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COMPLETE RETRACT MAPPINGS OF A COMPLETE LATTICE ORDERED GROUP

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Retracts of partially ordered sets were studied in [2]–[5]. Retracts of abelian lattice ordered groups were dealt with in [6]. In [7], retract varieties of abelian lattice ordered groups were investigated.

An endomorphism f of a lattice ordered group H is said to be a complete retract (cf. [6]) if it satisfies the following conditions:

(i) $f(f(h)) = h$ for each $h \in H$;

(ii) if $\{h_i\}_{i\in I} \subseteq H$, $h \in H$, $h = \bigvee\limits_{i\in I} h_i$ holds in H, then $f(h) = \bigvee\limits_{i\in I} f(h_i)$, and dually.

The following results concern the relations between complete retract mappings and direct decompositions of a lattice ordered group H.

(A) *Let H be an internal direct product of its l-subgroups A_1, A_2 and A_3. For $h \in H$ let h_i ($i \in \{1,2,3\}$) be the component of h in A_i. Assume that φ is a complete isomorphism of A_2 into A_3. For each $h \in H$ put*

$$f(h) = h_1 + h_2 + \varphi(h_2). \tag{1}$$

Then f is a complete retract mapping of H.

(B) *Let H be a complete lattice ordered group and let f be a complete retract mapping of H. Then there are convex l-subgroups A_1, A_2 and A_3 in H and a complete isomorphism φ of A_1 into A_2 such that*

(i) *H is an internal direct product of its l-subgroups A_i ($i = 1,2,3$);*

(ii) *for each $h \in H$ the relation (1) is valid (where h_1 and h_2 are the components of h in A_1 and in A_2, respectively).*

The assertion (A) is easy to verify; (B) will be proved below. Next, (B) will be applied to obtain a sharpening of a result established in [6]. Let us remark that if H

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fails to be complete, then the assertions of (B) need not be valid for H (cf. Example 1.3 below). Further, the notion of a complete retract variety will be introduced and the lattice of all complete retract varieties will be investigated.

1. PRELIMINARIES

An endomorphism f of a lattice ordered group H will be said to be a retract mapping of H, if $f(f(x)) = f(x)$ for each $x \in H$. If f is a retract mapping of H, then the l-subgroup $f(H)$ of H is called a retract of H (cf. [6]).

If f is a retract mapping of H and if, moreover, f is a complete endomorphism (i.e., if the above condition (ii) is satisfied), then f is said to be a complete retract of H.

The following example shows that a retract mapping need not be complete.

Example 1.1. Let R be the set of all reals and $R^+ = \{t \in R : t \geqslant 0\}$. Let H be the set of all real functions which are defined and continuous on R^+. The lattice operations and the operation $+$ in H are defined point-wise; hence H is an abelian lattice ordered group. For each $x \in H$ let $f(x) \in H$ be such that $f(x)(t) = x(0)$ for each $t \in R^+$. Then f is a retract mapping of H.

Let N be the set of all positive integers. For each $n \in N$ let x_n be an element of H such that $x_n(0) = 0$, $x_n(t) = 1$ for each $t \in R^+$ with $t \geqslant \frac{1}{n}$, and x_n is linear on the interval $[0, \frac{1}{n}]$ of R^+. Next, let $x \in H$ be such that $x(t) = 1$ for each $t \in R^+$, and let $\bar{0}$ be the neutral element of H. Then we have $f(x_n) = \bar{0}$ for each $n \in N$ and

$$\bigvee_{n \in N} x_n = x,$$

hence

$$\bigvee_{n \in N} f(x_n) = \bar{0} \neq x = f(x).$$

Thus f fails to be a complete retract mapping.

The question whether each retract mapping of a complete lattice ordered group must be complete remains open.

An isomorphism φ of a lattice ordered group H_1 into a lattice ordered group H_2 is said to be complete if, whenever $\{h_i\}_{i \in I} \subseteq H_1$, $h \in H_1$ and $\bigvee_{i \in I} h_i = h$ in H_1, then $\varphi(h) = \bigvee_{i \in I} \varphi(h_i)$, and dually.

