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DIMENSION AND ATTACHED PRIMES OF AN ARTINIAN MODULE

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§1 INTRODUCTION

One way to deal with an Artinian module $A$ is to follow the produce of Sharp [13]; namely write $A = \oplus \Gamma_M(A)$, where the sum is over all maximal ideals $M$, and $\Gamma_M(A) = \bigcup_{n \geq 1} (0 : A M^n)$ is zero for almost all $M$. The summand $\Gamma_M(A)$ is naturally a module over the completion $\hat{R}_M$ of $R_M$, and by Sharp's extension [13, (3.6)] of a theorem of Heinzer and Lanz [2, Proposition 4.3], $\hat{R}_M/\text{Ann}_{\hat{R}_M} \Gamma_M(A)$ is Noetherian. Thus Matlis duality [5] allows results for Artinian modules to be obtained from corresponding results for Noetherian modules over complete local rings.

Before I state my aim in this paper I give some useful concepts which help me to explain it. The phase "$(R, m)$ is quasi-local" will mean that $R$ has $m$ as its unique maximal ideal; by "$R$ is local" we shall mean that $R$ is both quasi-local and Noetherian.

I begin by recalling the notion of dimension due to Roberts [10] extended in the manner dual to that employed by Rentschler and Gabriel [10] to extend Krull dimension.

(1.1) Definition. [10]. The Krull dimension, $K - \dim_R A$ of an (Artinian) $R$-module $A$ is defined inductively as follows:

$$K - \dim_R A = -1 \iff A = 0.$$ 

Let $r \geq 0$ be an integer. Assume that those (Artinian) modules which have Krull dimension less than $r$ have been specified. If $A$ is an (Artinian) $R$-module which does not fall into this class then $A$ is said to have Krull dimension $r$ if, whenever $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$ is an ascending chain of submodules of $A$, then there exists an integer $n$ such that $K - \dim_R (A_{m+1}/A_m) < r$ for all $m \geq n$. If $A$ is an $R$-module
such that, for all integers \( r \geq 1 \) \( A \) does not have Krull dimension \( r \), we say \( A \) has infinite Krull dimension.

In [10] Roberts also defined the classical Krull dimension, denoted \( \text{cl} \ K - \dim_R(A) \), as to be \(-1\) if \( A = 0 \) and the least number of generators of a proper ideal \( I \) of \( R \) such that \((0 :_A I)\) has finite length if \( A \neq 0 \). Moreover he proved, [10, Theorem 6] that \( K - \dim_R(A) = \text{cl} \ K - \dim_R(A) \).

Now it is time to recall basic facts concerning a secondary module and a secondary representation of a module, for the details see [3], [5] and [8]. An \( R \)-module \( A \neq 0 \) is called secondary if for each \( r \in R \) the multiplication by \( r \) on \( A \) is either surjective or nilpotent. Then \( \text{Rad} \ Ann_R A = P \) is a prime ideal and \( A \) is called \( P \)-secondary. We say that \( A \) has a secondary representation if there is a finite number of secondary submodules \( A_1, \ldots, A_k \) such that \( A = A_1 + \ldots + A_k \). One may assume that the prime ideals \( P_i = \text{Rad} \ Ann_R A_i, \ i = 1, \ldots, k \) are all distinct and, by omitting redundant summands, that the representation is minimal. Then the set of prime ideals \( \{P_1, \ldots, P_k\} \) depends only on \( A \) and not on the minimal representation, see [5, (2.2)]). This set is called the set of attached prime ideals \( \text{Att}_R A \). Any Artinian \( R \)-module \( A \) has a secondary representation, see [5, (5.2)]).

