Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 43 (1993), No. 2, 265–269

Persistent URL: http://dml.cz/dmlcz/128398

Czechoslovak Mathematical Journal, 43 (118) 1993, Praha

DIMENSION AND ATTACHED PRIMES OF AN ARTINIAN MODULE

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(Received September 11, 1991)

§1 INTRODUCTION

One way to deal with an Artinian module A is to follow the produce of Sharp [13]; namely write $A = \oplus\Gamma_M(A)$, where the sum is over all maximal ideals M, and $\Gamma_M(A) = \bigcup\limits_{n\geqslant 1}(0 :_A M^n)$ is zero for almost all M. The summand $\Gamma_M(A)$ is naturally a module over the completion $\hat{R}_M$ of R_M, and by Sharp's extension [13, (3.6)] of a theorem of Heinzer and Lanz [2, Propositoin 4.3], $\hat{R}_M / \operatorname{Ann}_{\hat{R}_M} \Gamma_M(A)$ is Noetherian. Thus Matlis duality [5] allows results for Artinian modules to be obtained from corresponding results for Noetherian modules over complete local rings.

Before I state my aim in this paper I give some useful concepts which help me to explain it. The phase "(R, m) is quasi-local" will mean that R has m as its unique maximal ideal; by "R is local" we shall mean that R is both quasi-local and Noetherian.

I begin by recalling the notion of dimension due to Roberts [10] extended in the manner dual to that employed by Rentschler and Gabriel [10] to extend Krull dimension.

(1.1) Definition. [10]. The Krull dimension, $K - \dim_R A$ of an (Artinian) R-module A is defined inductively as follows:

$$K - \dim_R A = -1 \Leftrightarrow A = 0.$$

Let $r \geqslant 0$ be an integer. Assume that those (Artinian) modules which have Krull dimension less than r have been specified. If A is an (Artinian) R-module which does not fall into this class then A is said to have Krull dimension r if, whenever $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$ is an ascending chain of submodules of A, then there exists an integer n such that $K - \dim_R(A_{m+1}/A_m) < r$ for all $m \geqslant n$. If A is an R-module such that, for all integers $r \geqslant 1$ A does not have Krull dimension r, we say A has infinite Krull dimension.

In [10] Roberts also defined the classical Krull dimension, denoted $\mathrm{cl}\, K-\dim_R(A)$, as to be -1 if $A = 0$ and the least number of generators of a proper ideal I of R such that $(0 :_A I)$ has finite length if $A \neq 0$. Moreover he proved, [10, Theorem 6] that $K - \dim_R(A) = \mathrm{cl}\, K - \dim_R(A)$.

Now it is time to recall basic facts concerning a secondary module and a secondary representation of a module, for the details see [3], [5] and [8]. An R-module $A \neq 0$ is called secondary if for each $r \in R$ the multiplication by r on A is either surjective or nilpotent. Then $\mathrm{Rad}\ \mathrm{Ann}_R A = P$ is a prime ideal and A is called P-secondary. We say that A has a secondary representation if there is a finite number of secondary submodules $A_1, \ldots, A_k$ such that $A = A_1 + \ldots + A_k$. One may assume that the prime ideals $P_i = \mathrm{Rad}\,\mathrm{Ann}_R A_i$, $i = 1, \ldots, k$ are all distinct and, by ommiting redundand summands, that the representation is minimal. Then the set of prime ideals $\{P_1, \ldots, P_k\}$ depends only on A and not on the minimal representation, see [5, (2.2)]. This set is called the set of attached prime ideals $\mathrm{Att}_R A$. Any Artinian R-module A has a secondary representation, see [5, (5.2)].

