Ján Jakubík
Complete retract mappings of a complete lattice ordered group


Persistent URL: http://dml.cz/dmlcz/128396

Terms of use:

© Institute of Mathematics AS CR, 1993

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://dml.cz
COMPLETE RETRACT MAPPINGS
OF A COMPLETE LATTICE ORDERED GROUP

Ján Jakubík,* Košice

(Received November 18, 1991)

Retracts of partially ordered sets were studied in [2]–[5]. Retracts of abelian lattice ordered groups were dealt with in [6]. In [7], retract varieties of abelian lattice ordered groups were investigated.

An endomorphism \( f \) of a lattice ordered group \( H \) is said to be a complete retract (cf. [6]) if it satisfies the following conditions:

(i) \( f(f(h)) = h \) for each \( h \in H \);
(ii) if \( \{h_i\}_{i \in I} \subseteq H \), \( h \in H \), \( h = \bigvee h_i \) holds in \( H \), then \( f(h) = \bigvee_{i \in I} f(h_i) \), and dually.

The following results concern the relations between complete retract mappings and direct decompositions of a lattice ordered group \( H \).

(A) Let \( H \) be an internal direct product of its \( l \)-subgroups \( A_1 \), \( A_2 \) and \( A_3 \). For \( h \in H \) let \( h_i \) \( (i \in \{1, 2, 3\}) \) be the component of \( h \) in \( A_i \). Assume that \( \varphi \) is a complete isomorphism of \( A_2 \) into \( A_3 \). For each \( h \in H \) put

\[
(1) \quad f(h) = h_1 + h_2 + \varphi(h_2).
\]

Then \( f \) is a complete retract mapping of \( H \).

(B) Let \( H \) be a complete lattice ordered group and let \( f \) be a complete retract mapping of \( H \). Then there are convex \( l \)-subgroups \( A_1 \), \( A_2 \) and \( A_3 \) in \( H \) and a complete isomorphism \( \varphi \) of \( A_1 \) into \( A_2 \) such that

(i) \( H \) is an internal direct product of its \( l \)-subgroups \( A_i \) \( (i = 1, 2, 3) \);
(ii) for each \( h \in H \) the relation (1) is valid (where \( h_1 \) and \( h_2 \) are the components of \( h \) in \( A_1 \) and in \( A_2 \), respectively).

The assertion (A) is easy to verify; (B) will be proved below. Next, (B) will be applied to obtain a sharpening of a result established in [6]. Let us remark that if \( H \)

* Supported by grant GA SAV 362/91
fails to be complete, then the assertions of (B) need not be valid for \( H \) (cf. Example 1.3 below). Further, the notion of a complete retract variety will be introduced and the lattice of all complete retract varieties will be investigated.

1. Preliminaries

An endomorphism \( f \) of a lattice ordered group \( H \) will be said to be a retract mapping of \( H \), if \( f(f(x)) = f(x) \) for each \( x \in H \). If \( f \) is a retract mapping of \( H \), then the \( l \)-subgroup \( f(H) \) of \( H \) is called a retract of \( H \) (cf. [6]).

If \( f \) is a retract mapping of \( H \) and if, moreover, \( f \) is a complete endomorphism (i.e., if the above condition (ii) is satisfied), then \( f \) is said to be a complete retract of \( H \).

The following example shows that a retract mapping need not be complete.

Example 1.1. Let \( R \) be the set of all reals and \( R^+ = \{ t \in R : t \geq 0 \} \). Let \( H \) be the set of all real functions which are defined and continuous on \( R^+ \). The lattice operations and the operation + in \( H \) are defined point-wise; hence \( H \) is an abelian lattice ordered group. For each \( x \in H \) let \( f(x) \in H \) be such that \( f(x)(t) = x(0) \) for each \( t \in R^+ \). Then \( f \) is a retract mapping of \( H \).

Let \( N \) be the set of all positive integers. For each \( n \in N \) let \( x_n \) be an element of \( H \) such that \( x_n(0) = 0 \), \( x_n(t) = 1 \) for each \( t \in R^+ \) with \( t \leq \frac{1}{n} \), and \( x_n \) is linear on the interval \( [0, \frac{1}{n}] \) of \( R^+ \). Next, let \( x \in H \) be such that \( x(t) = 1 \) for each \( t \in R^+ \), and let \( 0 \) be the neutral element of \( H \). Then we have \( f(x_n) = 0 \) for each \( n \in N \) and

\[
\bigvee_{n \in N} x_n = x,
\]

hence

\[
\bigvee_{n \in N} f(x_n) = 0 \neq x = f(x).
\]

Thus \( f \) fails to be a complete retract mapping.

