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FREE ALGEBRAS ON A FIXED SET

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Free algebras play a great role in classes of algebras and it is common to consider those classes of algebras in which free algebras exist. Namely, in varieties there exists a free algebra on each set. In this paper relations between various free algebras on a fixed set will be considered. In our considerations the free algebra on a set X need not be generated by the set X by means of basic operations.

Definition. Let C be a class of similar algebras. The C-free algebra on a set X in an algebra $F(X)$ from the class C such that if $A \in C$ is an arbitrary algebra, then each mapping $X \to A$ can be uniquely extended to the homomorphism $F(X) \to A$. If $F(X)$ is generated by the set X by means of basic operations, then $F(X)$ is called a *proper free algebra*, otherwise it is called an improper free algebra.

Example 1. A free semigroup is a proper free algebra in the class of all semigroups.

Example 2. A free group is an improper free algebra in the class G of all groups regarded with one binary operation of multiplication. Observe that the G-free algebra on the one-element set $\{1\}$ is the set of all integers with one binary operation of addition, and the whole group is not generated by the set $\{1\}$ by means of addition.

All free algebras in varieties are proper. Now we shall deal with free algebras on a fixed set and both proper and improper free algebras will be considered.

Definition. Let A be an algebra and let $h\colon A^n \to A$ be a mapping for which $h(a_1, \ldots, a_n)\varphi = h(a_1\varphi, \ldots, a_n\varphi)$ holds, where $a_1, \ldots, a_n \in A$ are arbitrary elements and $\varphi\colon A \to A$ is an arbitrary homomorphism. Then h is called an *n-ary improper operation on A*.

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R e m a r k . Each basic operation and each polynomial is a improper operation. In the following example it is shown that there exist improper operations which are neither basic operations nor polynomials.

E x a m p l e 3. Let $A = \langle Z, \{f\}\rangle$ be an algebra where Z is the set of all integers and the only unary operation f is defined by $f(a) = a + 1$. Then the mapping h: $A \to A$ defined by $h(a) = a - 1$ is unary improper operation which is neither a basic operation nor a polynomial. First we show that h is preserved by all endomorphisms of the basic operations. If $a \in A$ is an arbitrary element and $\varphi \colon A \to A$ is a homomorphism, then $h(a)\varphi = hf[h(a)\varphi] = h[fh(a)\varphi] = h(a\varphi)$. It is obvious that $p(a) \geqslant a$ for an arbitrary polynomial p and $h(a) < a$.

In an improper free algebra on an infinite set X there can be elements whose homomorphic image depends on the homomorphic images of finitely many elements of X. As we shall see, these elements can be values of suitable improper operations with arguments from the set X. The other elements, whose homomorphic images depend on the homomorphic images of an infinite subset of the set X, are called transcendent elements. The existence of transcendent elements is shown in [2]. In this paper just free algebras without transcendent elements will be considered.

Definition. Let A and B be similar algebras and let A be the free algebra on a set X in the class $\{A, B\}$. Then the algebras A and B are in the relation F_X. In this case we shall write $AF_X B$.

The aim of this paper is to describe some relationships between proper and improper free algebras, the relationship between improper operations in various algebras, and to show that the system of all free algebras on a fixed set X is a partially ordered set with respect to the relation F_X.

We shall use the same notation for algebras and their basic sets when there is no danger of confusion.

Theorem 1. *Let $F(X)$ be a free algebra on the set X. If $a \in F(X)$ is not a transcendent element, then it is a value of a suitable improper operation with arguments from the set X.*

P r o o f . Let all homomorphic images of a depend on the homomorphic images of elements $x_i \in X$ for $i = 1, \ldots, n$. We can define an n-ary improper operation h by

$$h(x_1, \ldots, x_n) = a$$

and for arbitrary elements $a_1, \ldots, a_n \in F(X)$

$$h(a_1, \ldots, a_n) = a\varphi$$

where $\varphi\colon F(X) \to F(X)$ is an arbitrary homomorphism with the property $x_i\varphi = a_i$ for $i = 1, \ldots, n$.

