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ON A CLASS OF FUNCTIONAL BOUNDARY VALUE PROBLEMS FOR SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PARAMETER

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In this paper sufficient conditions concerning only operators $Q, F$ are given for the functional differential equation

$$y''(t) - Q[y, y'](t) \cdot y(t) = F[y, y', \mu](t)$$

depending on the parameter $\mu$ to admit, for a suitable value of $\mu$, a solution $y$ satisfying functional boundary conditions

$$\alpha_1(y(t_1) - y(t)|J_1) = 0, \; y(t_2) = 0, \; \alpha_2(y(t_3) - y(t)|J_2) = 0,$$

where $-\infty < t_1 < t_2 < t_3 < \infty, \alpha_i$ are continuous functionals and $y(t)|J_i$ denotes the restriction of $y$ to $J_i = (t_i, t_{i+1})$ ($i = 1, 2$). Next, sufficient conditions are given under which the above equation has, for a suitable value of the parameter $\mu$, a bounded solution $y$ on the halfline $(t_1, \infty)$ and $\alpha_1(y(t_1) - y(t)|J_1) = 0, y(t_2) = 0$.

1. INTRODUCTION

Let $-\infty < t_1 < t_2 < t_3 < \infty, -\infty < a < b < \infty, J = (t_1, t_3), J_1 = (t_1, t_2), J_2 = (t_2, t_3), I = (a, b)$ and $X (X_1; X_2)$ be the Banach space of the $C^0$-functions on $J (J_1; J_2)$ with the norm $\|y\| = \max \{|y(t)|; t \in J\}$ ($\|y\|_1 = \max \{|y(t)|; t \in J_1\}$; $\|y\|_2 = \max \{|y(t)|; t \in J_2\}$). Consider the functional differential equation

$$y''(t) - Q[y, y'](t) \cdot y(t) = F[y, y', \mu](t),$$

depending on a parameter $\mu$. Here $Q: X \times X \rightarrow X, F: X \times X \times I \rightarrow X$ are continuous operators, $Q[y, z](t) > 0$ on $J$ for all $[y, z] \in X \times X$. 
Let \( \alpha_i: X_i \to \mathbb{R} \quad (i = 1, 2) \) be continuous increasing (i.e. \( \alpha_i(x) < \alpha_i(y) \) for all \( x, y \in X_i \), \( x(t) < y(t) \) for \( t \in J_i \setminus \{t_{2i-1}\}, \quad x(t_{2i-1}) = y(t_{2i-1}) = 0 \) functionals, \( \alpha_i(0) = 0 \). The purpose of this paper is to obtain using the Schauder linearization technique and the Schauder fixed point theorem, sufficient conditions imposed on the operators \( Q, F \) under which equation (1) admits, for a suitable value of the parameter \( \mu \), a solution \( y \) satisfying the functional boundary conditions

\[
\alpha_1(y(t_1) - y(t)|J_1) = 0, \quad y(t_2) = 0, \quad \alpha_2(y(t_3) - y(t)|J_2) = 0,
\]

where \( y(t)|J_i \quad (i = 1, 2) \) denotes the restriction of \( y \) to the interval \( J_i \).

In Section 4, we use BVP (1)-(2) to consider bounded solutions of (1) on the halfline \( (t_1, \infty) \) satisfying the functional boundary conditions

\[
\alpha_1(y(t_1) - y(t)|J_1) = 0, \quad y(t_2) = 0.
\]

The paper generalizes the author's results in [1]–[3] and, in a special case, also his results in [4]. In [1] the existence of solutions of (1) satisfying for example the boundary conditions \( y(t_1) - y(t_2) = y(t_3) = y(t_4) - y(t_5) = 0 \quad (-\infty < t_1 < t_2 < t_3 < t_4 < t_5 < \infty) \) was studied.

In [2] sufficient conditions for the existence (and uniqueness) of solutions of the differential equation

\[
y'' - q(t)y = f(t, y, y', \mu)
\]

satisfying the boundary conditions

\[
y(t_1) = y(t_2) = y(t_3) = 0
\]

\((-\infty < t_1 < t_2 < t_3 < \infty) \) was established.

