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CYCLIC EXTENSIONS OF THE MEDVEDEV ORDERED GROUPS

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SECTION ONE: INTRODUCTION AND BACKGROUND

An \( \ell \)-variety is a class of lattice-ordered groups defined by a set of equations. Any \( \ell \)-group law can be expressed in the form "\( w(\overline{x}) = e \)," where \( w(\overline{x}) \) is an element of the free \( \ell \)-group on a countable set \( X \) of free generators; \( w(\overline{x}) \) has, then, a (nonunique) standard form \( w(\overline{x}) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \prod_{k=1}^{p} x_{ijk}^{\varepsilon_{ijk}} \), where \( \varepsilon_{ijk} = \pm 1 \) and \( x_{ijk} \in X \cup \{e\} \). An \( \ell \)-group \( G \) satisfies "\( w(\overline{x}) = e \)" if for any mapping of \( X \) into \( G \), letting \( g_{ijk} \) be the image of \( x_{ijk} \), \( w(\overline{g}) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \prod_{k=1}^{p} g_{ijk}^{\varepsilon_{ijk}} = e \).

Weinberg [W] showed that the \( \ell \)-variety \( \mathcal{A} \) of abelian \( \ell \)-groups is the smallest nontrivial \( \ell \)-variety. Since \( \mathcal{A} \) is finitely based, any \( \ell \)-variety properly containing \( \mathcal{A} \) contains an \( \ell \)-variety minimal with respect to properly containing \( \mathcal{A} \), called a cover of \( \mathcal{A} \). Scrimer [Sc] proved the existence of countably infinitely many solvable covers of \( \mathcal{A} \), one for each prime integer \( p \), known now as the Scrimer covers \( \mathcal{J}_{p} \). These \( \ell \)-varieties were generated by \( \ell \)-groups that are not representable: i.e., not representable as subdirect products of totally ordered groups. Subsequently, Gurchenkov–Kopytov [GK], Reilly [Rl], and Darnel [D] showed that the Scrimer covers were the only nonrepresentable covers of \( \mathcal{A} \). Medvedev [M] proved the existence of three solvable representable covers of \( \mathcal{A} \). Of these, one, herein denoted \( \mathcal{M}^{0} \), is generated by the free nil-2 group on two generators \( a \) and \( b \), where if \( c = [a, b] \), any element is of the (unique) form \( a^{k}b^{m}c^{n} \), ordered lexicographically from the left by \( k, m, \) and \( n \).

Describing the other Medvedev covers requires more explanation. Let \( A \) and \( B \) be totally ordered groups. The restricted wreath product \( A \wr B \) can be ordered in

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two ways. For \( g \in A \wr B, \ g = (\hat{f}, b), \) where \( \hat{f} : B \to A \) has finite support. Define 
\( (\hat{f}, b) > (e, e) \) if \( b > e \) or \( b = e \) and \( \hat{f}(b_0) > e \) for \( b_0 = \max(\text{supp}(\hat{f})) \). This gives
an \( o \)-group denoted by \( A \wr B. \) \( A \wr B \) is defined analogously, except \( (\hat{f}, b) > (e, e) \)
if \( b > e \) or if \( b = e \) and \( \hat{f}(b_1) > e \) for \( b_1 = \min(\text{supp}(\hat{f})) \). One Medvedev cover,
denoted \( \mathcal{A}^+ \), is generated by \( Z \wr Z \), where \( Z \) is the group of integers with the usual
order, and the order by \( Z \wr Z \).

Two other representable covers of \( \mathcal{A} \) are known at this time, based on orderings
of the free group of rank two. Bergman [B] and Kopytov [K] independently proved
the existence of one of these, \( \mathcal{A}^+ \), and by reversing the order, Kopytov [K] obtained
the other, \( \mathcal{A}^- \).

The lattice of all \( \ell \)-varieties is distributive, and thus if \( \mathcal{U} \) and \( \mathcal{V} \) are covers of \( \mathcal{A} \),
then \( \mathcal{U} \vee \mathcal{V} \) covers both \( \mathcal{U} \) and \( \mathcal{V} \). Gurchenkov [Gu1] proved that all \( \ell \)-varieties have
covers. Presently much more is known about the \( \ell \)-varieties containing the Scrimger
covers that those containing the Medvedev covers. Indeed, Holland and Reilly [HR]
and Gurchenkov [Gu2] independently described all \( \ell \)-metabelian \( \ell \)-varieties whose
intersections with the \( \ell \)-variety \( \mathcal{A} \) of representable \( \ell \)-groups is the abelian \( \ell \)-variety
\( \mathcal{A} \). (An \( \ell \)-group \( G \) is \( \ell \)-metabelian if there exists a convex \( \ell \)-subgroup \( A \trianglelefteq G \) such
that \( A \) and \( G/A \) are abelian. In this case, as indeed for all \( \ell \)-groups, there exists a
unique largest abelian convex \( \ell \)-subgroup [H] called the abelian radical and which is
denoted by \( \mathcal{A}^+(G) \). \( G \) is thus \( \ell \)-metabelian if and only if \( G/\mathcal{A}^+(G) \in \mathcal{A} \).)

Darnel [D2] showed that \( \mathcal{A}^+ \) is contained in the \( \ell \)-variety \( \mathcal{T}^+ \) generated by
all \( \ell \)-metabelian \( o \)-groups \( G \) having the positive infinite shifting property: for any 
\( e < h \in \mathcal{A}(G) \) and \( e < g \in G \setminus \mathcal{A}(G) \), \( g^{-1}h \gg h \) and produced laws for \( \mathcal{T}^+ \). From
these laws, results due to Huss [Hu] and Reilly [R2] that \( \mathcal{A}^+ \) is not closed with
respect to lex extensions by the ordered group of integers will be proven in Section
Three.

A convex \( \ell \)-subgroup \( C \) of an \( \ell \)-group \( G \) is a sublattice and a subgroup with the
property that if \( e \leq x \leq c \in C \), then \( x \in C \). The lattice order of \( G \) induces a lattice
order on the set of right cosets \( \mathcal{R}_G(C) \) of \( C \) by \( Cx \vee Cy = C(x \vee y) \). A convex
\( \ell \)-subgroup is prime if \( x \wedge y \in C \) implies \( x \in C \) or \( y \in C \); this is equivalent to \( \mathcal{R}_G(C) \)
being totally ordered. Note a convex \( \ell \)-subgroup \( P \) is prime if and only if for convex
\( \ell \)-subgroups \( A \) and \( B \), \( P \subseteq A \) and \( P \subseteq B \) implies \( P \subseteq A \cap B \).