Now I am able to explain my aim in this paper. The aim is to investigate whether there is any relation between \( K \)-dim of an Artinian module \( A \) and the attached primes of \( A \). Indeed, Professor R.Y. Sharp asked the author whether the following is always true or not: Let \((R, m)\) be a quasi-local ring. Let \( A \) be an Artinian \( R \)-module with \( K - \dim_R(A) = d \). Let \( A = A_1 + \ldots + A_r \) be a minimal secondary representation with \( \sqrt{0 : A_i} = P_i \) for \( i = 1, \ldots, r \). Then \( \text{Att}_R A = \{P_1, \ldots, P_r\} \). The question asked was whether if \( K - \dim_R(A_i) = d \) for some \( i, \ 1 \leq i \leq r \), then is \( P_i \) minimal among the primes corresponding to \( A \). Since the answer to the question is positive for complete local rings, I will start by trying to reduce the problem to this case and use Matlis duality. In case \( R \) is quasi-local complete ring we obtain a positive answer to the question by using Sharp’s method, [13]. But we are unable to say “yes” for the general case as will be shown in 2.7 we produce an example of a local domain for which the question has a negative answer.

\section*{\textsection 2 The results}

I begin with some useful concepts which will be helpful for me to prove what I have been aiming.

\textbf{(2.1) Lemma.} Let \( A \) be an Artinian module over the quasi-local ring \((R, m)\). Then \( K - \dim_{\hat{R}} A = K - \dim_R A \) where \( \hat{R} \) is the \( m \)-adic completion of \( R \).
Proof. This is immediate from [13,(1.11) and (1.12)]. □

Since the following lemma is very clear, I omit its proof.

(2.2) Lemma. Let $A$ be non-zero Artinian module over the quasi-local ring $(R, m)$. Regard $A$ as a module over the $m$-adic completion $\hat{R}$ of $R$ in the manner indicated in [13, (1.11)]. Let $R' = \hat{R}/0 :\hat{R} A$. Then $K - \dim_R(A) = K - \dim_{R'}(A)$.

(2.3) Proposition. Let $(R, m)$ be a complete local ring. Let $A$ be an Artinian module over $R$. Then

$$K - \dim_R(A) = \dim_R(A)$$

where "dim" refers to the classical Krull dimension.

Proof. Let $D$ denote Matlis duality which is available over $R$. Then

$$\dim_R(A) = \dim_R(R/(0 :_R D(A))) \quad \text{(by [13,(2.7)])}$$

$$= \dim_R(D(A))$$

$$= \text{the least number of } x_1, \ldots, x_n \in m \text{ such that}$$

the length of $D(A)/(x_1, \ldots, x_n)D(A)$ is finite

by [14,(15.24)].

Now the result follows from [13,(2.1)(v) and (2.4)(ii)]. □

Now I am able to give a positive answer to the question, which is mentioned in the introductory section, over a complete local ring.

(2.4) Theorem. Let $A$ be an Artinian module over a complete local ring $(R, m)$. Let $A = A_1 + \ldots + A_r$ be a minimal secondary representation for $A$ with $\sqrt{0 :_R A_i} = P_i$ for $i = 1, \ldots, r$. Let $K - \dim_R(A) = d$. Then if $K - \dim_R(A_i) = d$, for some $i$, $1 \leq i \leq r$, then $P_i$ is minimal among the attached primes of $A$.

Proof. Suppose that $K - \dim_R(A_i) = d$, for some $i$, $1 \leq i \leq r$, but $P_i$ is not a minimal member of $\text{Att}_R(A)$. Then there exists $P_j \in \text{Att}_R(A)$ such that $P_j \subset P_i$.

Let $A_j$ be the corresponding secondary component of $A$. Now by using the same argument as in (2.3) we get

$$K - \dim_R(A_j) > K - \dim_R(A_i).$$

This is a contradiction to the maximality of $K - \dim_R A$. This completes the proof. □
Let $A$ be a non-zero Artinian module over the quasi-local ring $(R, m)$. Let $\hat{R}$ and $R'$ be as in (2.2). Then by [13, (1.10)],

$$\text{Att}_{\hat{R}}(A) = \{ P/0 : \hat{R} A : P \in \text{Att}_{\hat{R}}(A) \}.$$ 

There is one more very nice relation between $R$ and $R'$. This is given in the following proposition without proof.

