i < j$. Then

$$d_2(x_3, x_7) = d_3(x_3, x_7) = 3$$

but

$$d_2(x_3, x_9) = d_2(x_9, x_7) = d_3(x_3, x_9) = d_3(x_9, x_7) = 1.$$

□

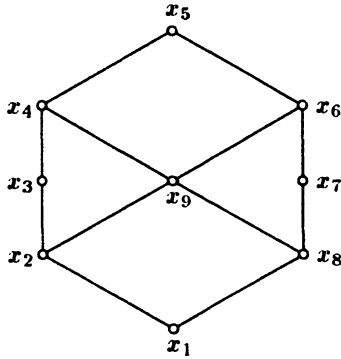


Figure 1

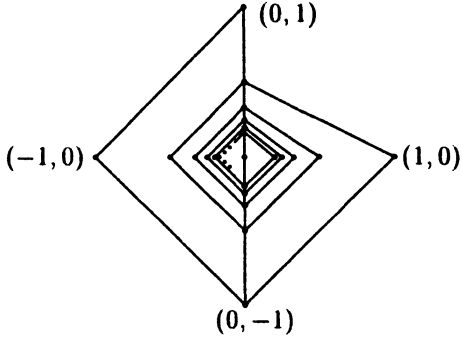


Figure 2

2. COUNTEREXAMPLES

Remark 2.1. In (1.2), the assumption that I_0 is finite is not superfluous.

Proof. Take $(X, u) = \mathbf{R}$ the reals with the usual uniformity, $I = I_0 = \mathbf{N}$ and $f_n(x, y) = nxy$ ($n \in \mathbf{N}$; $x, y \in \mathbf{R}$). Suppose that u is generated by a metric d such that

$$d(f_n(x, y), f_n(x', y)) \leq d(x, x') \text{ for all } n \in \mathbf{N} \text{ and } x, x', y \in \mathbf{R}.$$

Then

$$d(1, 0) = d\left(f_n\left(\frac{1}{n}, 1\right), f_n(0, 1)\right) \leq d\left(\frac{1}{n}, 0\right) \rightarrow 0 \quad (n \rightarrow \infty),$$

a contradiction. □

Remark 2.2. In (1.3), the assumption that the addition is associative is not superfluous.

Proof. Take $(X, u) = [0, 2]$ with the usual uniformity, $x \oplus y := \min\{2, xy\}$ for $x, y \in [0, 2]$. Suppose that u is generated by a metric d such that $d(x \oplus y, x' \oplus y) \leq d(x, x')$ for all $x, x', y \in [0, 2]$. Then $d(x, 0) = d((\frac{1}{2}x) \oplus 2, 0 \oplus 2) \leq d(\frac{1}{2}x, 0)$ and by induction $d(x, 0) \leq d(2^{-n}x, 0)$, hence $d(1, 0) = d(2^{-n}, 0) \rightarrow 0$ ($n \rightarrow \infty$), a contradiction. □

The examples given in (2.1) and (2.2) also show that in (1.2) one cannot replace $q > 1$ by $q = 1$.

The example (2.3) shows that in (1.6) we cannot dispense with the distributivity.

Example 2.3. (cf. Figure 2). Let $K_0 := \{x \in \mathbf{R} : x = 0 \text{ or } |x| = \frac{1}{n} \text{ for some } n \in \mathbf{N}\}$. Define on $L := (\{0\} \times K_0) \cup (K_0 \times \{0\}) \subset \mathbf{R}^2$ two real-valued functions f and g by $f(0, y) = g(0, y) = y$, $f(x, 0) = -|x|$; $g(x, 0) = |x|$ if $x \leq 0$ and $g(x, 0) = \frac{1}{n+1}$ if $x = \frac{1}{n}$ for an $n \in \mathbf{N}$.

For $a, b \in L$ define

$$a \leq b \text{ iff } g(a) \leq f(b) \text{ or } a = b.$$

Let u be the uniformity induced on L by the usual uniformity of \mathbf{R}^2 .

(a) Then (L, \leq) is a lattice, u is a compact metrizable uniformity and (L, u) is a uniform lattice.

(b) If d is any continuous pseudo-metric on L such that for all $x, y, z \in L$

$$d(x \vee z, y \vee z) \leq d(x, y) \quad \text{and} \quad d(x \vee z, y \vee z) \leq d(x, y),$$

then $d((0, -1), (1, 0)) = 0$.

In particular, u is not generated by a metric satisfying $(*)$.

Proof. (a) (L, \leq) is a lattice by the next lemma (2.4), applied for $K = \{0\} \times K_0$ with its natural order, $s(a) = (0, g(a))$ and $i(a) = (0, f(a))$ ($a \in L$).

By definition, u is metrizable. L is a closed, bounded subset of \mathbf{R}^2 , hence (L, u) is compact.

We prove now that \vee and \wedge are continuous. From that it follows that \vee, \wedge are uniformly continuous since (L, u) is compact. Since $(0, 0)$ is the only accumulation point of L , it is enough to show that

(i) $(a, b) \mapsto a \vee b$ and $(a, b) \mapsto a \wedge b$ are continuous in $((0, 0), (0, 0))$ and that

(ii) $a \mapsto a \vee b$ and $a \mapsto a \wedge b$ are continuous in $(0, 0)$ for every $b \in L$, $b \neq (0, 0)$.

