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On commuting isometries


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ON COMMUTING ISOMETRIES

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INTRODUCTION

Reflexive algebras have been studied intensively by many authors interested in invariant subspace problem. Among the most interesting results in this direction the reflexivity of the algebra generated by a single isometry, by a normal, and by a subnormal operator was proved by J. A. Deddens ([2]), D. Sarason ([9]), and by R. Olin and J. E. Thomson ([5]), respectively.

The reflexivity of algebra generated by two isometries was studied by M. Ptak in [6], [7]. He obtained the positive result for certain class of shifts which were called compatible (for definition see [4]) and which include doubly commuting shifts. The present authors conjecture that algebra generated by any family of commuting isometries is reflexive. The aim of this paper is to present some partial results in that direction.

Let $T = (T_\alpha)_{\alpha \in A}$ be a family of operators on a Hilbert space $\mathcal{H}$. As usual we denote by $\text{Lat } T$ the lattice of all subspaces of $\mathcal{H}$ invariant with respect to any $T_\alpha$ ($\alpha \in A$). Further $\text{AlgLat } T$ denotes the (weakly closed) algebra of operators leaving invariant every subspace from $\text{Lat } T$.

A weakly closed algebra generated by $T_\alpha$ is called reflexive if it is equal to $\text{AlgLat } T$. The commutant of family $T$ is the set of all operators which commute with every $T_\alpha$ ($\alpha \in A$) and is denoted by $T'$. The commutant $T''$ of $T'$ is called the double commutant of $T$.

The main result of this paper is that $\text{AlgLat } V$ is contained in the double commutant $V'$ for any family $V = (V_\alpha)_{\alpha \in A}$ of commuting isometries. Moreover, it is proved that $\tilde{T}$ belongs to the double commutant of $(\tilde{V}_\alpha)_{\alpha \in A}$ where $(\tilde{V}_\alpha)_{\alpha \in A}, V_\alpha \in B(\mathcal{K})$, are unitary extensions of given isometries $V_\alpha$ and $\tilde{T} \in B(\mathcal{K})$ the corresponding extension of $T \in \text{AlgLat } V$. This means that $\tilde{T}$ is a function of $\tilde{V}$.
Throughout the paper we shall use the following well-known Wold decomposition: If $V$ is an isometry on a Hilbert space $\mathcal{H}$ then $\mathcal{H}$ can be decomposed into two parts $M_+(\mathcal{L}) \oplus \mathcal{R}_V$ where $V|M_+(\mathcal{L})$ is the unilateral shift on $M_+(\mathcal{L}) = \mathcal{L} \oplus V\mathcal{L} \oplus V^2\mathcal{L} \oplus \ldots$, $\mathcal{L} = \mathcal{H} \ominus VH$ being the corresponding wandering subspace, and $V|\mathcal{R}_V$ is unitary on the (residue) subspace $\mathcal{R}_V = \bigcap_{n \geq 0} V^n\mathcal{H}$.

We begin with a simple lemma.

**Lemma 1.** Let $V \in B(\mathcal{H})$ be an isometry, let $x \in \mathcal{H} \ominus VH$ and let $\alpha$ be a complex number such that $|\alpha| < 1$. Then

$$(I - \bar{\alpha}V)^{-1}x \perp (\alpha - V)\mathcal{H}.$$  

**Proof.** The equality

$$\alpha - V = \alpha V^* V - V = (\alpha V^* - I)V$$

gives for $|\alpha| < 1$

$$V = (\alpha V^* - I)^{-1}(\alpha - V).$$

Let $h \in \mathcal{H}$. Then

$$0 = \langle x, Vh \rangle = \langle x, (\alpha V^* - I)^{-1}(\alpha - V)h \rangle = \langle ((\bar{\alpha} V - I)^{-1}x, (\alpha - V)h \rangle,$$

hence $(I - \bar{\alpha}V)^{-1}x \perp (\alpha - V)\mathcal{H}$. \hfill \Box

**Lemma 2.** Let $V \in B(\mathcal{H})$ be an isometry, and let $T \in B(\mathcal{H})$ be an operator that leaves invariant every subspace $(\alpha - V)\mathcal{H}$ with $|\alpha| < 1$. Then $(VT - TV)\mathcal{H} \subset \bigcap_{n \geq 0} V^n\mathcal{H}$.