**Section Two: O-groups of \( \ell \)-Varieties Generated by Ordered Groups**

The \( \ell \)-variety generated by a class \( \mathcal{C} \) of \( \ell \)-groups is the class of all \( \ell \)-groups \( G \)
that are \( \ell \)-homomorphic images of \( \ell \)-subgroups of cardinal products of numbers of \( \mathcal{C} \).
Thus if \( \mathcal{C} \) is a collection of \( o \)-groups, any \( o \)-group \( G \) in \( \ell \text{-Var}(\mathcal{C}) \) is the \( \ell \)-homomorphic
image of an \( \ell \)-subgroup \( S \) of a cardinal product \( \Pi_{\lambda}G_{\lambda} \) of \( o \)-groups \( \{ G_{\lambda} \} \subseteq \mathcal{C} \) by a prime subgroup \( P \) of \( S \). While \( P \) is always the intersection of a prime \( Q \) of \( \Pi_{\lambda}G_{\lambda} \) with \( S \), in general \( S \) need not be contained in the normalizer \( N_{\Pi}(Q) \) of \( Q \) in \( \Pi_{\lambda}G_{\lambda} \) and so we can not in general substitute \( SQ \) for \( S \) and \( Q \) for \( P \).

**Proposition 2.1.** Let \( G \) be a representable \( \ell \)-group, \( S \) be an \( \ell \)-group of \( G \), and \( P \) be a prime of \( S \). If \( P \triangleleft S \), then there exists a prime subgroup \( Q \) of \( G \) such that \( P = S \cap Q \), \( S \) is contained in the normalizer \( N_{G}(Q) \), and \( Q \triangleleft SQ \). In this case, \( SQ/Q \cong S/P \).

**Proof.** We start by replicating from [C] that there is always a prime subgroup \( Q \) of \( G \) such that \( S \cap Q = P \).

Let \( \mathcal{A} = \{ C \in \mathcal{C}(G) : C \cap S = P \} \). \( \mathcal{A} \neq \emptyset \) as the convex \( \ell \)-subgroup of \( G \) generated by \( P \) is in \( \mathcal{A} \). Let \( \mathcal{C} \) be a chain in \( \mathcal{A} \). Then \( P \subseteq S \cap \bigcup \mathcal{C} \). Suppose there exists \( g \in (\bigcup \mathcal{C} \cap S) \setminus P \). Then \( g \in C \in \mathcal{C} \) and \( g \in S \), implying \( g \in C \cap S = P \). Thus \( \mathcal{A} \) has maximal elements; let \( Q \) be one.

Now suppose \( e = a \wedge b \) where \( a, b \in G \setminus Q \). Then \( Q \subseteq T = G(Q, a) \) and \( Q \subseteq R = G(Q, b) \). So \( P \subseteq S \cap T \) and \( P \subseteq S \cap R \). But since \( P \) is prime in \( S \), \( P \subseteq S \cap T \cap (S \cap R) = S \cap (T \cap R) = S \cap Q = P \), an obvious contradiction. So \( Q \) is prime in \( G \).

Now suppose that \( G \) is representable and that \( P \triangleleft S \). Suppose by way of contradiction that \( S \not\subseteq N_{G}(Q) \). Choose \( s \in S \) such that \( s^{-1}Qs \neq Q \). Since \( G \) is representable, either \( Q \subseteq s^{-1}Qs \) or \( Q \subseteq sQs^{-1} \).

Assume \( Q \subseteq s^{-1}Qs \). Then \( P \subseteq S \cap s^{-1}Qs = s^{-1}(S \cap Q)s = s^{-1}Ps = P \) since \( P \triangleleft S \). So \( S \subseteq N_{G}(Q) \) which is an \( \ell \)-subgroup of \( G \) ([BKW, p. 77], [Mc], [R1], and [Dl]). The rest follows from the Second Isomorphism Theorem. \( \square \)

Proposition 2.1 allows us to consider only \( o \)-groups arising from quotients of \( \ell \)-subgroups of a cardinal product of \( o \)-groups by prime subgroups of that product.

We can further specify those primes.

Since the intersection of prime subgroups is prime, every prime subgroup contains a minimal prime subgroup. Conrad and McAlister [CM] showed that the minimal prime subgroups \( M \) of a cardinal product \( \Pi_{\lambda}G_{\lambda} \) of \( o \)-groups \( \{ G_{\lambda} \} \) are in a one-to-one correspondence with the set of ultrafilters \( \mathcal{U} \) on \( \Lambda \); the correspondence is, for \( M \), \( \mathcal{U}_{M} = \{ \lambda \in \Lambda : \text{there exists } g \in M \text{ such that } g_{\lambda} = e \} \), and for \( \mathcal{U} \), \( M_{\mathcal{U}} = \{ g \in \Pi_{\lambda}G_{\lambda} : \{ \lambda : g_{\lambda} = e \} \in \mathcal{U} \} \). Since minimal primes are normal in representable \( \ell \)-groups, this means that \( o \)-groups in \( \ell \)-varieties generated by \( o \)-groups arise in a very natural way from quotients of \( \ell \)-subgroups of ultraproducts of the generating \( o \)-groups.
SECTION THREE: REPRESENTABLE COVERS OF $\mathcal{M}^+$

In the Introduction, $\mathcal{M}^+$ was mentioned as being contained in the $\ell$-variety $\mathcal{F}^+$ generated by all $\ell$-metabelian $\alpha$-groups $G$ having the property that for any $e < h \in \mathcal{A}(G)$ and $e < g \in G \setminus \mathcal{A}(G)$, $g^{-1}hg \gg h$. In [D2], it was shown that laws for $\mathcal{F}^+$ are:

**Proposition 3.1.** $\mathcal{F}^+$ is defined by the laws:

(i) $y^{-1} x_+ y \wedge x_- = e$

(ii) $[|w_1| \wedge |x_1, y_1|, |w_2| \wedge |x_2, y_2|] = e$

(iii) for $e \leq u \leq v$, $v^{-1}[a, b]v \geq u^{-1}[a, b]u$

(iv) $[|x_3, y_3, t|] \wedge \left(\left([|x_3, y_3|]^{2}\left([x_3, y_3]^{-2} \wedge e\right)\right) \wedge \left([|z| |z|^{2} \wedge e\right)\right] = e$

(v) $e \leq y \leq x$, $x^{-1}([x, y] \wedge |w|)x \geq (|[x, y] \wedge |w|)^{n}$, $n = 1, 2, 3, \ldots$.