(2.5) **Proposition.** Let $R$ and $R'$ be as above. Let $A = A_1 + \ldots + A_r$ be a minimal secondary representation of $A$ as $R'$-module. Then $P/0 : \hat{R} A$ is a minimal prime ideal of $R'$ if and only if $P$ is a minimal member of $\text{Att}_{\hat{R}}(A)$.

(2.6) **Theorem.** Let $A$ be a non-zero Artinian module over a quasi-local complete ring $(R, m)$. Let $A = A_1 + \ldots + A_r$ be a minimal secondary representation of $A$ with $\sqrt{0 : R A_i} = P_i$ for $i = 1, \ldots, r$. $K = \dim_R(A) = d$. If $K = \dim_R(A_i) = d$, for some $i$, $1 \leq i \leq r$, then $P_i$ is minimal among the attached primes of $A$.

**Proof.** Let $R' = R/0 : \hat{R} A$. Then $R'$ is a complete local ring. Let $K = \dim_R(A_i) = d$, for some $i$, $1 \leq i \leq r$. Then by (2.1) and (2.2), $K = \dim_R(A_i) = K = \dim_{R'}(A_i) = d$. Now the result follows from (2.4) and (2.5).

Now it is time to produce an example of a local domain for which the question has a negative answer. Before doing this we want to note that we will need to use “contraction” of ideals under a ring homomorphism. For the details the reader is referred to [14, (2.41)].

(2.7) **Counter Example.** First note that in [1] Ferrand and Raynaud showed that there exists a 2-dimensional local domain $(R, m)$ such that $\hat{R}$, the $m$-adic completion of $R$, has exactly one embedded associated prime $a$.

It is known that if $P \in \text{Spec}(\hat{R})$, then depth $\hat{R}_P \geq \text{depth } R_{P^c}$ where “c” refers to the natural ring homomorphism $R_{P^c} \longrightarrow \hat{R}_P$ (see [7, p.181] or [11, (2.7)]). On the other hand, depth $R_\alpha = 0$ and depth $R_{\hat{m}^c} = \text{depth } R_m = \text{depth } R \geq 1$ (because $R$ is domain). And $\alpha \neq \hat{m}$ so $\text{ht}_{\hat{R}} \alpha = 1$. Also depth $R_{\alpha^c} = 0$. Therefore $\alpha^c = 0$. Now let us choose another prime ideal $P$ of $\hat{R}$ such that $\text{ht}_{\hat{R}} P = 1$ and $P^c$ contains a non-zero element $r$ of $R$ where “c” refers to the natural ring homomorphism $R \longrightarrow \hat{R}$.

Let $E$ be the injective hull of $\hat{R}/m$, i.e. $E = \hat{E}_{\hat{R}}(\hat{R}/m)$. Then $E$ is Artinian $\hat{R}$-module by [15,(4.30)]. Now by [12,(2.1)] we get the following Artinian $\hat{R}$-modules: $S = \text{Hom}_{\hat{R}}(\hat{R}/P, E)$, $P$-secondary with annihilator $P$ and $K = \dim_{\hat{R}} S = 1$, and $T = \text{Hom}_{\hat{R}}(\hat{R}/\alpha, E)$, $\alpha$-secondary with annihilator $\alpha$ and $K = \dim_{\hat{R}} T = 1$. Let $A = S \oplus T$. Then $A$ is Artinian with $K = \dim_{\hat{R}} A = 1$ by [10, Proposition 1]. Over
$R$, $A = S \oplus T$ is Artinian and that is still reduced secondary representation for $A$ and $K - \dim_R A = 1$. $K - \dim_R S = K - \dim_R A = 1$. By [13,(1.12)] $P^c \in \operatorname{Att}_R(A)$. But $P^c$ is not a minimal prime of $A$.

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References


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