The question whether each retract mapping of a complete lattice ordered group must be complete remains open.

An isomorphism \( \varphi \) of a lattice ordered group \( H_1 \) into a lattice ordered group \( H_2 \) is said to be complete if, whenever \( \{ h_i \}_{i \in I} \subseteq H_1 \), \( h \in H_1 \) and \( \bigvee_{i \in I} h_i = h \) in \( H_1 \), then \( \varphi(h) = \bigvee_{i \in I} \varphi(h_i) \), and dually.

The following example shows that an isomorphism need not be complete.
Example 1.2. Let $R$ be the additive group of all reals with the natural linear order. Put $H_1 = R$, $H_2 = R \circ R$, where $\circ$ denotes the operation of lexicographic product. For each $x \in H_1$ we put $\varphi(x) = (x, 0)$. Then $\varphi$ is an isomorphism of $H_1$ into $H_2$. Let $x_n = \frac{1}{n}$ for each positive integer $n$. We have $\bigwedge_{n \in \mathbb{N}} x_n = 0$, but $\bigwedge_{n \in \mathbb{N}} \varphi(x_n)$ does not exist in $H_2$. Hence the isomorphism $\varphi$ fails to be complete.

If $H$ is not complete, then the assertion of (B) need not hold.

Example 1.3. Put $H = R \circ R$ and for each $(x, y) \in H$ let $f((x, y)) = (x, 0)$. Then $f$ is a complete retract mapping and there exist no direct factors $A_1, A_2$ and $A_3$ of $H$ with the properties as in (B).

The notion of an internal direct decomposition of a lattice ordered group will be applied in the same sense as in [6] or [7].

2. DIRECT DECOMPOSITION CORRESPONDING TO A COMPLETE RETRACT MAPPING

In this section we assume that $H$ is a complete lattice ordered group and that $f$ is a complete retract mapping of $H$.

Denote $f^{-1}(0) = H_1$.

**Lemma 2.1.** $H_1$ is a closed $l$-ideal of $H$.

**Proof.** Because $f$ is an endomorphism of $H$, we obtain that $H_1$ is an $l$-ideal of $H$. Next, since $f$ is complete, $H_1$ is closed in $H$.

For each $X \subseteq H$ we put

$$X^\perp = \{h \in H : |h| \wedge |x| = 0 \text{ for each } x \in X\};$$

$X^\perp$ is a polar of $H$.

**Lemma 2.2.** $H_1$ is a polar of $H$.

**Proof.** This is a consequence of 2.1 and of the completeness of $H$ (cf., e.g., Birkhoff [1], Chap. XIII, Theorem 27).

Put $K = H_1^\perp$. Since each complete lattice ordered group is strongly projectable, we have

$$(1) \quad H = (i)K \times H_1.$$
In view of (1), each \( h \in H \) can be written as

\[
h = k + h_1 \quad (k \in K, \, h_1 \in H_1)
\]

and then \( f(h) = f(k) \). Hence for determining \( f \), it suffices to know all the values \( f(k) \) for \( k \) running over \( K \).

Put \( K_1 = \{ k \in K : f(k) \in K \} \).

**Lemma 2.3.** Let \( k \in K \). The following conditions are equivalent:

(i) \( k \in K_1 \);

(ii) \( f(k) = k \).

**Proof.** Clearly (ii) \( \Rightarrow \) (i). Let (i) hold. Since \( K \) is an \( l \)-subgroup of \( H \), we have \( f(k) - k \in K \). On the other hand,

\[
f(f(k) - k) = f(f(k)) - f(k) = 0.
\]

whence \( f(k) - k \in H_1 \). Therefore \( f(k) - k = 0 \). \( \square \)

**Lemma 2.4.** \( K_1 \) is a closed \( l \)-ideal of \( H \).

**Proof.** From the definition of \( K_1 \) it follows immediately that \( K_1 \) is an \( l \)-subgroup of \( H \). Let \( h \in H, k_1 \in K_1, 0 \leq h \leq k_1 \). Then \( 0 = f(0) \leq f(h) \leq f(k_1) = k_1 \). Since \( K \) is convex in \( H \), we obtain \( f(h) \in K \) and thus \( h \in K_1 \). Therefore \( K_1 \) is a convex \( l \)-subgroup of \( H \). Let \( k_i (i \in I) \) be elements of \( K_1 \) and let \( \bigvee_{i \in I} k_i = h \). In view of 2.1 we have \( h \in K \). Next, according to 2.3, \( f(k_i) = k_i \) for each \( i \in I \), whence

\[
f\left( \bigvee_{i \in I} k_i \right) = \bigvee_{i \in I} f(k_i) = \bigvee_{i \in I} k_i.
\]