In [4] the author considered the functional differential equation

\[
y''(t) - q(t)y(t) = f(t, y(t), y(h_0(t)), y'(t), y'(h_1(t)), \mu)
\]

with boundary conditions

\[
\sum_{i=1}^{m} \alpha_i y(t_i) = 0, \quad y(c) = 0, \quad \sum_{j=1}^{n} \beta_j y(x_j) = 0
\]

\((\alpha_i > 0, \beta_j > 0 \text{ constants}, \quad a = t_1 < \ldots < t_m < c < x_n < \ldots < x_1 = b)\).

In [3]—among other—sufficient conditions for the boundedness of solutions of (3) on a halfline \( (t_1, \infty) \) satisfying the boundary conditions \( y(t_1) = y(t_2) = 0 \quad (t_2 > t_1) \) were obtained.
A functional boundary value problem depending on one parameter was studied also in [5]. In this paper the retarded functional differential equation

\[ y'' - q(t)y = f(t, y_t, \mu) \]

with boundary conditions (4) was considered.

2. Notation, Lemmas

Let \( \varphi \in C^1(J) \) and let \( u_\varphi, v_\varphi \) be the solutions of the differential equation

\[ y'' = Q[\varphi', \varphi](t) y, \]

\[ u_\varphi(t_2) = 0, \quad u'_\varphi(t_2) = 1, \quad v_\varphi(t_2) = 1, \quad v'_\varphi(t_2) = 0. \]

For \((t, s) \in J \times J\) define \(r(t, s; \varphi)\) and \(r'_1(t, s; \varphi)\) by

\[ r(t, s; \varphi) = u_\varphi(t) v_\varphi(s) - u_\varphi(s) v_\varphi(t) \]
\[ r'_1(t, s; \varphi) = u'_\varphi(t) v_\varphi(s) - u_\varphi(s) v'_\varphi(t). \]

Then \( r(t, s; \varphi) > 0 \) for all \( t_1 \leq s < t \leq t_3, \) \( r(t, s; \varphi) < 0 \) for all \( t_1 \leq t < s \leq t_3, \)

\( r'_1(t, s; \varphi) > 1 \) for all \((t, s) \in J \times J\) and \( t \neq s, r'_1(t, t; \varphi) = 1 \) for all \( t \in J \) (for the proof, see e.g. [2]).

**Lemma 1.** Assume \( \varphi \in C^1(J), h \in C^0(J \times I), h(t, \cdot) \) is increasing on \( I \) for each fixed \( t \in J \) and

\[ h(t, a) h(t, b) \leq 0 \quad \text{for all} \quad t \in J. \]

Then there is a unique \( \mu_0 \in I \) such that the differential equation

\[ y'' = Q[\varphi', \varphi](t) y + h(t, \mu) \]

with \( \mu = \mu_0 \) admits a solution \( y \) satisfying (2). Moreover, this solution \( y \) is unique.

**Proof.** The function \( y(t; \mu, c) \) defined on \( J \times I \times R\) by

\[ y(t; \mu, c) = c u_\varphi(t) + \int_{t_2}^t r(t, s; \varphi) h(s, \mu) \, ds \]
is the general solution of (7) vanishing at the point \( t = t_2 \). Since

\[
y(t_1; \mu, c) - y(t; \mu, c) = c(u_\varphi(t_1) - u_\varphi(t)) + \\
+ \int_{t_2}^{t_1} [r(t_1, s; \varphi) - r(t, s; \varphi)] h(s, \mu) \, ds + \int_{t}^{t_1} r(t_1, s; \varphi) h(s, \mu) \, ds,
\]

\[
y(t_3; \mu, c) - y(t; \mu, c) = c(u_\varphi(t_3) - u_\varphi(t)) + \\
+ \int_{t_2}^{t_1} [r(t_3, s; \varphi) - r(t, s; \varphi)] h(s, \mu) \, ds + \int_{t}^{t_3} r(t_3, s; \varphi) h(s, \mu) \, ds
\]