(i) By (2.4), $a \vee b$ and $a \wedge b$ belong to $\{s(a), s(b), i(a), i(b), a, b\}$. Hence $\|a \vee b\|_\infty, \|a \wedge b\|_\infty \leq \max\{\|a\|_\infty, \|b\|_\infty\}$. This implies (i).

(ii) Let $b \in L$, $b \neq (0, 0)$. Put $U := \{a \in L : \|a\|_\infty < \frac{1}{3}\|b\|_\infty\}$. If $b = (0, y) \in K$ with $y > 0$, then $a \wedge b = a$ for $a \in U$, hence $a \mapsto a \wedge b$ is continuous in $(0, 0)$. Similarly, if $b = (0, y) \in K$ with $y < 0$, then $a \vee b = a$ for $a \in U$, hence $a \mapsto a \vee b$ is continuous in $(0, 0)$. In all other cases (for b) the functions $a \mapsto a \vee b$ and $a \mapsto a \wedge b$ are constant on U and therefore continuous in $(0, 0)$.

(b) Suppose that d is a pseudo-metric on L , which is continuous in $(0, 0)$ and satisfies $(*)$. For $n \in \mathbf{N}$, put

$$r_n = \left(\frac{1}{n}, 0\right), \quad l_n = \left(-\frac{1}{n}, 0\right), \quad a_n = \left(0, \frac{1}{n}\right), \quad b_n = \left(0, -\frac{1}{n}\right).$$

Then $d(b_n, r_n) = d(l_{n+1} \wedge r_n, a_{n+1} \wedge r_n) \leq d(l_{n+1}, a_{n+1}) = d(b_{n+1} \vee l_{n+1}, r_{n+1} \vee l_{n+1}) \leq d(b_{n+1}, r_{n+1})$, hence by induction $d(b_1, r_1) \leq d(b_n, r_n)$ for $n \in \mathbf{N}$. Since $d(b_n, r_n) \rightarrow 0$ ($n \rightarrow \infty$), it follows that $d((0, -1), (1, 0)) = d(b_1, r_1) = 0$. \square

Lemma 2.4. *Let K be a lattice, L a set, which contains K , and $i, s: L \rightarrow K$ two functions such that $i(x) = s(x) = x$ for $x \in K$ and $i(x) < s(x)$ for $x \in L \setminus K$. Then*

$$x \leq y \text{ (in } L) \text{ iff } s(x) \leq i(y) \text{ (in } K) \text{ or } x = y$$

defines a partial ordering on L . With respect to this partial ordering L becomes a lattice and K is a sublattice of L . Moreover, if x, y are incomparable elements of L , then

$$\sup_L \{x, y\} = \sup_K \{s(x), s(y)\} \quad \text{and} \quad \inf_L \{x, y\} = \inf_K \{i(x), i(y)\}.$$

Proof. Since $i(x) = s(x) = x$ for $x \in K$, the relation defined on L coincides on K with the given partial ordering on K . Obviously, \leq is reflexive on L .

\leq is antisymmetric: Suppose that $x, y \in L$ with $x \leq y$, $y \leq x$, $x \neq y$. Then $s(x) \leq i(y)$ and $s(y) \leq i(x)$. Since $i(z) \leq s(z)$ for all $z \in L$, one obtains $s(x) \leq i(y) \leq s(y) \leq i(x) \leq s(x)$, hence $s(x) = i(x) = s(y) = i(y)$. It follows that $x, y \in K$, since $s(z) \neq i(z)$ for $z \in L \setminus K$. Consequently, $x = s(x) = s(y) = y$.

\leq is transitive: Suppose that $x, y, z \in L$ with $x \leq y$, $y \leq z$ and $x \neq y \neq z$. Then $s(x) \leq i(y) \leq s(y) \leq i(z)$, hence $s(x) \leq i(z)$ and $x \leq z$.

Let x, y be incomparable elements of L . $a := \sup_K \{s(x), s(y)\}$ is the supremum of $\{x, y\}$ in L : Since $a \in K$, we have $s(a) = i(a) = a$. Therefore $s(x) \leq a = i(a)$, hence $x \leq a$ and just so $y \leq a$. Let $z \in L$ be an upper bound of $\{x, y\}$. Since x, y are incomparable, it follows that $z \neq x$ and $z \neq y$ and therefore $s(x) \leq i(z)$ and $s(y) \leq i(z)$, hence $s(a) = a \leq i(z)$. Consequently $a \leq z$. Similarly one gets that $\inf_K \{i(x), i(y)\}$ is the infimum of $\{x, y\}$. In particular, L is a lattice and K a sublattice of L . \square

In the example (2.3), (L, u) is a compact Hausdorff uniform lattice. It follows from some statements in [W2] that L is (as lattice) complete and that u is order continuous, exhaustive and satisfies (F) and (σ) (see [W2] for the definitions). Therefore (L, u) has strong topological properties. On the other hand, the lattice L is not modular. It would be of interest to decide, whether such an example exists also in the modular case.

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