Therefore \( h \in K_1 \). The dual condition can be verified analogously. Hence \( K_1 \) is closed in \( H \). \( \square \)

In view of 2.4, \( K_1 \) is an internal direct factor of \( H \). Moreover, since \( K_1 \subseteq K \), (1) implies that \( K_1 \) is an internal direct factor of \( K \). Thus there is an \( l \)-ideal \( K_2 \) in \( K \) such that

\[
K = (i)K_1 \times K_2.
\]

Each \( k \in K \) can be written as \( k = k_1 + k_2 \) with \( k_1 \in K_1, \, k_2 \in K_2 \). Then

\[
f(k) = f(k_1) + f(k_2) = k_1 + f(k_2).
\]

312
Hence for determining \( f \) it suffices to know the values \( f(k_2) \), where \( k_2 \) runs over \( K_2 \). For each \( k \in K_2 \) we put
\[
\varphi(k) = f(k)(H_1).
\]

**Lemma 2.5.** \( \varphi \) is a complete isomorphism of \( K_2 \) into \( H_1 \).

**Proof.** Since \( f \) is an endomorphism of \( H \) and since the mapping \( \psi: h \rightarrow h(H_1) \) is a homomorphism of \( H \) onto \( H_1 \) we infer that \( \varphi \) is a homomorphism of \( K_2 \) into \( H_1 \). Next, both \( f \) and \( \psi \) are complete and thus \( \varphi \) is complete as well.

Let \( k \in K_2 \) and assume that \( \varphi(k) = 0 \). Thus \( f(k)(H_1) = 0 \) and so in view of (1), \( f(k) \in K_2 \). Hence \( k \in K_1 \). Therefore according to (2) we have \( k = 0 \). We have obtained that \( \varphi^{-1}(0) = \{0\} \), hence \( \varphi \) is an isomorphism of \( K_2 \) into \( H_1 \).

**Lemma 2.6.** Let \( k \in K_2 \). Then \( f(k)(K_1) = 0 \).

**Proof.** By way of contradiction, suppose that \( f(k)(K_1) = k_1 \neq 0 \). Then \( f(|k|)(K_1) = |k_1| > 0 \). According to 2.3, \( f(|k_1|) = |k_1| \). In view of (2) we have \( |k_1| \land |k| = 0 \), hence \( f(|k_1|) \land f(|k|) = 0 \). Thus
\[
0 = (f(|k_1|) \land f(|k|))(K_1) = f(|k_1|)(K_1) \land f(|k|)(K_1)
\]
\[
= |k_1||(K_1) \land f(|k|)(K_1) = |k_1| \land f(|k|)(K_1) = |k_1|,
\]
which is a contradiction.

**Lemma 2.7.** Let \( k \in K_2 \). Then \( f(k)(K_2) = k \).

**Proof.** Denote \( f(k) - k = x \). Then \( f(x) = 0 \), whence \( x \in H_1 \). From \( f(k) = k + x \) and from (1) we obtain \( f(k)(K) = (k + x)(K) = k(K) + x(K) = k(K) = k \). Next, in view of (2),
\[
f(k)(K_2) = f(k)(K_1)(K_2) = k(K_2) = k.
\]

**Lemma 2.8.** For each \( k \in K_2 \) we have \( f(k) = k + \varphi(k) \).

**Proof.** In view of (1) and (2) the relation
\[
f(k) = f(k)(K_1) + f(k)(K_2) + f(k)(H_1)
\]
is valid. Hence in view of 2.6 and 2.7 we have \( f(k) = k + \varphi(k) \).
Proof of Theorem (B).

Denote $A = K, A_2 = K^2, A_3 = H$. For $h \in H$ let $h_i$ be the component of $h$ in $A_i$ ($i = 1, 2, 3$). In view of (1) and (2) we have $h = h_1 + h_2 + h_3$, whence $f(h) = f(h_1) + f(h_2) + f(h_3)$. According to 2.3, $f(h_1) = h_1$. Next, $\varphi$ is a complete isomorphism of $A_2$ into $A_1$ and in view of 2.8, $f(h_2) = h_2 + \varphi(h_2)$. Therefore

$$f(h) = h_1 + h_2 + \varphi(h_2).$$

The following result sharpens Theorem 4.13 of [6].