and \( u_\varphi(t_1) - u_\varphi(t) < 0 \) on \( (t_1, t_3) \), \( u_\varphi(t_3) - u_\varphi(t) > 0 \) on \( (t_1, t_3) \), \( r(t_1, s; \varphi) - r(t, s; \varphi) = r'_1(\xi, s; \varphi)(t_1 - t) < 0 \) for \( (t, s) \in J \times J, t \neq t_1 \) (where \( \xi \) lies between \( t_1 \) and \( t \)), \( r(t_3, s; \varphi) - r(t, s; \varphi) = r'_1(\eta, s; \varphi)(t_3 - t) > 0 \) for \( (t, s) \in J \times J, t \neq t_3 \) (where \( \eta \) lies between \( t_3 \) and \( t \)), we see that the functions \( p_i: I \times \mathbb{R} \rightarrow \mathbb{R}, p_i(\mu, c) = \alpha_i(y(t_{2i-1}; \mu, c) - y(t_i; \mu, c)) \) \((i = 1, 2)\) are continuous on \( I \times \mathbb{R}, p_i(\cdot, c) \) are increasing on \( I \) for each fixed \( c \in \mathbb{R}, p_1(\mu, \cdot), p_2(\mu, \cdot) \) is decreasing (increasing) on \( R \) for each fixed \( \mu \in I \). Finally, one can check that \( \lim_{c \rightarrow -\infty} p_i(\mu, c) > 0, \lim_{c \rightarrow \infty} p_i(\mu, c) < 0, \lim_{c \rightarrow -\infty} p_2(\mu, c) < 0, \lim_{c \rightarrow \infty} p_2(\mu, c) > 0 \) for each fixed \( \mu \in I \). Hence there are unique functions \( c_i: I \rightarrow R \) \((i = 1, 2)\) such that

\[
p_i(\mu, c_i(\mu)) = 0 \quad \text{for all } \mu \in I \text{ and } i = 1, 2,
\]

and \( c_1(\mu) (c_2(\mu)) \) is increasing (decreasing) on \( I \).

To prove that \( c_i \) \((i = 1, 2)\) are continuous functions on \( I \) we suppose there are sequences \( \{\mu'_n\}, \{\mu''_n\} \) from \( I \) such that \( \lim_{n \rightarrow \infty} \mu'_n = \lim_{n \rightarrow \infty} \mu''_n = \mu_0 \) and \( \lim_{n \rightarrow \infty} c_i(\mu'_n) = \lambda_1, \lim_{n \rightarrow \infty} c_i(\mu''_n) = \lambda_2, \lambda_1 < \lambda_2, \) for some \( i \in \{1, 2\} \). Then \( 0 = \lim_{n \rightarrow \infty} p_i(\mu'_n, c_i(\mu'_n)) = p_i(\mu_0, \lambda_1), 0 = \lim_{n \rightarrow \infty} p_i(\mu''_n, c_i(\mu''_n)) = p_i(\mu_0, \lambda_2), \) which is a contradiction to \( p_i(\mu_0, \lambda_1) \neq p_i(\mu_0, \lambda_2) \).

It remains to prove the existence of a unique \( \mu_0 \in I \) such that \( c_1(\mu_0) = c_2(\mu_0) \). Since \( h(t, a) \leq 0, h(t, b) \geq 0 \) on \( J \) (cf. (6)) we have \( y(t_1; a, 0) - y(t; a, 0) \leq 0, y(t_1; b, 0) - y(t; b, 0) \geq 0 \) for \( t \in (t_1, t_2) \), \( y(t_3; a, 0) - y(t; a, 0) \leq 0, y(t_3; b, 0) - y(t; b, 0) \geq 0 \) for \( t \in (t_2, t_3) \), and then \( p_i(a, 0) \leq 0, p_i(b, 0) \geq 0 \) \((i = 1, 2)\). Using the fact that \( p_1(a, \cdot), p_1(b, \cdot), p_2(a, \cdot), p_2(b, \cdot) \) are decreasing (increasing) on \( R \) and \( p_i(a, c_1(a)) = 0, p_i(b, c_1(b)) = 0 \) \((i = 1, 2)\), we get \( c_1(a) \leq 0, c_1(b) \geq 0, c_2(a) > 0, c_2(b) \leq 0 \), therefore \( c_1(a) - c_2(a) \leq 0, c_1(b) - c_2(b) \geq 0 \). Since \( c_1(\mu) - c_2(\mu) \) is continuous increasing on \( I \), the equation \( c_1(\mu) - c_2(\mu) = 0 \) has a unique solution on \( I \).
Next, we will suppose that there exist positive constants $r_0$, $r_1$ such that the operators $Q$, $F$ satisfy the following assumptions:

\[(H_1)\] \[|F[y, y', \mu](t)| \leq r_0 \cdot Q[y, y'](t) \quad \text{for all } t \in J \text{ and } [y, y', \mu] \in D \times I,\]

where $D = \{[y, y']; y \in C^1(J), ||y^{(i)}|| \leq r_i \text{ for } i = 0, 1\};$

\[(H_2)\] \[F[y, y', \mu_1](t) < F[y, y', \mu_2](t) \quad \text{for all } t \in J \text{ and } [y, y'] \in D, \mu_1, \mu_2 \in I, \mu_1 < \mu_2;\]

\[(H_3)\] \[F[y, y', a](t) \cdot F[y, y', b](t) \leq 0 \quad \text{for all } t \in J \text{ and } [y, y'] \in D;\]

\[(H_4)\] \[\min\{(A + r_0 B)\tau, 2\sqrt{r_0 A + r_0 B}\} \leq r_1,\]

where $A = \sup\{||F[y, y', \mu]||; [y, y', \mu] \in D \times I\}$,
\[B = \sup\{||Q[y, y']||; [y, y'] \in D\}, \quad \tau = \max\{t_2 - t_1, t_3 - t_2\}.$

**Lemma 2.** Let assumptions $\text{(H}_1\text{)}$–$\text{(H}_4\text{)}$ be fulfilled for positive constants $r_0$, $r_1$ and let $\varphi \in C^1(J)$, $||\varphi^{(i)}|| \leq r_i$ ($i = 0, 1$). Then there exists a unique $\mu_0 \in I$ such that the equation

\[(8)\] \[y'' = Q[\varphi, \varphi'](t) y + F[\varphi, \varphi', \mu](t)\]

with $\mu = \mu_0$ admits a (then unique) solution $y$ satisfying (2) and, moreover,

\[(9)\] \[||y^{(i)}|| \leq r_i \quad \text{for } i = 0, 1.\]

**Proof.** Setting $h(t, \mu) = F[\varphi, \varphi', \mu](t)$ for $(t, \mu) \in J \times I$, the function $h$ fulfills the assumptions of Lemma 1 and hence there is a unique $\mu_0 \in I$ such that equation (8) with $\mu = \mu_0$ admits a (then unique) solution $y$ satisfying (2).

Now we prove $||y|| \leq r_0$. Let $|y(\xi)| = ||y|| > r_0$ for some $\xi \in J$. If $\xi \in (t_1, t_3)$ then the function $y \cdot \text{sign } y(\xi)$ has a local maximum at the point $t = \xi$, which contradicts $y''(\xi) \cdot \text{sign } y(\xi) > 0$. The last inequality follows from assumption $(H_1)$. Hence $\xi \in \{t_1, t_3\}$. If $\xi = t_1$ ($\xi = t_3$) then due to $y(t_2) = 0$ and assumption $(H_1)$ we have $(y(t_1) - y(t)) \cdot \text{sign } y(t_1) > 0$ for all $t \in (t_1, t_2)$ ($y(t_3) - y(t)) \cdot \text{sign } y(t_3) > 0$ for all $t \in (t_2, t_3)$, which contradicts $\alpha_i(y(t_1) - y(t)|J_i) = 0$ ($\alpha_2(y(t_3) - y(t)|J_2) = 0$). Thus $||y|| \leq r_0$.

Since $\alpha_i(y(t_{2i-1}) - y(t)|J_i) = 0$, $\alpha_i$ are increasing functionals and $\alpha_i(0) = 0$ ($i = 1, 2$), there exist $\xi_1 \in (t_1, t_2)$, $\xi_2 \in (t_2, t_3)$ such that $y(t_{2i-1}) - y(\xi_i) = 0$ and therefore $y'(\eta_i) = 0$ for some $\eta_i \in (t_1, \xi_1)$, $\eta_2 \in (\xi_2, t_3)$. For the next part of the proof of the inequality $||y'|| \leq r_1$ see e.g. [2] and [4].
3. Existence theorem

Theorem 1. Assume assumptions (H1)-(H4) are fulfilled for positive constants $r_0$ and $r_1$. Then there exists $\mu_0 \in I$ such that equation (1) with $\mu = \mu_0$ admits a solution $y$ satisfying (2) and (9).