Containment of $\mathcal{M}^+$ in $\mathcal{F}^+$ is obvious since $\mathbb{Z} \wr \mathbb{Z}$ is an $\ell$-metabelian $\alpha$-group having the infinite shorting property.

These laws also show that $\mathcal{M}^+$ is not closed with respect to lex extensions involving the group of integers.

**Proposition 3.2.** a) (Huss [Hu]) $(\mathbb{Z} \wr \mathbb{Z} \times \mathbb{Z}) \notin \mathcal{M}^+$,

b) (Reilly [R2]) $\mathbb{Z} \bar{\times} (\mathbb{Z} \wr \mathbb{Z}) \notin \mathcal{M}^+$.

**Proof.**

(a) For $G = (\mathbb{Z} \wr \mathbb{Z} \times \mathbb{Z})$, any element is of the form $(f, m, n)$, where $f: \mathbb{Z} \rightarrow \mathbb{Z}$ and $m, n \in \mathbb{Z}$. Then $\mathcal{A}(G) = \{(f, m, n): m = n = 0\}$. So if $h = \chi_0$, the characteristic function of $\{0\}$, and $g = (0, 0, 1)$, then $g^{-1}hg = h$ is not infinitely greater than $h$.

(b) For $H = \mathbb{Z} \bar{\times} (\mathbb{Z} \wr \mathbb{Z})$, elements of the form $(m, f, n)$, where $m, n \in \mathbb{Z}$ and $f: \mathbb{Z} \rightarrow \mathbb{Z}$. $\mathcal{A}(H)$ is then $\{(m, f, n): n = 0\}$. If $h = (1, 0, 0)$ and $g = (0, 0, 1)$, then $g^{-1}hg = g$.

There is a third way to totally order $(\mathbb{Z} \wr \mathbb{Z}) \oplus \mathbb{Z}$. Define $((f, n), m) \in (\mathbb{Z} \wr \mathbb{Z}) \oplus \mathbb{Z}$ to be positive if $n > 0$, if $n = 0$ and $m > 0$, or if $n = m = 0$ and $\hat{f}(k) > 0$ for $k = \max(\text{supp}(f))$. We will denote this $\alpha$-group as $\mathbb{Z} \bar{\wr}_{(0 \times \mathbb{Z})} (\mathbb{Z} \bar{\times} \mathbb{Z})$, the subscript to denote that the wreath action is done by the upper component of $\mathbb{Z} \bar{\times} \mathbb{Z}$ while the lower component has a trivial action. To be consistent with the order on $\mathbb{Z} \bar{\wr}_{(0 \times \mathbb{Z})} (\mathbb{Z} \bar{\times} \mathbb{Z})$, we will write an element $((f, n), m)$ as $(\hat{f}, m, n)$.

**Proposition 3.3.** $\mathbb{Z} \bar{\wr}_{(0 \times \mathbb{Z})} (\mathbb{Z} \bar{\times} \mathbb{Z}) \notin \mathcal{M}^+$.

**Proof.** Let $g = (\hat{0}, 0, 1)$ and $h = (\hat{0}, 1, 0)$. Then $h$ is in the abelian radical while $g$ is not, and $g^{-1}hg = h$.

* Due to Andrew Glass
Remarks. This author originally had a proof based on ultrapowers of $Z \wr Z$ that showed $Z \wr (0 \times Z) (Z \times Z) \notin \mathcal{M}^+$. A. M. W. Glass then devised law (iv) of Proposition 3.1 that simultaneously excluded $Z \times (Z \wr Z)$, $(Z \wr Z) \times Z$, and $Z \wr (0 \times Z) (Z \times Z)$ from $\mathcal{M}^+$.

The following proposition, whose proof will be left to the reader, describes $Z \times (Z \wr Z)$, $(Z \wr Z) \times Z$, and $Z \wr (0 \times Z) (Z \times Z)$ in terms of generators.

**Proposition 3.4.** Let $G$ be an $o$-group generated by elements $a$, $b$, and $c$.

a) If $e < c \ll b \ll a$, $[a, c] = [c, b] = [b^m, b^n] = e$ for all integers $m$ and $n$, and if $b \ll b^a$, then $G \cong Z \times (Z \wr Z)$.

b) If $e < c \ll b \ll a$, $[a, c] = [c, b] = [b^m, b^n] = e$ for all integers $m$ and $n$, and if $c \ll c^b$, then $G \cong (Z \wr Z) \times Z$.

c) If $e < c \ll b \ll a$, $[a, c] = [c, b] = [b^m, b^n] = e$ for all integers $m$ and $n$, and if $c \ll c^b$, then $G \cong Z \wr (0 \times Z) (Z \times Z)$.

**Definition.** $Y^+_b$ will be the $\ell$-variety generated by $Z \times (Z \wr Z)$, $Y^+_m$ the $\ell$-variety generated by $Z \wr (0 \times Z) (Z \times Z)$, and $Y^+_t$ the $\ell$-variety generated by $(Z \wr Z) \times Z$.

Our goal is now to show that $Y^+_b$, $Y^+_m$, and $Y^+_t$ are distinct $\ell$-varieties and to discuss their containments. For the discussion, recall that elements $g$ and $h$ of an $\ell$-group are $a$-equivalent if there exist positive integers $m$ and $n$ such that $|h| \leq |g|^m$ and $|g| \leq |h|^n$.

**Proposition 3.5.** $Y^+_m \subset Y^+_t$.

**Proof.** In $(Z \wr Z) \times Z$, let $a = (0, 1, 0)$, $b = (0, 0, 1)$, and $c = (\chi_0, 0, 0)$. Then $b^{-1}cb = c$, while $a^{-1}ca(\chi_1, 0, 0)$.

Note $(ab)^{-1}c(ab) = a^{-1}ca$ and $ab$ is $a$-equivalent to $b$.