**Proof.** For $h \in \mathcal{H}$ let us denote $m_0 = Th$. As $T$ leaves $V\mathcal{H}$ invariant, $TVh \in V\mathcal{H}$, which means that $TVh = Vm_1$ for some $m_1 \in \mathcal{H}$. If $x \in \mathcal{H} \ominus VH$, and $|\alpha| < 1$, then $T(\alpha - V)h \in (\alpha - V)\mathcal{H}$ and by the preceding Lemma

$$0 = \langle T(\alpha - V)h, (I - \bar{\alpha}V)^{-1}x \rangle = \langle \alpha m_0 - Vm_1, \sum_{j=0}^{\infty} \bar{\alpha}^j V^j x \rangle =$$

$$= \sum_{j=0}^{2} \alpha^{j+1} \langle m_0, V^j x \rangle - \sum_{j=0}^{\infty} \alpha^j \langle Vm_1, V^j x \rangle =$$

$$= \sum_{j=0}^{2} \alpha^{j+1} \langle m_0 - m_1, V^j x \rangle.$$
Hence \( \langle m_0 - m_1, V^j x \rangle = 0 \) for any integer \( j \geq 0 \) and \( x \in \mathcal{H} \ominus V\mathcal{H} \), which implies that \( m_0 - m_1 \in \bigcap_{j \geq 0} V^j \mathcal{H} \), and \( (VT - TV)h = V(m_0 - m_1) \in \bigcap_{j \geq 0} V^j \mathcal{H} \). The inclusion is proved. \( \square \)

**Theorem 1.** Let \( \mathbf{V} = (V_\alpha)_{\alpha \in A} \) be a commuting system of isometries on a Hilbert space \( \mathcal{H} \). If \( T \in \mathrm{AlgLat} \mathbf{V} \) then \( TV_\alpha = V_\alpha T \) (\( \alpha \in A \)).

**Proof.** For any finite subset \( F = \{\alpha_1, \ldots, \alpha_k\} \subset A \) let us denote

\[
V_F = V_{\alpha_1} \cdots V_{\alpha_k}, \quad \mathcal{R}_F = \bigcap_{n \geq 0} V_n^F \mathcal{H}.
\]

The subspace \( \mathcal{R}_F \) reduces \( V_F \) and is invariant for any isometry \( V_\alpha, \alpha \in A \).

Further the common residue subspace \( \mathcal{R} \) of all \( V_\alpha \) is defined by

\[
\mathcal{R} = \bigcap_{\substack{F \subset A \mid F \neq \emptyset \atop |F| < \infty}} \mathcal{R}_F.
\]

Clearly, \( \mathcal{R} \) is invariant for any \( V_\alpha \) (\( \alpha \in A \)).

Let \( \alpha_0 \in A_0 \). For any finite subset \( F' \subset A \) containing \( \alpha_0 \), \( F' = \{\alpha_0, \alpha_1, \ldots, \alpha_k\} \), \( \mathcal{R}_{F'} \) also reduces \( V_{\alpha_0} \) as

\[
V_{\alpha_0}^* \mathcal{R}_{F'} = V_{\alpha_0}^* (V_{\alpha_1}^* \cdots V_{\alpha_n}^*) (V_{\alpha_1} \cdots V_{\alpha_n}) \mathcal{R}_{F'} \subset V_{\alpha_0}^* \mathcal{R}_{F'} \subset \mathcal{R}_{F'}. \]

Thus for any finite subset \( F \subset A \) we have

\[
V_{\alpha_0}^* \mathcal{R} \subset V_{\alpha_0}^* \mathcal{R}_{F \cup \{\alpha_0\}} \subset \mathcal{R}_{F \cup \{\alpha_0\}} \subset \mathcal{R}_F,
\]

\[
V_{\alpha_0}^* \mathcal{R} \subset \bigcap_{\substack{F \subset A \mid F \neq \emptyset \atop |F| < \infty}} \mathcal{R}_F = \mathcal{R}
\]

which proves that \( \mathcal{R} \) reduces every \( V_\alpha \) (\( \alpha \in A \)) and \( V_\alpha | \mathcal{R} \) are unitary operators.

By our assumption on \( T \), it follows that \( \mathcal{R} \) reduces \( T \), and by an application of the von Neumann double commutant theorem (see e.g. [3] or [1]) its restriction \( T | \mathcal{R} \) commutes with every \( V_\alpha | \mathcal{R} \), i.e. \( TV_\alpha | \mathcal{R} = V_\alpha T | \mathcal{R} \) for any \( \alpha \in A \).