Now I am able to explain my aim in this paper. The aim is to investigate whether there is any relation between K-dim of an Artinian module A and the attached primes of A. Indeed, Professor R.Y. Sharp asked the author whether the following is always true or not: Let (R, m) be a quasi-local ring. Let A be an Artinian R-module with $K - \dim_R(A) = d$. Let $A = A_1 + \ldots + A_r$ be a minimal secondary representation with $\sqrt{0 : A_i} = P_i$ for $i = 1, \ldots, r$. Then $\mathrm{Att}_R A = \{P_1, \ldots, P_r\}$. The question asked was whether if $K - \dim_R(A_i) = d$ for some i, $1 \leqslant i \leqslant r$, then is P_i minimal among the primes corresponding to A. Since the answer to the question is positive for complete local rings, I will start by trying to reduce the problem to this case and use Matlis duality. In case R is quasi-local complete ring we obtain a positive answer to the question by using Sharp's method, [13]. But we are unable to say "yes" for the general case as will be shown in 2.7 we produce an example of a local domain for which the question has a negative answer.

§2 The results

I begin with some useful concepts which will be helpful for me to prove what I have been aiming.

(2.1) Lemma. *Let A be an Artinian module over the quasi-local ring (R, m). Then $K - \dim_{\hat{R}} A = K - \dim_R A$ where $\hat{R}$ is the m-adic completion of R.*

Proof. This is immediate from [13,(1.11) and (1.12)]. □

Since the following lemma is very clear, I omit its proof.

(2.2) Lemma. *Let A be non-zero Artinian module over the quasi-local ring (R, m). Regard A as a module over the m-adic completion $\hat{R}$ of R in the manner indicated in [13, (1.11)]. Let $R' = \hat{R}/0 :_{\hat{R}} A$. Then $K-\dim_R(A) = K-\dim_{R'}(A)$.*

(2.3) Proposition. *Let (R, m) be a complete local ring. Let A be an Artinian module over R. Then*

$$K-\dim_R(A) = \dim_R(A)$$

where "dim" refers to the classical Krull dimension.

Proof. Let D denote Matlis duality which is available over R. Then

$$\begin{aligned}\dim_R(A) &= \dim_R(R/(0 :_R D(A))) \quad \text{(by [13,(2.7)])}\\ &= \dim_R(D(A))\\ &= \text{the least number of } x_1, \ldots, x_n \in m \text{ such that}\\ &\quad\ \text{the length of } D(A)/(x_1, \ldots, x_n)D(A) \text{ is finite}\\ &\quad\ \text{by [14,(15.24)]).}\end{aligned}$$

Now the result follows from [13,(2.1)(v) and (2.4)(ii)]. □

Now I am able to give a positive answer to the question, which is mentioned in the introductory section, over a complete local ring.

(2.4) Theorem. *Let A be an Artinian module over a complete local ring (R, m). Let $A = A_1 + \ldots + A_r$ be a minimal secondary representation for A with $\sqrt{0 :_R A_i} = P_i$ for $i = 1, \ldots, r$. Let $K-\dim_R(A) = d$. Then if $K-\dim_R(A_i) = d$, for some i, $1 \leqslant i \leqslant r$, then P_i is minimal among the attached primes of A.*

Proof. Suppose that $K-\dim_R(A_i) = d$, for some i, $1 \leqslant i \leqslant r$, but P_i is not a minimal member of $\mathrm{Att}_R(A)$. Then there exists $P_j \in \mathrm{Att}_R(A)$ such that $P_j \subset P_i$.

Let A_j be the corresponding secondary component of A. Now by using the same argument as in (2.3) we get

$$K-\dim_R(A_j) > K-\dim_R(A_i).$$

This is a contradiction to the maximality of $K-\dim_R A$. This completes the proof. □

Let A be a non-zero Artinian module over the quasi-local ring (R, m). Let $\hat{R}$ and R' be as in (2.2). Then by [13, (1.10)],

$$\operatorname{Att}_{R'}(A) = \{P/0 :_{\hat{R}} A : P \in \operatorname{Att}_{\hat{R}}(A)\}.$$

There is one more very nice relation between $\hat{R}$ and R'. This is given in the following proposition without proof.