The following example shows that an isomorphism need not be complete.

E x a m p l e 1.2. Let R be the additive group of all reals with the natural linear order. Put $H_1 = R$, $H_2 = R \circ R$, where $\circ$ denotes the operation of lexicographic product. For each $x \in H_1$ we put $\varphi(x) = (x, 0)$. Then φ is an isomorphism of H_1 into H_2. Let $x_n = \frac{1}{n}$ for each positive integer n. We have $\bigwedge\limits_{n \in N} x_n = 0$, but $\bigwedge\limits_{n \in N} \varphi(x_n)$ does not exist in H_2. Hence the isomorphism φ fails to be complete.

If H is not complete, then the assertion of (B) need not hold.

E x a m p l e 1.3. Put $H = R \circ R$ and for each $(x, y) \in H$ let $f\big((x, y)\big) = (x, 0)$. Then f is a complete retract mapping and there exist no direct factors A_1, A_2 and A_3 of H with the properties as in (B).

The notion of an internal direct decomposition of a lattice ordered group will be applied in the same sense as in [6] or [7].

2. Direct decomposition corresponding to a complete retract mapping

In this section we assume that H is a complete lattice ordered group and that f is a complete retract mapping of H.

Denote $f^{-1}(0) = H_1$.

Lemma 2.1. *H_1 is a closed l-ideal of H.*

P r o o f. Because f is an endomorphism of H, we obtain that H_1 is an l-ideal of H. Next, since f is complete, H_1 is closed in H.

For each $X \subseteq H$ we put

$$X^{\perp} = \{h \in H : |h| \wedge |x| = 0 \text{ for each } x \in X\};$$

$X^{\perp}$ is a polar of H. □

Lemma 2.2. *H_1 is a polar of H.*

P r o o f. This is a consequence of 2.1 and of the completeness of H (cf., e.g., Birkhoff [1], Chap. XIII, Theorem 27). □

Put $K = H_1^{\perp}$. Since each complete lattice ordered group is strongly projectable, we have

$$H = (i) K \times H_1. \tag{1}$$

In view of (1), each $h \in H$ can be written as

$$h = k + h_1 \quad (k \in K,\ h_1 \in H_1)$$

and then $f(h) = f(k)$. Hence for determining f, it suffices to know all the values $f(k)$ for k running over K.

Put $K_1 = \{k \in K : f(k) \in K\}$.

Lemma 2.3. *Let $k \in K$. The following conditions are equivalent:*

(i) $k \in K_1$;

(ii) $f(k) = k$.

Proof. Clearly (ii) $\Rightarrow$ (i). Let (i) hold. Since K is an l-subgroup of H, we have $f(k) - k \in K$. On the other hand,

$$f(f(k) - k) = f(f(k)) - f(k) = 0.$$

whence $f(k) - k \in H_1$. Therefore $f(k) - k = 0$. □

Lemma 2.4. *K_1 is a closed l-ideal of H.*

Proof. From the definition of K_1 it follows immediately that K_1 is an l-subgroup of H. Let $h \in H$, $k_1 \in K_1$, $0 \leqslant h \leqslant k_1$. Then $0 = f(0) \leqslant f(h) \leqslant f(k_1) = k_1$. Since K is convex in H, we obtain $f(h) \in K$ and thus $h \in K_1$. Therefore K_1 is a convex l-subgroup of H. Let k_i $(i \in I)$ be elements of K_1 and let $\bigvee_{i \in I} k_i = h$. In view of 2.1 we have $h \in K$. Next, according to 2.3, $f(k_i) = k_i$ for each $i \in I$, whence

$$f(\bigvee_{i \in I} k_i) = \bigvee_{i \in I} f(k_i) = \bigvee_{i \in I} k_i.$$

Therefore $h \in K_1$. The dual condition can be verified analogously. Hence K_1 is closed in H. □

In view of 2.4, K_1 is an internal direct factor of H. Moreover, since $K_1 \subseteq K$, (1) implies that K_1 is an internal direct factor of K. Thus there is an l-ideal K_2 in K such that

$$K = (i) K_1 \times K_2. \tag{2}$$

Each $k \in K$ can be written as $k = k_1 + k_2$ with $k_1 \in K_1$, $k_2 \in K_2$. Then

$$f(k) = f(k_1) + f(k_2) = k_1 + f(k_2).$$

Hence for determining f it suffices to know the values $f(k_2)$, where k_2 runs over K_2.

