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ON SOME DISCONTINUOUS FIXED POINT MAPPINGS IN CONVEX METRIC SPACES

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1. INTRODUCTION

Let X be a Banach space and C a closed convex subset of X. M. Greguš [7] proved the following result

Theorem 1 (Greguš [7]). *Let $T\colon C \to C$ be a mapping satisfying*

$$\text{(G)} \qquad \|Tx - Ty\| \leqslant a\|x - y\| + p\|Tx - x\| + p\|Ty - y\|$$

for all x, $y \in C$, where $0 < a < 1$, $p \leqslant 0$ and $a + 2p = 1$. Then T has a unique fixed point.

Many theorems which are closely related to Greguš's Theorem have appeared in recent years ([2]–[9]).

The purpose of this note is to define and to investigate a class of mappings (not necessarily continuous) which are defined on metric spaces and satisfy the following contractive condition.

$$\text{(1)} \quad d(Tx,Ty) \leqslant ad(x,y) + (1-a)\max\{d(x,Tx), d(y,Ty), b[d(x,Ty) + d(y,Tx)]\}$$

where $0 < a < 1$ and $b \leqslant \frac{1}{2} - \frac{1-a^2}{10+6a^2}$. We shall prove a fixed point theorem which is a double generalization of the above theorem of Greguš. Firstly the nonexpansive nature of the mapping is generalized, and secondly the underlying space is freed to a non-linear situation. An example is constructed to show that our Theorem is a genuine generalization of the theorems of Greguš [7] and Li [8].

We recall the following definition of a convex metric space.

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Definition 1. (Takahashi [10]). Let X be a metric space and $I = [0, 1]$ the closed unit interval. A continuous mapping $W \colon X \times X \times I \to X$ is said to be a *convex structure* on X if for all $x, y \in X$ and $\lambda \in I$, $d[u, W(x, y, \lambda)] \leqslant \lambda d(u, x) + (1 - \lambda) d(u, y)$ for all $u \in X$. X together with a convex structure is called a *convex metric space*. A subset $K \subseteq X$ is convex, if $W(x, y, \lambda) \in K$ whenever $x, y \in K$ and $\lambda \in I$.

Clearly a Banach space, or any convex subset of it, is a convex metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda) y$.

2. Main result

Now we are in a position to state our main result.

Theorem 2. *Let K be a closed convex subset of a complete convex metric space X and $T \colon K \to K$ a mapping satisfying (1) for all $x, y \in K$. Then T has a unique fixed point.*

Proof. Let $x = x_0$ be an arbitrary point and consider the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$; $n = 0, 1, 2, \ldots$ From (1) we have

$$\begin{aligned} d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n) &\leqslant ad(x_{n-1}, x_n) \\ &\quad + (1 - a) \max \left\{ d(x_{n-1}, x_n), d(x_n, Tx_n), b\left[d(x_{n-1}, Tx_n)\right] \right\}. \end{aligned}$$

Since $b < \frac{1}{2}$, by simple calculation we obtain

$$d(x_n, Tx_n) \leqslant d(x, Tx) \quad (n = 1, 2, \ldots). \tag{2}$$

We shall show that

$$d(Tx_k, T^3 x_k) \leqslant \left[1 + a + \frac{(1 - a)^3}{3}\right] d(x, Tx) \tag{3}$$

for some $k \geqslant 0$. Using (1), (2) and the triangle inequality we have $d(Tx_n, T^3 x_n) \leqslant ad(Tx_{n-1}, T^3 x_{n-1}) + (1 - a) \max \left\{ d(x, Tx), b\left[2d(x, Tx) + d(Tx_{n-1}, T^3 x_{n-1})\right] \right\}$. If for some $n = k$

$$d(Tx_k, T^3 x_k) \leqslant ad(Tx_{k-1}, T^3 x_{k-1}) + (1 - a) d(x, Tx), \tag{4}$$

then (3) holds, since by (2) and the triangle inequality we have

$$d(Tx_{k-1}, T^3 x_{k-1}) = d(x_k, Tx_{k+1}) \leqslant d(x_k, Tx_k) + d(x_{k+1}, Tx_{k+1}) \leqslant 2d(x, Tx).$$