(2.5) Proposition. *Let $\hat{R}$ and R' be as above. Let $A = A_1 + \ldots + A_r$ be a minimal secondary representation of A as R'-module. Then $P/0 :_{\hat{R}} A$ is a minimal prime ideal of R' if and only if P is a minimal member of $\operatorname{Att}_{\hat{R}}(A)$.*

(2.6) Theorem. *Let A be a non-zero Artinian module over a quasi-local complete ring (R, m). Let $A = A_1 + \ldots + A_r$ be a minimal secondary representation of A with $\sqrt{0 :_R A_i} = P_i$ for $i = 1, \ldots, r$. $K - \dim_R(A) = d$. If $K - \dim_R(A_i) = d$, for some i, $1 \leqslant i \leqslant r$, then P_i is minimal among the attached primes of A.*

Proof. Let $R' = R/0 :_R A$. Then R' is a complete local ring. Let $K - \dim_R(A_i) = d$, for some i, $1 \leqslant i \leqslant r$. Then by (2.1) and (2.2), $K - \dim_R(A_i) = K - \dim_{R'}(A_i) = d$. Now the result follows from (2.4) and (2.5). □

Now it is time to produce an example of a local domain for which the question has a negative answer. Before doing this we want to note that we will need to use "contraction" of ideals under a ring homomorphism. For the details the reader is referred to [14, (2.41)].

(2.7) Counter Example. First note that in [1] Ferrand and Raynaud showed that there exists a 2-dimensional local domain (R, m) such that $\hat{R}$, the m-adic completion of R, has exactly one embedded associated prime α.

It is known that if $P \in \operatorname{Spec}(\hat{R})$, then depth $\hat{R}_P \geqslant$ depth R_{P^c} where "c" refers to the natural ring homomorphism $R_{P^c} \longrightarrow \hat{R}_P$ (see [7, p. 181] or [11, (2.7)]. On the other hand, depth $\hat{R}_\alpha = 0$ and depth $R_{\hat{m}^c} =$ depth $R_m =$ depth $R \geqslant 1$ (because R is domain). And $\alpha \neq \hat{m}$ so $\operatorname{ht}_{\hat{R}} \alpha = 1$. Also depth $R_{\alpha^c} = 0$. Therefore $\alpha^c = 0$. Now let us choose another prime ideal P of $\hat{R}$ such that $\operatorname{ht}_{\hat{R}} P = 1$ and P^c contains a non-zero element r of R where "c" refers to the natural ring homomorphism $R \longrightarrow \hat{R}$.

Let E be the injective hull of $\hat{R}/\hat{m}$, i.e. $E = E_{\hat{R}}(\hat{R}/\hat{m})$. Then E is Artinian $\hat{R}$-module by [15,(4.30)]. Now by [12,(2.1)] we get the following Artinian $\hat{R}$-modules: $S = \operatorname{Hom}_{\hat{R}}(\hat{R}/P, E)$, P-secondary with annihilator P and $K - \dim_{\hat{R}} S = 1$, and $T = \operatorname{Hom}_{\hat{R}}(\hat{R}/\alpha, E)$, α-secondary with annihilator α and $K - \dim_{\hat{R}} T = 1$. Let $A = S \oplus T$. Then A is Artinian with $K - \dim_{\hat{R}} A = 1$ by [10, Proposition 1]. Over

R, $A = S \oplus T$ is Artinian and that is still reduced secondary representation for A and $K - \dim_R A = 1$. $K - \dim_R S = K - \dim_R A = 1$. By [13,(1.12)] $P^c \in \mathrm{Att}_R(A)$. But P^c is not a minimal prime of A. □

I am extremely grateful to Professor R.Y. Sharp, The University of Sheffield-England, for his valuable advice and suggestions on this work. I also wish to thank Hacettepe University, Turkey, for financial support.

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