Now we show that h is preserved by all homomorphisms of basic operations. Let $b_1, \ldots, b_n \in F(X)$ be arbitrary elements and let $\psi\colon F(X) \to F(X)$ be an arbitrary homomorphism. By the definition of h we have $h(b_1, \ldots, b_n)\psi = h(x_1, \ldots, x_n)\varphi\psi = h(x_1\varphi\psi, \ldots, x_n\varphi\psi) = h(b_1\psi, \ldots, b_n\psi)$ where $\varphi\colon F(X) \to F(X)$ is an arbitrary homomorphism with the property $x_i\varphi = b_i$ for $i = 1, \ldots, n$. □

It is obvious that each improper operation can be a basic operation or a polynomial on an algebra with the same basic set regarded as an algebra of a suitable, as a rule extended, type.

Example 4. An improper free group on any set regarded with one binary operation of multiplication can be regarded as a proper free group on the same set, namely with one binary operation of multiplication, one nullary operation of the unit and one unary operation of the inverse element.

Definition. Let A be an improper free algebra on a set X. Then $\bar{A}$ will denote a proper free algebra on the same set.

Remark. A proper free algebra $\bar{A}$ always exists but it need not be unique. For the improper free group in Example 4 there exists another proper free group with one binary operation $*$ defined by $x * y = x \cdot y^{-1}$.

Definition. Let A and B be similar algebras, let h be an n-ary improper operation on A and let k be an n-ary improper operation on B. We say that h is in the relation Hom with k ($h \operatorname{Hom} k$) if

$$h(a_1, \ldots, a_n)\varphi = k(a_1\varphi, \ldots, a_n\varphi)$$

for an arbitrary homomorphism $\varphi\colon A \to B$ and arbitrary elements $a_1, \ldots, a_n \in A$.

The following two theorems establish the relationship between the relations F_X and Hom.

Theorem 2. *Let $F(X)F_XA$ where X is an infinite set and let h be an n-ary improper operation of $F(X)$. Then there exists an n-ary improper operation k on A with the property $h \operatorname{Hom} k$.*

Proof. Let $x_1, \ldots, x_n \in X$ be arbitrary elements. Then for arbitrary elements $a_1, \ldots, a_n \in A$ we put

$$k(a_1, \ldots, a_n) = h(x_1, \ldots, x_n)\varphi$$

where $\varphi\colon F(X) \to A$ is an arbitrary homomorphism with the property $x_i\varphi = a_i$ for $i = 1, \ldots, n$.

First we show that the homomorphic image $h(x_1, \ldots, x_n)$ depends only on the elements $x_i\varphi$ for $i = 1, \ldots, n$. Let us suppose that there exists a homomorphism $\psi\colon F(X) \to A$ with the property $x_i\varphi = x_i\psi$ for $i = 1, \ldots, n$ and $h(x_1, \ldots, x_n)\varphi \neq h(x_1, \ldots, x_n)\psi$. Since X is infinite, the set $Y = X - \{x_1, \ldots, x_n\}$ is obviously also infinite, therefore there exist two disjoint sets Z and T with the same cardinality as X, with the property $Y = Z \cup T$. Thus there exist bijections $\xi\colon Y \to Z$ and $\eta\colon Y \to T$. Denote $U = Z \cup \{x_1, \ldots, x_n\}$ and $V = T \cup \{x_1, \ldots, x_n\}$. Extend the bijection $\xi\colon Y \to Z$ to the bijection $\mu\colon X \to U$ by

$$x\mu = \begin{cases} x\xi, & \text{for } x \in Y, \\ x, & \text{for } x \in \{x_1, \ldots, x_n\} \end{cases}$$

and define a mapping $\lambda\colon U \to A$ by

$$u\lambda = u\mu^{-1}\varphi \quad \text{for } u \in U.$$

Similarly extend the bijection $\eta\colon Y \to T$ to the bijection $\nu\colon X \to V$ by

$$x\nu = \begin{cases} x\eta, & \text{for } x \in Y, \\ x, & \text{for } x \in \{x_1, \ldots, x_n\} \end{cases}$$

and define a mapping $\varrho\colon V \to A$ by

$$v\varrho = v\nu^{-1}\psi \quad \text{for } v \in V.$$

Since μ is a bijection, the mapping $\lambda\colon U \to A$ can be uniquely extended to a homomorphism $\lambda\colon F(X) \to A$ with the property $h(x_1, \ldots, x_n)\lambda = h(x_1, \ldots, x_n)\varphi$. Similarly since ν is a bijection, the mapping $\varrho\colon V \to A$ can be uniquely extended to a homomorphism $\varrho\colon F(X) \to A$ with the property $h(x_1, \ldots, x_n)\varrho = h(x_1, \ldots, x_n)\psi$.