Suppose now that \( h \in \mathcal{R}^\perp = \mathcal{H} \ominus \mathcal{R} \), and \( \alpha \in A \) are given. For any finite subset \( F \subset A \) we have by Lemma 2 (for the isometry \( V_F \))

\[
(TV_F - V_F T) \mathcal{H} \subset \mathcal{R}_F,
\]

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and for the isometry $V_{\alpha}V_F$ we obtain analogously
\[
\mathcal{R}_{F \cup \{ \alpha \}} = \bigcap_{k \geq 0} (V_FV_{\alpha})^k \mathcal{H} \subset \bigcap_{k \geq 0} V_F^k \mathcal{H} \subset \mathcal{R}_F,
\]
which gives the inclusion
\[
(V_FV_{\alpha}T - TV_FV_{\alpha})\mathcal{H} \subset \mathcal{R}_F.
\]
Then
\[
(V_FV_{\alpha}T - V_FTV_{\alpha})h = (V_FV_{\alpha}T - TV_FV_{\alpha})h + (TV_F - V_FT)V_{\alpha}h \in \mathcal{R}_F
\]
which implies that
\[
(V_{\alpha}T - TV_{\alpha})h \in V_F^* \mathcal{R}_F = \mathcal{R}_F
\]
for any finite subset $F \subset A$, $h \in \mathcal{R}^\perp$, $\alpha \in A$. Hence
\[
(V_{\alpha}T - TV_{\alpha})h \in \bigcap_{F \subset A, |F| < \infty} \mathcal{R}_F = \mathcal{R}.
\]

On the other hand, as $\mathcal{R}$ reduces any $V_{\alpha}$ and $T$ leaves invariant every subspace from $\bigcap_{\alpha \in A} \text{Lat} V_{\alpha}$, we have $TR^\perp \subset \mathcal{R}^\perp$ and $V_{\alpha}R^\perp \subset \mathcal{R}^\perp$ which implies that $(V_{\alpha}T - TV_{\alpha})h \in \mathcal{R}^\perp$, hence $V_{\alpha}Th = TV_{\alpha}h$ for any $h \in \mathcal{R}^\perp$, $\alpha \in A$. This finishes the proof. \hfill \Box

**Lemma 3.** Let $V$, $T_{\alpha} \in B(\mathcal{H})$ ($\alpha \in A$) be commuting operators, $V$ an isometry. Then there exist commuting extensions $\tilde{V}$, $\tilde{T}_{\alpha}$ on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $\tilde{V}|\mathcal{H} = V$, $\tilde{T}_{\alpha}|\mathcal{H} = T_{\alpha}$, $\tilde{V}$ is unitary, $\|\tilde{T}_{\alpha}\| = \|T_{\alpha}\|$, and $\tilde{T}_{\alpha}$ is an isometry (unitary, resp.) if $T_{\alpha}$ is.

**Proof.** Taking $\tilde{V}$ the minimal unitary extension of $V$ on some Hilbert space $\mathcal{K} \supset \mathcal{H}$ (where the condition of minimality gives $\mathcal{K} = \bigvee_{k \geq 0} \tilde{V}^*k(\mathcal{H})$) we define
\[
\tilde{T}_{\alpha} \left( \sum_{k=0}^{m} \tilde{V}^*k h_k \right) = \sum_{k=0}^{m} \tilde{V}^*k T_{\alpha} h_k,
\]
for $\alpha \in A, m \geq 0, h_0, \ldots, h_m \in \mathcal{H}$. The definition is correct as
\[
\left\| \sum_{k=0}^{m} \tilde{V}^*k T_{\alpha} h_k \right\| = \left\| \sum_{k=0}^{m} \tilde{V}^{m-k} T_{\alpha} h_k \right\| = \left\| \sum_{k=0}^{m} V^{m-k} T_{\alpha} h_k \right\| =
\]
\[
= \left\| T_{\alpha} \sum_{k=0}^{m} V^{m-k} h_k \right\| \leq \left\| T_{\alpha} \right\| \left\| \sum_{k=0}^{m} \tilde{V}^{m-k} h_k \right\| \leq \left\| T_{\alpha} \right\| \left\| \sum_{k=0}^{m} \tilde{V}^*k h_k \right\|.
\]

(1)
Clearly, $\tilde{T}_\alpha$ can be extended to $\mathcal{K}$ in such a way that

$$||\tilde{T}_\alpha|| = ||T_\alpha||, \quad \tilde{T}_\alpha \tilde{V} = \tilde{V} \tilde{T}_\alpha, \quad \tilde{T}_\alpha \tilde{T}_\beta = \tilde{T}_\beta \tilde{T}_\alpha \quad (\alpha, \beta \in A).$$