For each $k \in K_2$ we put

$$\varphi(k) = f(k)(H_1).$$

Lemma 2.5. *φ is a complete isomorphism of K_2 into H_1.*

Proof. Since f is an endomorphism of H and since the mapping $\psi\colon h \to h(H_1)$ is a homomorphism of H onto H_1 we infer that φ is a homomorphism of K_2 into H_1. Next, both f and ψ are complete and thus φ is complete as well.

Let $k \in K_2$ and assume that $\varphi(k) = 0$. Thus $f(k)(H_1) = 0$ and so in view of (1), $f(k) \in K$. Hence $k \in K_1$. Therefore according to (2) we have $k = 0$. We have obtained that $\varphi^{-1}(0) = \{0\}$, hence φ is an isomorphism of K_2 into H_1. □

Lemma 2.6. *Let $k \in K_2$. Then $f(k)(K_1) = 0$.*

Proof. By way of contradiction, suppose that $f(k)(K_1) = k_1 \neq 0$. Then $f(|k|)(K_1) = |k_1| > 0$. According to 2.3, $f(|k_1|) = |k_1|$. In view of (2) we have $|k_1| \wedge |k| = 0$, hence $f(|k_1|) \wedge f(|k|) = 0$. Thus

$$\begin{aligned} 0 &= \big(f(|k_1|) \wedge f(|k|)\big)(K_1) = f(|k_1|)(K_1) \wedge f(|k|)(K_1) \\ &= |k_1|(K_1) \wedge f(|k|)(K_1) = |k_1| \wedge f(|k|)(K_1) = |k_1|, \end{aligned}$$

which is a contradiction. □

Lemma 2.7. *Let $k \in K_2$. Then $f(k)(K_2) = k$.*

Proof. Denote $f(k) - k = x$. Then $f(x) = 0$, whence $x \in H_1$. From $f(k) = k + x$ and from (1) we obtain $f(k)(K) = (k + x)(K) = k(K) + x(K) = k(K) = k$. Next, in view of (2),

$$f(k)(K_2) = f(k)(K)(K_2) = k(K_2) = k.$$

□

Lemma 2.8. *For each $k \in K_2$ we have $f(k) = k + \varphi(k)$.*

Proof. In view of (1) and (2) the relation

$$f(k) = f(k)(K_1) + f(k)(K_2) + f(k)(H_1)$$

is valid. Hence in view of 2.6 and 2.7 we have $f(k) = k + \varphi(k)$. □

Proof of Theorem (B).

Denote $A_1 = K_1$, $A_2 = K_2$, $A_3 = H_1$. For $h \in H$ let h_i be the component of h in A_i $(i = 1,2,3)$. In view of (1) and (2) we have $h = h_1 + h_2 + h_3$, whence $f(h) = f(h_1) + f(h_2) + f(h_3)$. According to 2.3, $f(h_1) = h_1$. Next, φ is a complete isomorphism of A_2 into A_1 and in view of 2.8, $f(h_2) = h_2 + \varphi(h_2)$. Therefore

$$f(h) = h_1 + h_2 + \varphi(h_2).$$

□

The following result sharpens Theorem 4.13 of [6].

Proposition 2.9. *Let H be a complete lattice ordered group, $H = (i)A \times B$, and let f be a complete retract mapping of H. Then there exist internal decompositions*

$$A = (i)A_1 \times A_2, \quad B = (i)B_1 \times B_2,$$
$$A_1 = (i)A_{11} \times A_{12} \times A_{13}, \quad B_1 = (i)B_{11} \times B_{12} \times B_{13}$$

and complete isomorphisms $\varphi_{10}\colon A_{12} \to A_{13}$, $\varphi_{20}\colon B_{12} \to B_{13}$, $\varphi_1\colon A_2 \to B_1$, $\varphi_2\colon A_2 \to A_1$, $\psi_1\colon B_2 \to A_1$, $\psi_2\colon B_2 \to B_1$ such that