Consider the elements $\hat{a} = (a, b, (a, b)^2, (a, b)^3, \ldots)$, $\hat{b} = (b, b, b, \ldots)$, and $\hat{c} = (c, c, c, \ldots)$. For any nonprincipal ultrafilter $\mathcal{U}$ on $\omega = \{0, 1, 2, \ldots\}$, the images $\hat{a}$, $\hat{b}$, and $\hat{c}$ of $a$, $b$, and $c$, respectively, in $[(Z \wr Z) \times Z]^\omega / \mathcal{U}$, have the properties that $\hat{a} \gg \hat{b} \gg \hat{c}$, $(\hat{a})^{-1}\hat{c}(\hat{a}) \gg \hat{c}$, $(\hat{b})^{-1}\hat{c}(\hat{b}) = \hat{c}$, and $\hat{a}\hat{b} = \hat{b}\hat{a}$. So the $\ell$-subgroup generated by $\hat{a}$, $\hat{b}$, and $\hat{c}$ is $o$-isomorphic to $Z \wr (0 \times Z) (Z \times Z)$. Thus $Y^+_m \subseteq Y^+_t$.

To show $Y^+_m \subset Y^+_t$, it suffices to note that $Z \wr (0 \times Z) (Z \times Z)$ satisfies (iii) of Proposition 3.1 while $(Z \wr Z) \times Z$ does not.

**Proposition 3.6.** $Y^+_b$ is incomparable to $Y^+_m$.

**Proof.** For any integer $n$, it is easy to verify that $Z \wr (0 \times Z) (Z \times Z)$ satisfies $(v-n)$ of Proposition 3.1 while $Z \times (Z \wr Z)$ does not. So $Y^+_b \not\subset Y^+_m$. 

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The following law showing $\mathcal{Y}_m^+ \not\subseteq \mathcal{Y}_b^+$ is due to A. M. W. Glass and again replaces a proof by this author that used ultraproducts.

$\mathcal{Y}_b^+$ satisfies the law:

$$a \left( (|x, y| \cap |z|^{-1} \land \varepsilon) \right) \land \left( (|x, y| |z^2||x, y|^{-2} \land \varepsilon) \right)$$

For in $\mathbb{Z}^x(\mathbb{Z} \wr \mathbb{Z})$, if $[x, y, t] \neq \varepsilon$, then $[x, y] \neq \varepsilon$ and $t$ is of the form $(m_1, f_1, n_1)$ where $n_1 \neq 0$. So if $z$ is of the form $(m_2, 0, 0)$, then $|([x, y]| |z|^{-1} \land \varepsilon) = \varepsilon$, while if $z$ is of the form $(m_2, f_2, 0)$, then $|z|^{-1} \gg |z|$ and so $|([x, y]| |z|^{-2} \land \varepsilon) = \varepsilon$. Finally if $z$ is of the form $(m_2, f_2, n_2)$, where $n_2 \neq 0$, then $|[x, y]| |z^2| \gg |x, y|$ and so $|([x, y]| |z^2||x, y|^{-2} \land \varepsilon) = \varepsilon$.

Now $\mathbb{Z} \wr_{(0, Z)}(\mathbb{Z} \wr Z)$ does not satisfy this law as can be seen by using the substitution $x = (\chi_0, 0, 0)$, $y = t = (0, 0, 1)$, and $z = (0, 1, 0)$. So $\mathcal{Y}_m^+ \not\subseteq \mathcal{Y}_b^+$. 

Proposition 3.7. $\mathcal{Y}_b^+ \not\subseteq \mathcal{Y}_t^+$.

Proof. It is easy to verify that $(\mathbb{Z} \wr Z) \mathbb{Z}$ satisfies the law:

$$|([x, y]| |c|)^z \lor (|[x, y]| |c|)^{z^{-1}} \geq (|[x, y]| |c|)^2$$

which fails in $\mathbb{Z}^x(\mathbb{Z} \wr \mathbb{Z})$.

With the aid of two lemmas, we will show that if $G$ is an $o$-group in $\mathcal{Y}_b^+ \setminus \mathcal{M}^+$, then $G$ contains a copy of $\mathbb{Z}^x(\mathbb{Z} \wr \mathbb{Z})$, which will then prove $\mathcal{Y}_b^+$ covers $\mathcal{M}^+$.

Lemma 3.8. $\mathcal{Y}_b^+ \cap \mathcal{T}^+ = \mathcal{M}^+$.

Proof. Let $G$ be an $o$-group in $\mathcal{Y}_b^+ \cap \mathcal{T}^+$; if $G \in \mathcal{A}$, then $G \in \mathcal{M}^+$. So assume $G$ is not abelian. Since $G \in \mathcal{Y}_b^+$, there exists a set $S$, an $\ell$-subgroup $S$ of $\Pi_{\lambda}(\mathbb{Z}^x(\mathbb{Z} \wr \mathbb{Z}))$, and prime subgroup $P$ of $\Pi_{\lambda}(\mathbb{Z}^x(\mathbb{Z} \wr \mathbb{Z}))$ such that $G \cong S/P$. Let $M$ be the minimal prime subgroup of $H = \Pi_{\lambda}(\mathbb{Z}^x(\mathbb{Z} \wr \mathbb{Z}))$ contained in $P$, and let $\mathcal{U}$ be the ultrafilter defined by $\mathcal{M}$.

Suppose $P < Ps \in \mathcal{A}(S/P)$. Then $s_\lambda = (m_\lambda, f_\lambda, n_\lambda)$, where $m_\lambda, n_\lambda \in \mathbb{Z}$, and $f_\lambda : \mathbb{Z} \to \mathbb{Z}$. Since $S/P$ is not abelian, there exists $P < Ps < Pt \notin \mathcal{A}(S/P)$. $S/P \in \mathcal{T}^+$ implies $P[t^{-1}, s^{-1}] < Ps < P[t, s^{-1}]$ and hence $M[t^{-1}, s^{-1}] < Ms < Mt[t, s^{-1}]$, giving us that $\{\lambda : [t_\lambda^{-1}, s_\lambda^{-1}] < s_\lambda < [t_\lambda, s_\lambda^{-1}]\} \in \mathcal{U}$. Thus $\{\lambda : n_\lambda = 0 \land f_\lambda > 0\} \in \mathcal{U}$. Clearly if $P < Ps \notin \mathcal{A}(S/P)$, $\{\lambda : n_\lambda > 0\} \in \mathcal{U}$.