Now define a mapping $\chi\colon X \to A$ in the following way:

$$x\chi = \begin{cases} x\lambda, & \text{for } x \in U, \\ x\varrho, & \text{for } x \in V. \end{cases}$$

Note that the definition of the mapping χ is correct, because $U \cap V = \{x_1, \ldots, x_n\}$ and $x_i\lambda = x_i\mu^{-1}\varphi = x_i\varphi = x_i\psi = x_i\nu^{-1}\psi = x_i\varrho$ for $i = 1, \ldots, n$.

Extend the mappings $\mu\colon X \to F(X)$, $\nu\colon X \to F(X)$ and $\chi\colon X \to A$ to homomorphisms $\mu\colon F(X) \to F(X)$, $\nu\colon F(X) \to F(X)$ and $\chi\colon F(X) \to A$, respectively. Since μ and ν are endomorphisms and h is an improper operation on $F(X)$, the homomorphic image of $h(x_1, \ldots, h_n)$ depends just on the homomorphic images of the elements $x_1, \ldots, x_n$. While $x_i\mu = x_i\nu = x_i$ for $i = 1, \ldots, n$, thus $h(x_1, \ldots, x_n)\mu = h(x_1, \ldots, x_n)\nu = h(x_1, \ldots, x_n)$ holds.

Now consider the homomorphism $\mu\chi = \mu\lambda = \varphi$. Since $h(x_1, \ldots, x_n)\mu = h(x_1, \ldots, x_n)$, it follows that $h(x_1, \ldots, x_n)\chi = h(x_1, \ldots, x_n)\varphi$. Similarly $\nu\chi = \nu\varrho = \psi$ and since $h(x_1, \ldots, x_n)\nu = h(x_1, \ldots, x_n)$, it follows that $h(x_1, \ldots, x_n)\chi = h(x_1, \ldots, x_n)\psi$ and we get $h(x_1, \ldots, x_n)\varphi = h(x_1, \ldots, x_n)\psi$, which contradicts the assumption.

Now we show that k is an improper operation. Let $\psi\colon A \to A$ be an arbitrary homomorphism. Then $k(a_1, \ldots, a_n)\psi = h(x_1, \ldots, x_n)\varphi\psi = k(x_1\varphi\psi, \ldots, x_n\varphi\psi) = k(a_1\psi, \ldots, a_n\psi)$.

Finally we show that $h \operatorname{Hom} k$. Let $b_1, \ldots, b_n \in F(X)$ be arbitrary elements and let $\varphi\colon F(X) \to A$ be an arbitrary homomorphism. Define a homomorphism ω: $F(X) \to F(X)$ with the property $x_i\omega = b_i$ for $i = 1, \ldots, n$. Then $h(b_1, \ldots, b_n)\varphi = h(x_1\omega, \ldots, x_n\omega)\varphi = k(x_1\omega\varphi, \ldots, x_n\omega\varphi) = k(b_1\varphi, \ldots, b_n\varphi)$. □

A similar theorem holds also in the case when X is a finite set, but in this case the arity of the improper operation should be equal to the cardinality of X.

Theorem 3. *Let $F(X)F_X A$ where $X = \{x_1, \ldots, x_n\}$. If h is an n-ary operation on $F(X)$, then there exists an n-ary operation k on A with the property $h \operatorname{Hom} k$.*

Proof. Let $a_1, \ldots, a_n \in A$ be arbitrary elements. Define

$$k(a_1, \ldots, a_n) = h(x_1, \ldots, x_n)\varphi$$

where $\varphi\colon F(X) \to A$ is the homomorphism for which $x_i\varphi = a_i$ for $i = 1, \ldots, n$. It is obvious that there is a single homomorphism $F(X) \to A$ with this property. The proof that k is an improper operation and the proof of $h \operatorname{Hom} k$ are quite similar to the respective parts of the proof of Theorem 2. □

It is obvious that the relation F_X is reflexive for every set X.