(i) *for each $a_2 \in A_2$ and each $b_2 \in B_2$ the relations*

$$f_2(\varphi_1(a_2)) = 0 = f_1(\varphi_2(a_2)), \quad f_1(\psi_1(b_2)) = 0 = f_2(\psi_2(b_2))$$

are valid;

(ii) *for each $h \in H$ the relation*

$$\begin{aligned} f(h) &= f_1(h(A_1)) + \varphi_2(h(A_2)) + h(A_2) + \varphi_1(h(A_1)) \\ &\quad + f_2(h(B_1)) + \psi_2(h(B_2)) + h(B_2) + \psi_1(h(B_2)) \end{aligned}$$

holds, where $f_1(h_1) = h_1(A_{11}) + f_1(A_{12}) + \varphi_{10}(h_1(A_{12}))$ and $f_2(h_2) = h_2(B_{11}) + h_2(B_{12}) + \varphi_{20}(h_2(B_{12}))$ for each $h_1 \in A_1$ and each $h_2 \in B_1$.

Proof. The assertion follows from Theorem 4.13 in [6] and from (B). □

Proposition 2.10. *Let H be a lattice ordered group, $H = (i) \prod_{i \in I} H_i$. Let f be a complete retract mapping of H. Then*

(i) $f(H) = (i) \prod_{i \in I} f(H_i)$;

(ii) *for each $i \in I$, the mapping $\varphi_i(h_i) = f(h_i)(H_i)$ is a complete retract mapping of H_i and the lattice ordered group $f(H_i)$ is isomorphic to $f(H_i)(H_i)$.*

Proof. The assertion (i) was proved in [7], Theorem 2.4. Let $i \in I$. Since f is a complete endomorphism of H and since the mapping $\psi(h) = h(H_i)$ is a complete endomorphism of H as well, we infer that φ_i is a complete endomorphism of H_i. The remaining part of (ii) was proved in [6] (Lemmas 2.6 and 2.7). □

Corollary 2.11. *Let H be as in 2.10. Then each complete retracts of H is isomorphic to a direct product of complete retract of the factors H_i $(i \in I)$.*

Next, 2.10 and (B) yield:

Theorem 2.12. *Let H be a complete lattice ordered group and let f be a complete retract mapping of H. Let A_1, A_2 and A_3 be as in* (B). *Then the complete retract $f(H)$ of H is isomorphic to the direct product $A_1 \times A_2 \times A_2$.*

3. Complete retract varieties

A retract variety of abelian lattice ordered groups is defined to be a nonempty class of abelian lattice ordered groups which is closed under direct product and retracts. (Cf. [7].)

Definition 3.1. A nonempty class of abelian lattice ordered groups is said to be a complete retract variety if it is closed under direct products and complete retracts.

Let $\overline{0}$ be the class of all one-element lattice ordered groups. Further, let C be the class of all complete lattice ordered groups.

Lemma 3.2. *Let $H \in C$ and let $f(H)$ be a complete retract of H. Then $f(H) \in C$.*

Proof. Let us apply the notation from (B). Since H is complete, each direct factor of H is complete; hence A_1 and A_2 are complete. Thus in view of 2.12, $f(H)$ is complete as well. □

Corollary 3.3. *C is a complete retract variety.*

Let us denote by R_c the collection of all complete retract varieties; next, let R_c^0 be the collection of all elements X of R_c with $X \subseteq C$. Both the collections R_c and R_c^0 will be considered to be partially ordered by inclusion. Let $\mathcal{G}$ be the class of all abelian lattice ordered groups. Hence $\overline{0}$ and $\mathcal{G}$ is the least element or the greatest element of R_c, respectively.

When considering a class X of lattice ordered groups we always assume that X is closed with respect to isomorphisms.