Proposition 2.9. Let $H$ be a complete lattice ordered group, $H = (i)A \times B$, and let $f$ be a complete retract mapping of $H$. Then there exist internal decompositions

$$A = (i)A_1 \times A_2, \quad B = (i)B_1 \times B_2,$$

$$A_1 = (i)A_{11} \times A_{12} \times A_{13}, \quad B_1 = (i)B_{11} \times B_{12} \times B_{13}$$

and complete isomorphisms $\varphi_1: A_2 \to A_{13}, \varphi_2: B_2 \to B_{13}, \varphi_1: A_2 \to A_1, \varphi_2: B_2 \to B_1$ such that

(i) for each $a_2 \in A_2$ and each $b_2 \in B_2$ the relations

$$f_2(\varphi_1(a_2)) = 0 = f_1(\varphi_2(a_2)), \quad f_1(\psi_1(b_2)) = 0 = f_2(\psi_2(h_2))$$

are valid;

(ii) for each $h \in H$ the relation

$$f(h) = f_1(h(A_1)) + \varphi_2(h(A_2)) + h(A_2) + \varphi_1(h(A_1))$$

$$+ f_2(h(B_1)) + \psi_2(h(B_2)) + h(B_2) + \psi_1(h(B_2))$$

holds, where $f_1(h_1) = h_1(A_{11}) + f_1(A_{12}) + \varphi_{10}(h_1(A_{12}))$ and $f_2(h_2) = h_2(B_{11}) + h_2(B_{12}) + \varphi_{20}(h_2(B_{12}))$ for each $h_1 \in A_1$ and each $h_2 \in B_1$.

Proof. The assertion follows from Theorem 4.13 in [6] and from (B).

Proposition 2.10. Let $H$ be a lattice ordered group, $H = (i)\prod_{i \in I} H_i$. Let $f$ be a complete retract mapping of $H$. Then

(i) $f(H) = (i)\prod_{i \in I} f(H_i)$;

(ii) for each $i \in I$, the mapping $\varphi_i(h_i) = f(h_i)(H_i)$ is a complete retract mapping of $H_i$ and the lattice ordered group $f(H_i)$ is isomorphic to $f(H_i)(H_i)$.

Proof. The assertion (i) was proved in [7], Theorem 2.4. Let $i \in I$. Since $f$ is a complete endomorphism of $H$ and since the mapping $\psi(h) = h(H_i)$ is a complete endomorphism of $H$ as well, we infer that $\varphi_i$ is a complete endomorphism of $H_i$. The remaining part of (ii) was proved in [6] (Lemmas 2.6 and 2.7).
Corollary 2.11. Let $H$ be as in 2.10. Then each complete retracts of $H$ is isomorphic to a direct product of complete retract of the factors $H_i$ ($i \in I$).

Next, 2.10 and (B) yield:

Theorem 2.12. Let $H$ be a complete lattice ordered group and let $f$ be a complete retract mapping of $H$. Let $A_1$, $A_2$ and $A_3$ be as in (B). Then the complete retract $f(H)$ of $H$ is isomorphic to the direct product $A_1 \times A_2 \times A_2$.

3. Complete retract varieties

A retract variety of abelian lattice ordered groups is defined to be a nonempty class of abelian lattice ordered groups which is closed under direct product and retracts. (Cf. [7].)

Definition 3.1. A nonempty class of abelian lattice ordered groups is said to be a complete retract variety if it is closed under direct products and complete retracts.

Let $\mathcal{O}$ be the class of all one-element lattice ordered groups. Further, let $C$ be the class of all complete lattice ordered groups.

Lemma 3.2. Let $H \in C$ and let $f(H)$ be a complete retract of $H$. Then $f(H) \in C$.

Proof. Let us apply the notation from (B). Since $H$ is complete, each direct factor of $H$ is complete; hence $A_1$ and $A_2$ are complete. Thus in view of 2.12, $f(H)$ is complete as well.

Corollary 3.3. $C$ is a complete retract variety.

Let us denote by $R_c$ the collection of all complete retract varieties; next, let $R^0_c$ be the collection of all elements $X$ of $R_c$ with $X \subseteq C$. Both the collections $R_c$ and $R^0_c$ will be considered to be partially ordered by inclusion. Let $\mathcal{G}$ be the class of all abelian lattice ordered groups. Hence $\mathcal{O}$ and $\mathcal{G}$ is the least element or the greatest element of $R_c$, respectively.

When considering a class $X$ of lattice ordered groups we always assume that $X$ is closed with respect to isomorphisms.

Theorem 3.4. Let $\emptyset \neq X \subseteq C$. Then the following conditions are equivalent:

(i) $X$ is a complete retract variety.
(ii) $X$ is closed under direct products and direct factors.
Proof. Since each direct factor of a lattice ordered group is a complete retract, we infer that (i) \(\Rightarrow\) (ii) holds. Let (ii) be valid and let \(H \in X\). Let \(f(H)\) be a complete retract of \(H\). We apply the notation from (B); then \(A_1\) and \(A_2\) are direct factors of \(H\). Thus in view of 2.12, \(f(H) \in X\). Hence (i) holds.