Suppose that (4) does not hold. Then

$$d(Tx_n, T^3x_n) \leqslant ad(Tx_{n-1}, T^3x_{n-1}) + (1-a)b\big[2d(x,Tx) + d(Tx_{n-1}, T^3x_{n-1})\big]$$

holds for each $n = 1, 2, \ldots$ Hence

$$d(Tx_n, T^3x_n) \leqslant \big[1 - (1-a)(1-b)\big]d(Tx_{n-1}, T^3x_{n-1}) + 2(1-a)b\, d(x,Tx).$$

Hence, by induction we get

$$d(Tx_n, T^3x_n) \leqslant h^n d(Tx, T^3x) + \frac{2b}{1-b} d(x,Tx), \tag{5}$$

where $h = 1 - (1-a)(1-b) < 1$. Since $d(Tx, T^3x) \leqslant 2d(x,Tx)$ and by hypothesis for b we have

$$\frac{2b}{1-b} \leqslant \frac{4(1+a^2)}{3+a^2} = 1 + a + \frac{(1-a)^3}{3+a^2} < 1 + a + \frac{(1-a)^3}{3},$$

we may choose k such that $2h^k + \frac{2b}{1-b} \leqslant 1 + a + \frac{(1-a)^3}{3}$. For such k, (5) implies (3). Therefore, we proved (3).

Let k be such that (3) holds and put $y = x_k$. Since K is convex, by Definition 1 $W(T^2y, T^3y, \frac{1}{2}) = z \in K$. Then, using Definition 1 and (2) and (3), we have

$$d(z, T^2y) \leqslant \frac{1}{2} d(T^2y, T^3y) \leqslant \frac{1}{2} d(x,Tx),$$

$$d(z, T^3y) \leqslant \frac{1}{2} d(T^2y, T^3y) \leqslant \frac{1}{2} d(x,Tx),$$

$$d(z, Ty) \leqslant \frac{1}{2}\big[d(Ty, T^2y) + d(Ty, T^3y)\big] \leqslant \frac{7 + 3a^2 - a^3}{6} \cdot d(x,Tx), \tag{6}$$

$$d(z, Tz) \leqslant \frac{1}{2}\big[d(Tz, T^2y) + d(Tz, T^3y)\big]. \tag{7}$$

Now we shall show that there is a real number λ, such that

$$d(z,Tz) \leqslant \lambda \cdot d(x,Tx); \quad 0 \leqslant \lambda < 1. \tag{8}$$

Put

$$M = M(x,z) = \max\big\{d(x,Tx), d(z,Tz)\big\}$$

and suppose $M > 0$. Using (1) again, from (2) we have

$$d(Tz, T^3y) \leqslant \frac{a}{2} \cdot M + (1-a)\max\left\{M, b\left[\frac{1}{2} \cdot M + d(Tz, T^2y)\right]\right\}, \tag{9}$$

$$d(Tz, T^2y) \leqslant a\,\frac{7 + 3a^2 - a^3}{6}\,M$$

$$(10) \qquad\qquad + (1 - a)\max\left\{M, b\left[\frac{1}{2}\,M + d(Tz, Ty)\right]\right\}.$$

Since $2b < 1$, using the triangle inequality we get

$$b\left[\frac{1}{2}\,M + d(Tz, T^2y)\right] \leqslant b\left[\frac{1}{2}\,M + d(z, Tz) + d(z, T^2y)\right] \leqslant 2b\,M < M.$$

Therefore, from (9) we have

$$d(Tz, T^3y) \leqslant \frac{a}{2}\,M + (1 - a)\,M = \left(1 - \frac{a}{2}\right)\,M.$$