Proof. Let $Y$ be the Banach space of the $C^1$-functions on $J$ with the norm $\|y\|_Y = \|y\| + \|y'\|$ for $y \in Y$ and $K = \{ y; y \in Y, \|y^{(i)}\| \leq r_i \text{ for } i = 0, 1 \}$. $K$ is a bounded convex closed subset of $Y$. Let $\varphi \in K$. By Lemma 2 there is a unique $\mu_0 \in I$ such that equation (8) with $\mu = \mu_0$ admits a (then unique) solution $y$ satisfying (2) and $y \in K$. Setting $T(\varphi) = y$ we obtain an operator $T: K \rightarrow K$. To prove Theorem 1 it is sufficient to show that $T$ has a fixed point.

First we prove that $T$ is a continuous operator. Let $\{y_n\} \subset K$ be a convergent sequence, $\lim y_n = y$ and let $z_n = T(y_n), z = T(y)$. Then there are sequences $\{\mu_n\} \subset I, \{c_n\} \subset R$ and $\mu_0 \in I, c_0 \in R$ such that we have (see the proof of Lemma 1)

$$z_n(t) = c_n u_{y_n}(t) + \int_{t_2}^t r(t, s; y_n)F[y_n, y'_n, \mu_n](s) \, ds \text{ for all } t \in J \text{ and } n \in \mathbb{N},$$

$$z(t) = c_0 u_{y}(t) + \int_{t_2}^t r(t, s; y)F[y, y', \mu_0](s) \, ds \text{ for all } t \in J,$$

and

$$\alpha_1(z_n(t_1) - z_n(t)|J_1) = 0, \quad z_n(t_2) = 0, \quad \alpha_2(z_n(t_3) - z_n(t)|J_2) = 0 \text{ for all } n \in \mathbb{N},$$

$$\alpha_1(z(t_1) - z(t)|J_1) = 0, \quad z(t_2) = 0, \quad \alpha_2(z(t_3) - z(t)|J_2) = 0.$$

The sequence $\{c_n\}$ is bounded since $\lim y_n = y$ and $\|z_n\| \leq r_0$ for all $n \in \mathbb{N}$. If $\{c_n\}$ is not convergent there are convergent subsequences $\{c_{k_n}\}, \{c_{r_n}\}$ and convergent subsequences $\{\mu_{k_n}\}, \{\mu_{r_n}\}$ of $\{\mu_n\}$ such that $\lim c_{k_n} = c^{(1)}$, $\lim c_{r_n} = c^{(2)}$, $\lim \mu_{k_n} = \mu^{(1)}$, $\lim \mu_{r_n} = \mu^{(2)}$, $c^{(1)} < c^{(2)}$ and $\mu^{(1)}, \mu^{(2)}$ are either equal or not. Then

$$(k_1(t) :=) \lim_{n \rightarrow \infty} z_{k_n}(t) = c^{(1)} u_{y}(t) + \int_{t_2}^t r(t, s; y)F[y, y', \mu^{(1)}](s) \, ds,$$

$$(k_2(t) :=) \lim_{n \rightarrow \infty} z_{r_n}(t) = c^{(2)} u_{y}(t) + \int_{t_2}^t r(t, s; y)F[y, y', \mu^{(2)}](s) \, ds$$

uniformly on $J$ and

$$\alpha_1(k_1(t_1) - k_1(t)|J_1) = 0, \quad k_1(t_2) = 0,$$

$$\alpha_2(k_1(t_3) - k_1(t)|J_2) = 0 \quad \text{for } i = 1, 2.$$
The equalities \((i = 1, 2)\)

\[
\begin{align*}
    k_i(t_1) - k_i(t) &= c^{(i)}(u_y(t_1) - u_y(t)) + \int_{t_2}^{t} (r(t_1, s; y) - r(t, s; y)) \\
    &\times F[y, y', \mu^{(i)}](s) \, ds + \int_{t}^{t_1} r(t_1, s; y) F[y, y', \mu^{(i)}](s) \, ds, \\
    k_i(t_3) - k_i(t) &= c^{(i)}(u_y(t_3) - u_y(t)) + \int_{t_2}^{t} (r(t_3, s; y) - r(t, s; y)) \\
    &\times F[y, y', \mu^{(i)}](s) \, ds + \int_{t}^{t_3} r(t_3, s; y) F[y, y', \mu^{(i)}](s) \, ds
\end{align*}
\]