Let $Q = \{g = (m_\lambda, f_\lambda, n_\lambda) \in \Pi_{\lambda}(\mathbb{Z}^x(\mathbb{Z} \wr \mathbb{Z})) : \{\lambda : f_\lambda > 0 \land n_\lambda = 0\} \in \mathcal{U}\}$; since $M \subset Q$, $Q$ is prime and so is comparable to $P$ because $M \subseteq P$. Suppose $P \subset Q$. Now from the proof of Proposition 2.1, we can assume that $P$ is maximal.
with respect to \( P \cap S \) being a prime subgroup of \( S \), and so \( P = P \cap S \subseteq Q \cap S \).

Let \( e < s \in (S \cap Q) \setminus P \). Then if \( P \subseteq \mathcal{A}(S/P) \), we have seen that \( \{ \lambda : \hat{f}_\lambda > 0 \text{ and } n_\lambda = 0 \} \in \mathcal{U} \), while if \( P \notin \mathcal{A}(S/P) \), \( \{ \lambda : n_\lambda > 0 \} \in \mathcal{U} \). So \( Q \subseteq P \) and consequently \( S/P \) is an \( \ell \)-homomorphic image of \( S/Q \).

Define \( \sigma : \Pi_\Lambda(\mathbb{Z} \times (\mathbb{Z} \wr \mathbb{Z})) \to \Pi_\Lambda(\mathbb{Z} \wr \mathbb{Z}) : (m_\lambda, \hat{f}_\lambda, n_\lambda) \sigma_\lambda = (\hat{f}_\lambda, n_\lambda) \). Then \( \sigma \) is an \( \ell \)-homomorphism. Now for the diagram

\[
\begin{array}{ccc}
S/Q & \to & \Pi_\Lambda(\mathbb{Z} \wr \mathbb{Z})/\mathcal{U} \\
\alpha & & \beta \\
S & \to & \Pi_\Lambda(\mathbb{Z} \wr \mathbb{Z}) \\
\sigma & & \\
\end{array}
\]

with \( \alpha \) and \( \beta \) natural. Clearly \( Q \sigma \subseteq \text{Ker} \beta \) and so \( \sigma \) lifts to an \( \ell \)-homomorphism \( \bar{\sigma} : S/Q \to \Pi_\Lambda(\mathbb{Z} \wr \mathbb{Z})/\mathcal{U} \).

Let \( t \in \text{Ker} \beta \) and \( s \in S \cap \{ t \} \sigma^{-1} \). Then \( \{ \lambda : t_\lambda = (0,0,0) \} \in \mathcal{U} \) and so \( \{ \lambda : s_\lambda = (m_\lambda, \hat{f}_\lambda, n_\lambda) \} \) with \( \hat{f}_\lambda = \hat{0}_\lambda \) and \( n_\lambda = 0 \) \( \in \mathcal{U} \). Thus \( s \in Q \) and so \( \bar{\sigma} \) is an \( \ell \)-isomorphism.

**Lemma 3.9.** For every positive integer \( n \), \( \gamma_b^+ \) satisfies the law:

\[
\left( [(r,s,t)] \land ([(a,b)]^{11} [(a,b)]^{-n} \land e) \right) \land ([(z)]^{11} |z|^{-n} \land e)
\]

\[
\land ([(a,b)] |z|^{-1} \land e) = e.
\]

**Proof.** Assume \( [r,s,t] \neq e \neq [a,b] \) in \( \mathbb{Z} \times (\mathbb{Z} \wr \mathbb{Z}) \). If \( |z| = (m,\hat{0},0) \), then \( |[a,b]| |z|^{-1} > e \). If \( |z| = (m,\hat{f},0) \), where \( \hat{f} > 0 \), then for any \( n \), \( |[a,b]| |z|^{11} \gg |z| \). Finally, if \( |z| = (m,\hat{f},k) \), when \( k > 0 \), \( |[a,b]| |z|^{11} \gg |[a,b]| \).

**Theorem 3.10.** \( \gamma_b^+ \) covers \( \mathcal{M}^+ \).

**Proof.** Let \( G \) be an \( \sigma \)-group in \( \gamma_b^+ \setminus \mathcal{M}^+ \). Then there exist \( e < h \in \mathcal{A}(G) \) and \( e < g \in G \setminus \mathcal{A}(G) \) such that \( g^{-1}hg \) is \( a \)-equivalent to \( h \).

We show first that if \( [a,b] \neq e \), then \( h \ll |[a,b]| \ll h \). Since \( G \) is not nil-2, there exist \( r,s,t \in G \) such that \( [r,s,t] \neq e \). Then \( t \in G \setminus \mathcal{A}(G) \); and since \( G/\mathcal{A}(G) \) is abelian, \( |g|^{11} |g|^{-n} < e \) for all \( n \geq 2 \). Likewise, since \( g \gg |[a,b]| \), \( |[a,b]| |g|^{-1} < e \) and so for the law of Lemma 3.9 to hold for \( z = g \), \( |[a,b]|^{11} > |[a,b]|^{11} \) for all \( n \geq 0 \). Thus \( g^{-1} |[a,b]| |g| \gg |[a,b]| \).

So \( |[a,b,g]| \neq e \). For \( n \geq 2 \) and any integer \( k \), \( h^{-k} |[a,b]| h^k |[a,b]|^{-n} < e \), while \( g^{-1} (h^k g) h^{-n} < e \). So again for Lemma 3.9 to hold with \( r = a, s = b, t = g \), and \( z = h \), we must have \( |[a,b]| > h^k \) for all \( k \).

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In particular, \([g, h] = e\), since \(g^{-1}hg\) being \(a\)-equivalent to \(h\) implies either \([g, h] \ll h\) or \([g, h] \) is \(a\)-equivalent to \(h\).

Thus the \(a\)-subgroup of \(G\) generated by \(h\), \([a, b]\), and \(g\) is \(a\)-isomorphic to \(\text{Zx}(\text{Z wr Z})\).

The proof that \(\mathcal{V}^+_m\) covers \(\mathcal{M}^+\) is much the same except in one key step which will be pointed out later.