Theorem 3.4. *Let $\emptyset \neq X \subseteq C$. Then the following conditions are equivalent:*
(i) *X is a complete retract variety.*
(ii) *X is closed under direct products and direct factors.*

P r o o f. Since each direct factor of a lattice ordered group is a complete retract, we infer that (i) $\Rightarrow$ (ii) holds. Let (ii) be valid and let $H \in X$. Let $f(H)$ be a complete retract of H. We apply the notation from (B); then A_1 and A_2 are direct factors of H. Thus in view of 2.12, $f(H) \in X$. Hence (i) holds. □

E x a m p l e s 3.5. For each infinite cardinal α let $X(\alpha)$ be the class of all complete lattice ordered groups which are α-distributive. In view of 3.4, $X(\alpha)$ is a complete retract variety.

Next, for each infinite cardinal α let $Y(\alpha)$ be the class of all complete lattice ordered groups H which have the following property: if $\{h_i\}_{i\in I}$ is a disjoint subset of H with card $I \leqslant \alpha$, then $\bigvee_{i\in I} h_i$ does exist in H. Again, in view of 3.4, the class $Y(\alpha)$ is a retract variety; if α and β are infinite cardinals with $\alpha < \beta$, then $Y(\alpha) \subset Y(\beta)$. Hence the mapping $\alpha \rightarrow Y(\alpha)$ is an order-preserving injection of the class of all infinite cardinals into the collection R_c^0.

Let $\emptyset \neq X \subseteq \mathcal{G}$; we denote by

r_cX—the class of all complete retracts of elements of X;

ΦX—the class of all internal direct factors of elements of X;

πX—the class of all direct product of elements of X.

Lemma 3.6. *Let $\emptyset \neq X \subseteq \mathcal{G}$. Then*

(i) *$\pi r_c X$ is a complete retract variety;*

(ii) *if $Y \in R_c$ and $X \subseteq Y$, then $\pi r_c X \subseteq Y$;*

(iii) *if $X \subseteq C$, then $\pi\Phi X = \pi r_c X$.*

P r o o f. The assertion (i) is a consequence of 2.10; (ii) is obvious. Finally, (iii) follows from 3.4. □

In view of 3.6 (i) and (ii), the complete retract variety $\pi r_c X$ will be said to be generated by the class X.

Let I be a nonempty class and for each $i \in I$ let X_i be an element of R_c. Put $Y = \bigcap_{i\in I} X_i$ and $Z = \pi \bigcup_{i\in I} X_i$.

Lemma 3.7. *Let X_i, Y and Z be as above. Then*

(i) *$Y, Z \in R_c$;*

(ii) *$Y = \bigwedge_{i\in I} X_i$ in R_c;*

(iii) *$Z = \bigvee_{i\in I} X_i$ in R_c.*

P r o o f. The relation $Y \in R_c$ is obvious. Hence (ii) is valid. Since $r_c X_i = X_i$ for each $i \in I$, we have $Z \in R_c$. Then clearly (iii) holds. □

In view of 3.7, the terminology of the lattice theory will be applied for R_c.

Theorem 3.8. *R_c is a Brouwer lattice.*

Proof. In view of 3.7, R_c is a complete lattice. The remaining part of the proof can be done analogously as in [7], Lemma 3.5 (where the lattice of all retract varieties was dealt with). □

Since R_c^0 is the interval $[0, C]$ of R_c, we obtain

Corollary 3.9. *R_c^0 is a Brouwer lattice.*

The notion of a large lexicographic factor of a linearly ordered group was introduced in [6]. It is obvious that if G is a large lexicographic factor of a linearly ordered group H, then G is a complete retract of H. Hence from 3.4 in [7] and from 3.6 we infer:

Proposition 3.10. *Let $\emptyset \neq X$ be a class of linearly ordered groups. Then the complete retract variety generated by X coincides with the retract variety generated by X.*

Corollary 3.11. *Let $\emptyset \neq X$ be a class of linearly ordered groups and let $T(X)$ be the retract variety generated by X. If $T(X)$ is an atom in R, then $T(X)$ is an atom in R_c.*

Thus 5.3 in [7] yields

Proposition 3.12. *There is an injective mapping of the class of all infinite cardinals into the collection of all atoms of the lattice R_c.*

By the same method as in [7], 5.6–5.8 we can verify that R_c has no dual atom; similarly, R_c^0 has no dual atom.

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