imply (see the proof of Lemma 1)

\[
\begin{align*}
    k_1(t_1) - k_1(t) > k_2(t_1) - k_2(t) \quad &\text{for} \quad t \in (t_1, t_2) \quad \text{and} \quad \mu^{(1)} > \mu^{(2)}, \\
    k_2(t_3) - k_2(t) > k_1(t_3) - k_1(t) \quad &\text{for} \quad t \in (t_2, t_3) \quad \text{and} \quad \mu^{(1)} \leq \mu^{(2)},
\end{align*}
\]

which contradicts (10). Hence \(\{c_n\}\) is convergent, and let \(\lim_{n \to \infty} c_n = c^*\). If \(\{\mu_n\}\) is not convergent there are convergent subsequences \(\{\mu_{j_n}\}, \{\mu_{i_n}\}\), \(\lim_{n \to \infty} \mu_{j_n} = \lambda^{(1)}, \lim_{n \to \infty} \mu_{i_n} = \lambda^{(2)}, \lambda^{(1)} < \lambda^{(2)}\). Then

\[
\begin{align*}
    (p_1(t) :=) \lim_{n \to \infty} z_{j_n}(t) &= c^* u_y(t) + \int_{t_2}^{t} r(t, s; y) F[y, y', \lambda^{(1)}](s) \, ds, \\
    (p_2(t) :=) \lim_{n \to \infty} z_{i_n}(t) &= c^* u_y(t) + \int_{t_2}^{t} r(t, s; y) F[y, y', \lambda^{(2)}](s) \, ds
\end{align*}
\]

uniformly on \(J\) and

\[
\begin{align*}
    \alpha_1(p_i(t_1) - p_i(t)|J_1) &= 0, \quad p_i(t_2) = 0, \\
    \alpha_2(p_i(t_3) - p_i(t)|J_2) &= 0 \quad \text{for} \quad i = 1, 2.
\end{align*}
\]

As above we may verify

\[
\begin{align*}
    p_2(t_1) - p_2(t) > p_1(t_1) - p_1(t) \quad &\text{for all} \quad t \in (t_1, t_2), \\
    p_2(t_3) - p_2(t) > p_1(t_3) - p_1(t) \quad &\text{for all} \quad t \in (t_2, t_3),
\end{align*}
\]

which contradicts (11). Hence \(\{\mu_n\}\) is convergent, and let \(\lim_{n \to \infty} \mu_n = \mu^*\). Then

\[
\begin{align*}
    (z^*(t) :=) \lim_{n \to \infty} z_n(t) &= c^* u_y(t) + \int_{t_2}^{t} r(t, s; y) F[y, y', \mu^*](s) \, ds
\end{align*}
\]
uniformly on \( J \), and consequently, \( z^* \) is a solution of the differential equation
\[
 w'' = Q[y, y'](t) w + F[y, y', \mu^*](t)
\]
and
\[
 \alpha_1(z^*(t_1) - z^*(t)|J_1) = 0, \quad z^*(t_2) = 0, \quad \alpha_2(z^*(t_3) - z^*(t)|J_2) = 0.
\]
By Lemma 2 it is necessary that \( z = z^* \) and \( \mu_0 = \mu^* \). Since \( \lim_{n \to \infty} z^{(i)}_n(t) = z^{(i)}(t) \) uniformly on \( J \) for \( i = 0, 1 \), we have \( z = \lim_{n \to \infty} z_n = \lim_{n \to \infty} T(y_n) = T(y) \) and therefore \( T \) is a continuous operator. Let \( \varphi \in K \) and \( T(\varphi) = y \). Then the equality
\[
y''(t) = Q[\varphi, \varphi'](t) y(t) + F[\varphi, \varphi', \mu_0](t)
\]
holds on \( J \) for some \( \mu_0 \in I \), thus \( ||y''|| \leq A + r_0 B (:= r_2) \) and \( K \subset L = \{ y; y \in C^2(J), ||y^{(i)}|| \leq r_i \text{ for } i = 0, 1, 2 \} \). Since \( L \) is a compact subset of \( Y \), \( K \) is a relative compact subset of \( Y \).