**Lemma 3.11.** \(\mathcal{V}^+_m \cap \mathcal{Y}^+ = \mathcal{M}^+\).

*Proof.* Let \(G\) be an \(a\)-group in \(\mathcal{V}^+_m \cap \mathcal{Y}^+\). If \(G\) is abelian, then \(G \in \mathcal{M}^+\). Otherwise, for any \(e < h \in \mathcal{A}(G)\) and \(e < g \in G \setminus \mathcal{A}(G)\), \(g^{-1}hg \gg h\).

\(G \in \mathcal{V}^+_m\) implies, by Proposition 2.1, that there is a set \(A\), an \(\ell\)-subgroup \(S\) of \(H = \Pi_{\Lambda}(\text{Z wr}_{(0 \times Z)}(\text{Z x Z}))\), and a prime subgroup \(P\) of \(H\) such that \(P < S\) and \(G \cong S/P\).

Let \(M\) be the minimal prime subgroup of \(H\) contained in \(P\). Then \(M < S\) and since \(S/P\) is an \(a\)-homomorphic image of \(S/M\), it suffices to show that \(S/M \in \mathcal{M}^+\).

We first show \(S/M \in \mathcal{Y}^+\). To do so, we must show for any \(M < M_s \in \mathcal{A}(S/M)\) and any \(M < M_t \notin \mathcal{A}(S \setminus M), M_s \ll M_t \) st.

Suppose by way of contradiction that \(P_t \in \mathcal{A}(S/P)\). Since \(S/P\) is nonabelian, there exists \(r \in S\) such that \(P < P_t < P_r \notin \mathcal{A}(S/P)\). Since \(S/P \in \mathcal{Y}^+\), \(P_t \ll P_r^{-1}tr\) and so \(M_t \ll M_r^{-1}tr\). Now \(\mathcal{A}(S/M) = Q/M\) for some prime \(Q < S\). Since \(S/M\) is \(\ell\)-metabelian, \(S/Q\) is abelian and so there is \(q \in Q\) such that \(qt = r^{-1}tr\). Since \(M_t \notin \mathcal{A}(S/M) = Q/M\), \(M \ll M|q| \ll M_t \ll M_r^{-1}tr\), implying \(Mqt \ll M_r^{-1}tr\) \[BCD,\ Prop. 1.4\], a contradiction since \(qt = r^{-1}tr\). So \(M_t \notin \mathcal{A}(S/M)\) implies \(P_t \notin \mathcal{A}(S/M)\).

So \(P \ll P_s \ll P_t\) implies \(P_s \ll P_t^{-1}st\), and hence \(M_s \ll M_t^{-1}st\).

Let \(\mathcal{U}\) be the ultrafilter on \(\Lambda\) such that \(M = \{g \in H : \{\lambda : g_\lambda = e\} \in \mathcal{U}\}\). For \(s \in S\), \(s_\lambda\) will be written \((f_\lambda, m_\lambda, n_\lambda)\) as before.

\(M < M_s \notin \mathcal{A}(S/M)\) implies \(\{\lambda : n_\lambda > 0\} \in \mathcal{U}\), as if \(\{\lambda : n_\lambda = 0\} \in \mathcal{U}\), then \(M_s \in \mathcal{A}(H/M)\) and so is in \(\mathcal{A}(S/M)\).

On the other hand, \(M < M_s \in \mathcal{A}(S/M)\) implies that for any \(M < M_t \in (S/M) \setminus \mathcal{A}(S/M)\), \(M_s \ll M_t^{-1}st\); so \(M_s < M[t, s^{-1}]\) and thus \(\{\lambda : s_\lambda < [t_\lambda, s_\lambda^{-1}]\} \in \mathcal{U}\), \(t_\lambda < s_\lambda^{-1}\) in \(\mathcal{U}\).

Thus \(M \leq M_s \in \mathcal{A}(S/M)\) implies \(\{\lambda ; m_\lambda = n_\lambda = 0\} \in \mathcal{U}\).

Define \(\sigma ; \Pi_{\Lambda}(\text{Z wr}_{(0 \times Z)}(\text{Z x Z})) \rightarrow \Pi_{\Lambda}(\text{Z wr Z})\) by \((f_\lambda, m_\lambda, n_\lambda)\sigma_\lambda = (f_\lambda, n_\lambda)\). \(\sigma\) is a group homomorphism but, even on \(S\), need not be an \(\ell\)-homomorphism. Consider,
though, the diagram:

\[
\begin{array}{ccc}
S/M & \longrightarrow & \Pi_\Lambda(\mathbb{Z} \text{ wr } \mathbb{Z})/\mathcal{U} \\
\alpha \downarrow & & \beta \\
S & \longrightarrow & \Pi_\Lambda(\mathbb{Z} \text{ wr } \mathbb{Z})
\end{array}
\]

where \( \alpha \) and \( \beta \) are natural. Then \( M \sigma \subseteq \text{Ker } \beta \) and so \( \sigma \) lifts to a homomorphism \( \bar{\sigma} \) from \( S/M \) into \( (\mathbb{Z} \text{ wr } \mathbb{Z})^\Lambda/\mathcal{U} \).

Now \( t = (\ldots, (f_\lambda, n_\lambda), \ldots) \in \text{Ker } \beta \) implies \( \{ \lambda: (f_\lambda, n_\lambda) = (0, 0) \} \in \mathcal{U} \). So \( s = (\ldots, (f_\lambda, m_\lambda, n_\lambda), \ldots) \in t \bar{\sigma}^{-1} \) implies \( Ms \in \mathcal{A}(S/M) \). Since then \( \{ \lambda: m_\lambda = 0 \} \cap \{ \lambda: f = 0 \text{ and } n_\lambda = 0 \} \subseteq \mathcal{U} \), \( Ms = M \) and thus \( \bar{\sigma} \) is an isomorphism.