Using the triangle inequality and (2) we get

$$d(Tz, Ty) \leqslant d(Tz, T^2y) + d(T^2y, Ty) \leqslant M + d(Tz, T^2y).$$

Therefore, from (10) we have

$$(12) \quad d(Tz, T^2y) \leqslant a\frac{7 + 3a^2 - a^3}{6}\,M + (1 - a)\max\left\{M, b\left[\frac{3}{2}\,M + d(Tz, T^2y)\right]\right\}.$$

Case I. Suppose that from (12) we have

$$(12') \qquad d(Tz, T^2y) \leqslant a\frac{7 + 3a^2 - a^3}{6}\,M + (1 - a)\,M = \left[1 + a\frac{1 + 3a^2 - a^3}{6}\right]M.$$

Then by (7), (11) and (12′) we have

$$d(z, Tz) \leqslant \frac{1}{2}\left[1 - \frac{a}{2} + 1 + a\frac{1 + 3a^2 - a^3}{6}\right]M$$

$$(13) \qquad\qquad = \left[1 - a\frac{2 - 3a^2 + a^3}{12}\right]\max\{d(x, Tx), d(z, Tz)\}.$$

Since $0 < a < 1$ implies $\lambda_1 = 1 - a\frac{2 - 3a^2 + a^3}{12} < 1$, from (13) we have

$$(14) \qquad d(z, Tz) \leqslant \lambda_1\,d(x, Tx); \quad 0 < \lambda_1 < 1.$$

Case II. Assume now that (12) implies

$$d(Tz, T^2y) \leqslant a\frac{7 + 3a^2 - a^3}{6}\,M + (1 - a)b\left[\frac{3}{2}\,M + d(Tz, T^2y)\right].$$

Then, as by hypothesis $b \leqslant \frac{2(1+a^2)}{5+3a^2}$, we have

$$\begin{aligned}
&[5 + 3a^2 - (1 - a)2(1 + a^2)]\, d(Tz, T^2y) \\
&\qquad \leqslant a(7 + 3a^2 - a^3)\frac{5 + 3a^2}{6} M + 3(1 - a)(1 + a^2) M \\
&\qquad < a(6 + 5a^2 + a^3) M + 3(1 - a)(1 + a^2) M.
\end{aligned}$$

After some computations we get

$$(12'') \quad (3+2a+a^2+2a^3)\, d(Tz, T^2y) \leqslant \left[\left(1 + \frac{a}{2}\right)(3+2a+a^2+2a^3) - \frac{1}{2} a(1-a)^2 \right] M.$$

Now from (7), (11) and (12″) we have

$$(15) \qquad d(z, Tz) \leqslant \lambda_2 \max\{d(x, Tx), d(z, Tz)\},$$

where $\lambda_2 = 1 - a\frac{(1-a)^2}{12+8a+4a^2+8a^3}$. Since $\lambda_2 < 1$, from (15) we have

$$(16) \qquad d(z, Tz) \leqslant \lambda_2\, d(x, Tx); \quad \lambda_2 < 1.$$

Put $\lambda = \max\{\lambda_1, \lambda_2\}$. Then from (14) and (16) we conclude that (8) holds in any case.

Now it is easy to prove that (8) implies

$$(17) \qquad \inf\{d(x, Tx) \colon x \in K\} = m = 0.$$

Indeed, since $\lambda^{-\frac{1}{2}} > 1$, there exists some $x' \in K$ such that $d(x', Tx') \leqslant \lambda^{-\frac{1}{2}} m$. Then, as above, there is $z' = z'(x') \in K$ such that (8) holds, i.e. such that $d(z', Tz') \leqslant \lambda d(x', Tx')$. Then we have $m \leqslant d(z', Tz') \leqslant \lambda(\lambda^{-\frac{1}{2}} m) = \lambda^{\frac{1}{2}} m$. Hence $m = 0$.