By the Schauder fixed point theorem there is a fixed point of \( T \). This completes the proof. \( \square \)

Remark 1. If \( \alpha_1(z) = \alpha_2(z) = z(t_2) \), then Theorem 1 in [2] and Theorem 1 in [4] (where \( m = n = 1 \)) follow from Theorem 1.

Let \( t_1 < x_1 < t_2 < x_2 < t_3 \). If \( \alpha_1(z) = z(x_1), \alpha_2(z) = z(x_2) \), then Theorem 1 in [1] follows from Theorem 1.

Example 1. Consider the functional differential equation
\[
y''(t) = y(t) \exp \{ |y(y'(t))| \} + \frac{1}{2} \cos \left( t + y'(y(t)) \right) + \mu
\]
on the interval \( J = (0, t_3) \), where \( t_3 \geq 2\sqrt{1 + e} \). Let \( t_2 \in (0, t_3) \). Assumptions (H1)–(H4) are fulfilled with \( r_0 = 1, r_1 = 2\sqrt{1 + e} \) and \( I = \langle -\frac{1}{2}, \frac{1}{2} \rangle \). Let \( \alpha_1(z) = \int_0^{t_2} z^3(s) \, ds \) for \( z \in C^0([0, t_2]) \) and \( \alpha_2(z) = \max \{ z(t); t \in \langle t_2, \frac{1}{2}(t_2 + t_3) \rangle \} \) for \( z \in C^0([t_2, t_3]) \). Then by Theorem 1 there is \( \mu_0 \in \langle -\frac{1}{2}, \frac{1}{2} \rangle \) such that equation (12) with \( \mu = \mu_0 \) admits a solution \( y \) satisfying
\[
 \int_0^{t_2} (y(t_1) - y(s))^3 \, ds = 0, y(t_2) = 0, \max \{ y(t_3) - y(t); t \in \langle t_2, \frac{1}{2}(t_2 + t_3) \rangle \} = 0
\]
and
\[
 ||y|| \leq 1, \quad ||y'|| \leq 2\sqrt{1 + e}.
\]
4. BOUNDED SOLUTIONS ON A HALFLINE

In this section BVP (1)-(2) is applied to the investigation of bounded solutions of a functional differential equation of type (1) with functional boundary conditions

\[(13) \quad \alpha_1 (y(t_1) - y(t))|J_1| = 0, y(t_2) = 0.\]

Let \(Y\) be the space of bounded \(C^0\)-functions on the halfline \((t_1, \infty)\) with the topology of uniform convergence on compact subintervals of \((t_1, \infty)\). Consider the functional differential equation

\[(14) \quad y''(t) - U[y, y'](t)y(t) = V[y, y', \mu](t),\]

where \(U : Y \times Y \rightarrow Y, V : Y \times Y \times I \rightarrow Y\) are continuous operators, \(U[y, z](t) > 0\) for all \(t \geq t_1\) and \([y, z] \in Y \times Y\). Further we shall assume that there exists an increasing sequence \(\{x_n\} \subset R, x_1 > t_2, \lim_{n \to \infty} x_n = \infty\) such that the functions \(U[y, z](t), V[y, z, \mu](t)\) are defined on \((t_1, x_n)\) only by the restrictions of \(y, z\) to the interval \((t_1, x_n)\) \((n = 1, 2, \ldots)\), that is

\[U : Y_n \times Y_n \rightarrow Y_n, \quad V : Y_n \times Y_n \times I \rightarrow Y_n \quad (n = 1, 2, \ldots),\]

where \(Y_n\) is the Banach space of the \(C^0\)-functions on \((t_1, x_n)\) with the sup norm. The differential equation \(y'' - q(t, y, y')y = f(t, y, y', \mu)\), where \(q \in C^0((t_1, \infty) \times R^2), f \in C^0((t_1, \infty) \times R^2 \times I)\), is a special case of (14).