\( Ms > M \) implies, if \( Ms \notin \mathcal{A}(S/M) \), \( \{ \lambda: n_\lambda > 0 \} \subseteq \mathcal{U} \) and so \( (Ms)\bar{\sigma} \) is positive in \( (\mathbb{Z} \text{ wr } \mathbb{Z})^\Lambda/\mathcal{U} \) or, if \( Ms \in \mathcal{A}(S/M) \), \( \{ \lambda: n_\lambda = m_\lambda = 0 \text{ and } f_\lambda \geq 0 \} \subseteq \mathcal{U} \) and again \( (Ms)\bar{\sigma} \) is positive. So \( \bar{\sigma} \) is an \( \omega \)-isomorphism and thus \( S/M \in \mathcal{M}^+ \).

\[ \text{Lemma 3.12. For every positive integer } n, \, \mathcal{V}_m^+ \text{ satisfies} \]

\[ "[[r, s, t]] \wedge \left( [[x, y]]^{11} [[x, y]]^{-n} \wedge e \right) = e." \]

Proof. Suppose \( [r, s], [r, s, t], \) and \( [x, y] \) are all nonidentity elements of \( \mathbb{Z} \text{ wr}_{(0 \times \mathbb{Z})} (\mathbb{Z} \text{ wr } \mathbb{Z}) \). Then \( t = (f_1, m, n) \) where \( n \neq 0 \) since \( [r, s, t] \neq e \) and \( ||[x, y]| = (f_2, 0, 0) \). Then \( ||[x, y]|^{11} > ||[x, y]| \) and so the law is true. \( \square \)

(The above laws were proposed by Reilly \[R2\] as part of a set of laws that might define \( \mathcal{M}^+ \).)

In showing that \( \mathcal{V}_m^+ \) covers \( \mathcal{M}^+ \), we showed that any \( \omega \)-group \( G \in \mathcal{V}_m^+ \setminus \mathcal{M}^+ \) contains a copy of \( \mathbb{Z} \text{ wr}_{(0 \times \mathbb{Z})} (\mathbb{Z} \text{ wr } \mathbb{Z}) \). Unfortunately, it is not true that every \( \omega \)-group \( K \in \mathcal{V}_m^+ \setminus \mathcal{M}^+ \) contains a copy of \( \mathbb{Z} \text{ wr}_{(0 \times \mathbb{Z})} (\mathbb{Z} \text{ wr } \mathbb{Z}) \). Indeed, let \( H \) be the \( \omega \)-subgroup of \( \mathbb{Z} \text{ wr}_{(0 \times \mathbb{Z})} (\mathbb{Z} \text{ wr } \mathbb{Z}) \) generated by \((0, 0, 1)\) and \((-\chi_0, 1, 0)\). Then \( H \) does not contain a copy of \( \mathbb{Z} \text{ wr}_{(0 \times \mathbb{Z})} (\mathbb{Z} \text{ wr } \mathbb{Z}) \). We will, however, show that every \( \omega \)-group \( K \in \mathcal{V}_m^+ \setminus \mathcal{M}^+ \) contains a copy of \( H \) and so \( \ell \)-Var(H) does cover \( \mathcal{M}^+ \). The next lemma shows that \( \ell \)-Var(H) = \( \mathcal{V}_m^+ \).

**Lemma 3.13.** Let \( G = \mathbb{Z} \text{ wr}_{(0 \times \mathbb{Z})} (\mathbb{Z} \text{ wr } \mathbb{Z}) \) and \( H \) be the \( \ell \)-subgroup of \( G \) generated by \( \bar{a} = (0, 0, 1) \) and \( \bar{b} = (-\chi_0, 1, 0) \). Then \( \ell \)-Var(H) = \( \mathcal{V}_m^+ \).

Proof. It suffices to show that if \( G \) does not satisfy an \( \ell \)-group law \( "w(\bar{x}) = e," \) neither does \( H \).
So suppose \( \tilde{g} = \{g_{ijk}\} \) is a substitution for \( w(\overline{x}) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \bigcap_{k=1}^{p} x_{ij}^{f_{ijk}} \) into \( G \) such that \( w(\overline{\tilde{g}}) \neq e \). Since \( G \) is totally ordered, we can, by using \( w^{-1}(\overline{x}) \) if necessary, assume \( w(\overline{x}) > e \). Furthermore, if \( "w(\overline{x}) = e" \) is not a law for abelian \( \ell \)-groups, it obviously is not a law for \( \ell \)-Var(\( H \)) \( [W] \) and so we can assume \( w(\overline{\tilde{g}}) = (\overline{f}, 0, 0) \).

Note that \( g_{ijk} = (\tilde{f}_{ijk}, m_{ijk}, n_{ijk}) \).

For \( H, [H, H] \) is freely generated as an abelian \( o \)-group by \( \{[\tilde{a}, \tilde{b}]^{n} : n \in \mathbb{Z} \} \) and \( [H, H] \) is convex in \( H \). Thus any element of \( H \) can be written uniquely in the form \( ([\tilde{f}, \tilde{b}^{m}, \tilde{a}^{n}] : \tilde{f} \in [H, H]) \) and so we can assume \( w(g) = (\overline{f}, \overline{b}, \overline{a}) \) and so \( w(g) = (\overline{f}, \overline{b}, \overline{a}) \).

Note that \( g_{ijk} = (\tilde{f}_{ijk}, m_{ijk}, n_{ijk}) \).

For \( II, [H, H] \) is freely generated as an abelian \( o \)-group by \( \{[\tilde{a}, \tilde{b}]^{n} : n \in \mathbb{Z} \} \) and \( [H, H] \) is convex in \( H \). Thus any element of \( H \) can be written uniquely in the form \( ([\tilde{f}, \tilde{b}^{m}, \tilde{a}^{n}] : \tilde{f} \in [H, H]) \) and so we can assume \( w(g) = (\overline{f}, \overline{b}, \overline{a}) \).

For any \( s, g_{ij}(s) = g_{ij}(s) - g_{ij}(s+1) \). So \( g_{ij} \) is positive or negative as \( g_{ij} \) is. If \( g_{ij} = u_{ij} = 0 \) and \( g_{ij} = 0 \), then \( g_{ij} = 0 \) as well. So \( w(\overline{t}) \neq e \).

Unfortunately, \( t \) may not be contained in \( H \). A naive substitution (that almost works) is to substitute for \( t_{ijk} \) the element \( h_{ijk} = \tilde{f}_{ijk} \tilde{b}^{m_{ijk}} \tilde{a}^{n_{ijk}} = (\tilde{f}_{ijk} - m_{ijk} \tilde{a})^{n_{ijk}} \).

Note that \( g_{ijk} = (\tilde{f}_{ijk}, m_{ijk}, n_{ijk}) \).