Examples 3.5. For each infinite cardinal \(\alpha\) let \(X(\alpha)\) be the class of all complete lattice ordered groups which are \(\alpha\)-distributive. In view of 3.4, \(X(\alpha)\) is a complete retract variety.

Next, for each infinite cardinal \(\alpha\) let \(Y(\alpha)\) be the class of all complete lattice ordered groups \(H\) which have the following property: if \(\{h_i\}_{i \in I}\) is a disjoint subset of \(H\) with \(\text{card } I \leq \alpha\), then \(\bigvee_{i \in I} h_i\) does exist in \(H\). Again, in view of 3.4, the class \(Y(\alpha)\) is a retract variety; if \(\alpha\) and \(\beta\) are infinite cardinals with \(\alpha < \beta\), then \(Y(\alpha) \subseteq Y(\beta)\). Hence the mapping \(\alpha \rightarrow Y(\alpha)\) is an order-preserving injection of the class of all infinite cardinals into the collection \(R_c^0\).

Let \(\emptyset \neq X \subseteq \mathfrak{X}\); we denote by

- \(r_c X\) — the class of all complete retracts of elements of \(X\);
- \(\Phi X\) — the class of all internal direct factors of elements of \(X\);
- \(\pi X\) — the class of all direct product of elements of \(X\).

Lemma 3.6. Let \(\emptyset \neq X \subseteq \mathfrak{X}\). Then

(i) \(\pi r_c X\) is a complete retract variety;
(ii) if \(Y \in R_c\) and \(X \subseteq Y\), then \(\pi r_c X \subseteq Y\);
(iii) if \(X \subseteq C\), then \(\pi \Phi X = \pi r_c X\).

Proof. The assertion (i) is a consequence of 2.10; (ii) is obvious. Finally, (iii) follows from 3.4.

In view of 3.6 (i) and (ii), the complete retract variety \(\pi r_c X\) will be said to be generated by the class \(X\).

Let \(I\) be a nonempty class and for each \(i \in I\) let \(X_i\) be an element of \(R_c\). Put \(Y = \bigcap_{i \in I} X_i\) and \(Z = \pi \bigcup_{i \in I} X_i\).

Lemma 3.7. Let \(X_i, Y\) and \(Z\) be as above. Then

(i) \(Y, Z \in R_c\);
(ii) \(Y = \bigwedge_{i \in I} X_i\) in \(R_c\);
(iii) \(Z = \bigvee_{i \in I} X_i\) in \(R_c\).

Proof. The relation \(Y \in R_c\) is obvious. Hence (ii) is valid. Since \(r_c X_i = X_i\) for each \(i \in I\), we have \(Z \in R_c\). Then clearly (iii) holds.
In view of 3.7, the terminology of the lattice theory will be applied for \( R_c \).

**Theorem 3.8.** \( R_c \) is a Brouwer lattice.

**Proof.** In view of 3.7, \( R_c \) is a complete lattice. The remaining part of the proof can be done analogously as in [7], Lemma 3.5 (where the lattice of all retract varieties was dealt with).

Since \( R^0_c \) is the interval \([0, C]\) of \( R_c \), we obtain

**Corollary 3.9.** \( R^0_c \) is a Brouwer lattice.

The notion of a large lexicographic factor of a linearly ordered group was introduced in [6]. It is obvious that if \( G \) is a large lexicographic factor of a linearly ordered group \( H \), then \( G \) is a complete retract of \( H \). Hence from 3.4 in [7] and from 3.6 we infer:

**Proposition 3.10.** Let \( \emptyset \neq X \) be a class of linearly ordered groups. Then the complete retract variety generated by \( X \) coincides with the retract variety generated by \( X \).

**Corollary 3.11.** Let \( \emptyset \neq X \) be a class of linearly ordered groups and let \( T(X) \) be the retract variety generated by \( X \). If \( T(X) \) is an atom in \( R \), then \( T(X) \) is an atom in \( R_c \).

Thus 5.3 in [7] yields

**Proposition 3.12.** There is an injective mapping of the class of all infinite cardinals into the collection of all atoms of the lattice \( R_c \).

By the same method as in [7], 5.6–5.8 we can verify that \( R_c \) has no dual atom; similarly, \( R^0_c \) has no dual atom.

**References**


Author's address: Matematický ústav SAV, dislokované pracovisko, Grešákova 6, 040 01 Košice, Slovakia.