Suppose there are positive constants \(r_0, r_1\) such that the operators \(U, V\) satisfy the following assumptions:

(C1) \(|V[y, y', \mu](t)| \leq r_0 U[y, y'](t)\) for all \(t \geq t_1\) and \([y, y', \mu] \in H \times I, H = \{[y, y'] : y \in C^1((t_1, \infty)), |y^{(i)}(t)| \leq r_i \text{ for } t \geq t_1, i = 0, 1\}\);

(C2) \(V[y, y', \mu_1](t) < V[y, y', \mu_2](t)\) for all \(t \geq t_1\), \([y, y'] \in H\) and \(\mu_1, \mu_2 \in I, \mu_1 < \mu_2\);

(C3) \(V[y, y', a](t)V[y, y', b](t) \leq 0 \) for all \(t \geq t_1\) and \([y, y'] \in H\);

(C4) \(2\sqrt{r_0}A + r_0B \leq r_1\), where \(A = \sup_{t \geq t_1} \sup_{[y, y', \mu] \in H \times I} |V[y, y', \mu](t)|, B = \sup_{t \geq t_1} \sup_{[y, y'] \in H} |U[y, y'](t)|\).

Lemma 3. Assume assumptions (C1)-(C4) are fulfilled with positive constants \(r_0, r_1\). Then for any \(x_n\) \((n = 1, 2, \ldots)\) there exists a \(\mu_n \in I\) such that equation (14) with \(\mu = \mu_n\) admits a solution \(y_n\) defined on the interval \((t_1, x_n)\) and satisfying the boundary conditions

\[(15) \quad \alpha_1 (y_n(t_1) - y_n(t))|J_1| = 0, \quad y_n(t_2) = 0, \quad y_n(x_n) = 0 \quad (n = 1, 2, \ldots),\]
and, moreover,

\[
|y_n(t)| \leq r_0, \quad |y_n'(t)| \leq r_1,
\]

\[
|y_n''(t)| \leq A + r_0 B \quad \text{for } t \in (t_1, x_n), \quad (n = 1, 2, \ldots).
\]

**Proof.** The proof follows immediately from Theorem 1 if we set \( t_3 = x_n \) and \( \alpha_2(z) = z(t_2) \). The last inequality in (16) is evident. \( \square \)

**Theorem 2.** Assume assumptions \((C_1)-(C_4)\) are fulfilled with positive constants \( r_0, r_1 \). Then there exists a \( \mu_0 \in I \) such that equation (14) with \( \mu = \mu_0 \) admits a solution \( y \) satisfying (13) and

\[
|y(t)| \leq r_0, \quad |y'(t)| \leq r_1 \quad \text{for } t \geq t_1.
\]

**Proof.** According to Lemma 3 there exists a sequence \( \{y_n\} \) of solutions of equation (14) with \( \mu = \mu_n \in I \) on the intervals \((t_1, x_n)\) satisfying (15) and (16). Using the Ascoli-Arzela theorem, a diagonal process of Cantor and the fact that \( \{\mu_n\} \) is a bounded sequence, we may assume without loss of generality that \( \{y_n(t)\} \) and \( \{y_n'(t)\} \) are locally uniformly convergent on \((t_1, \infty)\) and \( \{\mu_n\} \) is convergent. Setting \( \lim_{n \to \infty} y_n(t) = y(t) \) for \( t \in (t_1, \infty) \) and \( \lim_{n \to \infty} \mu_n = \mu_0 \), then \( y \) is a solution of equation (14) with \( \mu = \mu_0 \) satisfying (13) and (17). \( \square \)

**Example 2.** Consider the functional differential equation

\[
y''(t) = 6\pi y(t) \exp \{ |y(t + (\sin t)^2)| \} + \ln (e + |y'(\sqrt{t})|) \arctan t + (1 + y^2(t)) \mu.
\]

The assumptions of Theorem 2 are satisfied with \( t_1 \geq 1, r_0 = 1, r_1 = e^3 \) and \( I = (-2\pi, 0) \). Therefore there exists a \( \mu_0 \in (-2\pi, 0) \) such that equation (18) with \( \mu = \mu_0 \) has a solution \( y \) defined on \((t_1, \infty)\), and (13) and \( |y(t)| \leq 1, |y'(t)| \leq e^3 \) for \( t \geq t_1 \) hold.

**References**


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