Theorem 4. *The relation F_X is transitive for every set X.*

Proof. First we show the transitivity of the relation F_X for proper free algebras. Let $F(X)$, $G(X)$ and $H(X)$ be proper free algebras on the set X and let $F(X)F_X G(X)$ and $G(X)F_X H(X)$. Let $\varphi\colon X \to H(X)$ be an arbitrary mapping. Extend the mapping φ to a homomorphism $\varphi\colon G(X) \to H(X)$ and extend the insertion $\iota\colon X \to G(X)$ to a homomorphism $\iota\colon F(X) \to G(X)$. The composition $\psi = \iota\varphi\colon F(X) \to H(X)$ is an extension of the mapping $\varphi\colon X \to H(X)$ to a homomorphism $\psi\colon F(X) \to H(X)$. This extension is unique because $F(X)$ is generated by X by means of the basic operations and the basic operations are preserved by all homomorphisms.

Now we show the transitivity of the relation F_X for improper free algebras. Since there are no transcendent elements, $F(X)$ is generated by X by means of basic and improper operations. According to Theorem 1 for each improper operation h on $F(X)$ there exists an improper operation k on $G(X)$ with the property $h \operatorname{Hom} k$.

Similarly, for each improper operation k on $G(X)$ there exists an improper operation l on $H(X)$ with the property $k \operatorname{Hom} l$. The system of all homomorphisms between two algebras is the same in both cases, when all operations are basic or polynomials and when only some operations are basic or polynomial and the other operations are improper. Therefore when the algebras $F(X)$, $G(X)$ and $H(X)$ are regarded as algebras of such a type that all operations are either basic or polynomials, the system of all homomorphisms between each pair of algebras is the same as in the case when the algebras are regarded with improper operations. Since the relation F_X is essentially based on the existence of homomorphisms, then $AF_X B$ if and only if $\bar{A} F_X \bar{B}$ for any algebras A and B. According to the first part of the proof, $F(X) F_X H(X)$ holds. □

Theorem 5. *The relation F_X is antisymmetric for any set X.*

Proof. Let $F(X)$ and $G(X)$ be such free algebras that $F(X) F_X G(X)$ and $G(X) F_X F(X)$. Let $\varphi\colon X \to G(X)$ and $\psi\colon X \to F(X)$ be insertions and let $\varphi\colon F(X) \to G(X)$ and $\psi\colon G(X) \to F(X)$ be their extensions to homomorphisms. The compositions $\varphi\psi\colon F(X) \to F(X)$ and $\psi\varphi\colon G(X) \to G(X)$ can be regarded as extensions of the identity mappings $\iota_X = \varphi\psi\colon X \to X$ and $\iota_X = \psi\varphi\colon X \to X$ to the homomorphisms $\iota_{F(X)} = \varphi\psi\colon F(X) \to F(X)$ and $\iota_{G(X)} = \psi\varphi\colon G(X) \to G(X)$, respectively. Since the identity mappings $\iota_{F(X)}\colon F(X) \to F(X)$ and $\iota_{G(X)}\colon G(X) \to G(X)$ are homomorphisms, the unique extensions of the insertions $\iota_X : X \to F(X)$ and $\iota_X : X \to G(X)$ to homomorphisms $F(X) \to F(X)$ and $G(X) \to G(X)$ must be $\varphi\psi$ and $\psi\varphi$, respectively, and they have the property $\varphi\psi = \iota_{F(X)}$ and $\psi\varphi = \iota_{G(X)}$. Therefore $\varphi\colon F(X) \to G(X)$ and $\psi\colon G(X) \to F(X)$ are bijections, consequently they are isomorphisms. □

Corollary. *F_X is the relation of partial ordering.*

Proof. It follows immediately from Theorems 4 and 5. □

References

[1] *G. Grätzer*: Universal Algebra, London, 1968.
[2] *I. Žembery*: Almost equational classes of algebras, Algebra Universalis *23* (1986), 293–307.

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