Now we shall show that

$$(18) \qquad \max\{d(Tx, Ty), d(x, y)\} \leqslant \left[2 + \frac{5 + 3a^2}{(1 - a)^2}\right] \max\{d(x, Tx), d(y, Ty)\}.$$

Let $M = \max\{d(x, Tx), d(y, Ty)\}$. Then from (1) and the triangle inequality we have

$$\begin{aligned}
d(Tx, Ty) &\leqslant a[d(x, Tx) + d(Tx, Ty) + d(Ty, y)] \\
&\qquad + (1 - a) \max\{M, b[d(x, Tx) + 2d(Tx, Ty) + d(y, Ty)]\} \\
&\leqslant 2aM + ad(Tx, Ty) + (1 - a)[M + 2bd(Tx, Ty)].
\end{aligned}$$

Hence, as $b \leqslant \frac{2+2a^2}{5+3a^2}$, we get

$$d(Tx,Ty) \leqslant \left[\frac{5+3a^2}{(1-a)^2}\right] M.$$

This and $d(x,y) \leqslant 2M + d(Tx,Ty)$ imply (18).

Now by (17) we can choose a sequence $\{x_n\}$ in K such that $d(x_n,Tx_n) \leqslant \frac{1}{n}$ $(n = 1, 2, \ldots)$. It follows from (18) that

$$\max\left\{d(Tx_m,Tx_n), d(x_m,x_n)\right\} \leqslant \frac{2+\frac{5+3a^2}{1-a^2}}{m} \qquad \text{for } 1 \leqslant m < n.$$

Therefore, both $\{x_n\}$ and $\{Tx_n\}$ are Cauchy sequences, and moreover they have a common limit, say u. By (1)

$$d(Tx_n,Tu) \leqslant ad(x_n,u)+(1-a)\max\left\{d(x_n,Tx_n), d(u,Tu), b\left[d(x_n,Tu)+d(u,Tx_n)\right]\right\}.$$

Taking the limit as $n \to \infty$ in this inequality, we get

$$d(u,Tu) \leqslant (1-a)d(u,Tu),$$

which implies that $Tu = u$. The uniqueness of a fixed point follows from (1). □

Remark 1. If in Theorem 2 $b = \frac{1}{2}$, then T may be without fixed points, as the following simple example shows it.

Example 1. Let K be the set of real numbers with usual metric and let $T\colon K \to K$ be defined by $Tx = x+1$. Then for any $0 < a < 1$

$$d(Tx,Ty) = d(x,y) = a\,d(x,y) + (1-a)\,\frac{1}{2}\left[d(x,y) - 1 + d(x,y) + 1\right] = d(x,y).$$

Remark 2. If $b = 0$, we obtain the result which was established by Fisher [5]. That result also appears in [2], [4], [6], and [9] as a corollary of common fixed point theorems.

Theorem 3. *Let K be as in Theorem 2 and $T\colon K \to K$ a mapping satisfying*

(19) $$d(Tx,Ty) \leqslant ad(x,y) + b\left[d(x,Ty)+d(y,Tx)\right] + c\max\left\{d(x,Tx), d(y,Ty)\right\}$$

for all $x,y \in K$, where $0 \leqslant a < 1$, $b \geqslant 0$, $c \geqslant 0$, $a+b > 0$ and

(20) $$a + \frac{5+a^2}{2+a^2}b + c \leqslant 1$$

Then T has a unique fixed point.

Proof. We have

$$ad(x,y)+b\frac{5+a^2}{2+a^2}\cdot\frac{2+a^2}{5+a^2}\big[d(x,Ty)+d(y,Tx)\big]+c\max\{d(x,Tx),d(y,Ty)\}$$
$$\leqslant ad(x,y)+\left[\frac{5+a^2}{2+a^2}b+c\right]\max\left\{d(x,Tx),d(y,Ty),\frac{2+a^2}{5+a^2}\left[d(x,Ty)+d(y,Tx)\right]\right\}$$
$$\leqslant ad(x,y)+(1-a)\max\left\{d(x,Tx),d(y,Ty),\frac{2+a^2}{5+a^2}\left[d(x,Ty)+d(y,Tx)\right]\right\}.$$

Therefore, (19) and (20) imply (1) with $0\leqslant a<1$ and

$$b=\frac{2+a^2}{5+a^2}<\frac{1}{2}-\frac{1-a^2}{10+6a^2}$$

and so we can apply Theorem 2 in the case $a>0$.