For \( II, [H, H] \) is freely generated as an abelian \( o \)-group by \( \{[\tilde{a}, \tilde{b}]^{n} : n \in \mathbb{Z} \} \) and \( [H, H] \) is convex in \( H \). Thus any element of \( H \) can be written uniquely in the form \( ([\tilde{f}, \tilde{b}^{m}, \tilde{a}^{n}] : \tilde{f} \in [H, H]) \) and so we can assume \( w(g) = (\overline{f}, \overline{b}, \overline{a}) \).

Theorem 3.14. \( \mathcal{V}^{+} \) covers \( \mathcal{M}^{+} \).
Proof. Let $G$ be an $o$-group in $\mathcal{V}_m^+ \setminus \mathcal{M}^+$. Since $G \notin \mathcal{T}^+$ (by Lemma 3.11), there exist $e < h \in \mathcal{A}(G)$ and $e < g \in G \setminus \mathcal{A}(G)$ such that $g^{-1}hg$ is $a$-equivalent to $h$.

There exists $e < c < d \leq g$ such that $|[c,d]| \neq e$. So $g^{-1}[|c,d]|g \geq d^{-1}[|c,d]|d \gg |[c,d]|$. Now for any $e < a < b$ such that $|[a,b]| \neq e$, by Lemma 3.12, we have $|[c,d,g]| \gg |[a,b]| |[a,b]|^{-n} \gg e$; implying $g^{-1}[|a,b]|g \gg |[a,b]|$. We must have, then, that $h \gg |[a,b]|$ for any $a$ and $b$, since if $h \ll |[a,b]|$ for some $e < a < b$, $b^{-1}hb = b^{-1}(h \wedge |[a,b]|)b \gg h \wedge |[a,b]| = h$, and thus $[h^{-1},b^{-1}] \gg h \gg [h^{-1},b^{-1}] > e$.

So $g^{-1}[b,h^{-1}]g \gg g^{-1}[h^{-1},b^{-1}]g \gg [h^{-1},b^{-1}]$ which is $a$-equivalent to $h$, a contradiction to $g^{-1}hg$ and $h$ being $a$-equivalent. Hence $h \gg |[g,h]|$.

If $[g,h] = e$, then the $o$-subgroup of $G$ generated by $g$, $h$, and $|[c,d]|$ is $o$-isomorphic to $Z \, \text{wr} (0 \times Z) \, (Z \times Z)$.

Suppose $[g,h] \neq e$; first assume $[g,h] > e$. Then the $o$-subgroup of $G$ generated by $g$ and $h$ is $o$-isomorphic to the $o$-subgroup of $Z \, \text{wr} (0 \times Z) \, (Z \times Z)$ generated by $a = (0,0,1)$ and $b = (-\dot{\lambda}_0, 1,0)$. By Lemma 3.13, this $o$-subgroup generates $\mathcal{V}_m^+$.

If $[g,h] < e$, then

$$
[g,[g,h]h] = g^{-1}h^{-1}[g,h]^{-1}g[g,h]h = g^{-1}h^{-1}gg^{-1}[g,h]^{-1}gh[g,h]
$$

$$
= g^{-1}h^{-1}ghg^{-1}g[g,h]^{-1}g[g,h] = [g,h]^2[g,h]^{-g} > e.
$$

So the $o$-subgroup generated by $g$ and $[g,h]h$ is $o$-isomorphic to $H$.

For the other Medvedev $\ell$-variety $\mathcal{M}^-$, we likewise obtain $\mathcal{V}_b^-$ covering $\mathcal{M}^-$, $\mathcal{V}_m^-$ covering $\mathcal{M}^-$, $\mathcal{V}_b^- \neq \mathcal{V}_m^-$ and $\mathcal{V}_m^- \subset \mathcal{V}_i^-$. A surprising result, due to Huss [Hu] but with a proof simpler than hers, is:

**Proposition 3.15.** $\mathcal{V}_i^+ = \mathcal{V}_i^-$.  

**Proof.** Let $H = \prod_{n=0}^{\infty} [(Z \, \text{wr} Z) \, \hat{Z} Z]$ and let $a$ be the element $((\dot{\lambda}_0, 0, 1), (\dot{\lambda}_0, 0, 2), (\dot{\lambda}_0, 0, 3), \ldots)$, $b$ be the element $((\dot{\lambda}_0, 0, 1), (\dot{\lambda}_0, 0, 1), (\dot{\lambda}_0, 0, 1), \ldots)$, $c$ be the element $((\dot{\lambda}_0, 1, 0), (\dot{\lambda}_0, 1, 0), (\dot{\lambda}_0, 1, 0), \ldots)$ and $d = ((\dot{\lambda}_0, 0, 0), (\dot{\lambda}_0, 0, 0), (\dot{\lambda}_0, 0, 0), \ldots)$.

Let $\mathcal{U}$ be any nonprincipal ultrafilter on $\omega = \{0,1,2,\ldots\}$ and $\check{a}$, $\check{b}$, $\check{c}$, and $\check{d}$ be the respective images of $a$, $b$, $c$, and $d$ in $H/\mathcal{U}$. Then $\check{a} \gg \check{b} \gg \check{c} \gg \check{d} > e$ and $\check{a}$, $\check{b}$ are central elements. Then the $o$-subgroup generated by $\check{a}$, $\check{b}(\check{c})^{-1}$, and $\check{d}$ is $o$-isomorphic to $(Z \, \text{wr} Z) \, \hat{Z} Z$.

Finally, we prove:

**Proposition 3.16.** $\mathcal{M}^0 \nsubseteq \mathcal{V}_i^+$.  

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Proof. We leave it to the reader to check that \((\mathbf{Z} \wr \mathbf{Z}) \mathbf{x} \mathbf{Z}\) satisfies the law:

\[
\text{"for } e \leq y \leq x, \quad |[x, y]|^x \vee |[x, y]|^{x^{-1}} \geq |[x, y]|^2.\]

Now for the free nil-2 o-group on generators \(a\) and \(b\) with \(a^kb^m[a, b]^n \geq e \) if \(k > 0, k = 0\) and \(m > 0\), or \(k = m = 0\) and \(n \geq 0\), it is clear that \([a, b] = [a, b]^a \vee [a, b]^{a^{-1}} < [a, b]^2\). \(\square\)

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