If $a=0$, then $a+b>0$ implies $b>0$, and then from (20) we have

$$0<2b+c\leqslant 1-\frac{b}{2}<1.$$

So in the case $a=0$ Theorem 3 reduces to a special case of Theorem 2.5 of [1]. □

Corollary 2 (Li [8]). *Let K be a closed convex subset of a convex metric space X and $T\colon K\longrightarrow K$ a mapping satisfying*

$$d(Tx,Ty)\leqslant ad(x,y)+b\big[d(x,Ty)+d(y,Tx)\big]+c\big[d(x,Tx)+d(y,Ty)\big]$$

for all $x,y\in K$, where $0\leqslant a<1$, $b\geqslant 0$, $c\geqslant 0$, $a+b>0$ and

$$a+3b+2c\leqslant 1. \tag{22}$$

if X has the property that every decreasing sequence of non-empty closed subsets of X with diameters tending to zero has non-empty intersection, then T has a unique fixed point in K.

Proof. It is clear that the inequalities (19) and (20) are more general than corresponding inequalities (21) and (22). Since the property of X, stated in Corollary 2 is equivalent to the completeness of X, we see that all assumptions of Theorem 3 are satisfied. □

The following simple example shows that our Theorems 2 and 3 are genuine generalizations of the Theorems of Greguš [7] and Li [8].

Example 2. Let $K = [-4, 4]$ be a closed convex subset of the real line and $T: K \to K$ a mapping defined by

$$Tx = \frac{x}{6}, \text{ if } -2 \leqslant x \leqslant 4; \quad Tx = 4, \text{ if } -4 \leqslant x < -2.$$

It is clear that if $x, y \in [-2, 4]$ or $x, y \in [-4, -2)$, then $d(Tx, Ty) \leqslant \frac{1}{6} d(x, y)$. Let now $x \in [-2, 4]$ and $y \in [-4, -2)$. Then we have

$$d(Tx, Ty) \leqslant 4 + \frac{1}{3} < \frac{5}{6} 6 \leqslant \frac{5}{6} d(y, Ty) \leqslant \frac{5}{6} \max\{d(y, Ty), d(x, Tx)\}.$$

Therefore, T satisfies the condition (19) with $a = \frac{1}{6}$, $c = \frac{5}{6}$ and $b = 0$, and the condition (1) with $a = \frac{1}{6}$ and any $0 \leqslant b < \frac{1}{2} - \frac{5}{38}$. Since K is compact, hence complete, all assumptions of Theorems 2 and 3 are satisfied and $u = 0$ is the unique fixed point of T. But T does not satisfy (21) with $a + 3b + 2c \leqslant 1$, and hence (G), since for all $x \in [-1, 0]$ and $y \in [-3, -2)$ we have

$$\begin{aligned} d(Tx, Ty) &\geqslant 4 > 4 - \frac{1}{12} = \max\left\{3, \frac{1}{3}(5 + 3), \frac{1}{2}\left(\frac{5}{6} + 7\right)\right\} \\ &\geqslant \max\left\{d(x, y), \frac{1}{3}\left[d(x, Ty) + d(y, Tx)\right], \frac{1}{2}\left[d(x, Tx) + d(y, Ty)\right]\right\} \\ &\geqslant ad(x, y) + b\left[d(x, Ty) + d(y, Tx)\right] + c\left[d(x, Tx) + d(y, Ty)\right] \end{aligned}$$

for any $a, b, c \geqslant 0$ with $a + 3b + 2c \leqslant 1$.

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