If $T_\alpha$ is an isometry then the equality in (1) holds and $\tilde{T}_\alpha$ is also an isometry. Analogously, if $T_\alpha$ is unitary then $\tilde{T}_\alpha$ is an isometry with range dense in $\mathcal{K}$ as $\tilde{T}_\alpha \mathcal{K} \supset \tilde{T}_\alpha \tilde{V}^* k \mathcal{H} = \tilde{V}^* k \mathcal{T}_\alpha \mathcal{H} = \tilde{V}^* k \mathcal{H}$ for any nonnegative integer $k$. Therefore $\tilde{T}_\alpha \mathcal{K} = \mathcal{K}$ and $\tilde{T}_\alpha$ is unitary.

**Corollary.** Let $V_\alpha$, $T \in B(\mathcal{H})$ be a commuting system of operators on a Hilbert space $\mathcal{H}$, $V_\alpha$ isometries ($\alpha \in A$). Then there exist commuting extensions $\tilde{V}_\alpha, \tilde{T} \in B(\mathcal{K})$ on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $\tilde{V}_\alpha | \mathcal{H} = V_\alpha$, $\tilde{T} | \mathcal{H} = T$, and $\tilde{V}_\alpha$ are unitary for any $\alpha \in A$.

**Proof.** Take a good ordering $\{\alpha_1, \alpha_2, \ldots\}$ of $A$. Using the previous Lemma we construct the space $\mathcal{K}$ and operators $\tilde{V}_\alpha, \tilde{T} \in B(\mathcal{K})$ by the transfinite induction.

Note that the Hilbert space $\mathcal{K}$ constructed above depends on isometries $V_\alpha$ only, not on $T$.

Let $\mathbf{V} = (V_\alpha)_{\alpha \in A}$ be a commuting system of isometries on a Hilbert space $\mathcal{H}$, let $T \in \text{AlgLat} \mathbf{V}$ and let $\tilde{\mathbf{V}} = (\tilde{V}_\alpha)_{\alpha \in A}, \tilde{T}$ be the extensions to the Hilbert space $\mathcal{K} \supset \mathcal{H}$ constructed above. Let $E(\cdot)$ be the spectral measure of the commuting system of unitary operators $\tilde{V}_\alpha$ ($\alpha \in A$) ($E$ is the projection-valued function on the Borel subsets of $A$). For $x \in \mathcal{H}$ let us denote by $\mathcal{Z}_+(x)$ ($\mathcal{Z}(x)$) the smallest subspace containing $x$ which is invariant (reducing) with respect to all $\tilde{V}_\alpha$ ($\alpha \in A$).

Clearly, $\mathcal{Z}_+(x) \subset \mathcal{Z}(x)$, and $\mathcal{Z}_+(x)$ is the closure in $\mathcal{H}$ of all $p(\mathbf{V})x$, $p \in \mathcal{P}$, where $\mathcal{P}$ is set of all polynomials with $|A|$ commuting variables.

As $T \in \text{AlgLat} \mathbf{V}$, $T \mathcal{Z}_+(x) \subset \mathcal{Z}_+(x)$. The extensions $\tilde{V}_\alpha$ are unitary on $\mathcal{K}$, hence $\tilde{T}$ commutes with all $\tilde{V}_\alpha, \tilde{V}_\alpha^*$ ($\alpha \in A$). It follows that

$$\tilde{T} \mathcal{Z}(x) \subset \mathcal{Z}(T x) \subset \mathcal{Z}(x).$$

Further let us denote $\mu_x = ||E(\cdot)x||^2$ the positive scalar measure (spectral measure) corresponding to $x \in \mathcal{H}$.

**Lemma 4.** If $x, y \in \mathcal{H}$ then there exists a complex number $\lambda$ such that the measures $\mu_x \vee \mu_y$ and $\mu_{x+\lambda y}$ are equivalent (i.e. absolutely continuous with respect to each other).
Proof. Let us denote \( \mu = \mu_x \vee \mu_y \). As

\[
\mu_{x+\lambda y}(B) = \|E(B)(x + \lambda y)\|^2 = \|E(B)x + \lambda E(B)y\|^2
\]

for any complex \( \lambda \) and a Borel subset \( B \subset \mathbb{T}^A \), \( \mu_{x+\lambda y} \ll \mu \). Hence there exists a measurable function (the spectral density) \( f_\lambda \in L^1(\mu) \) such that \( d\mu_{x+\lambda y} = f_\lambda \, d\mu \).

If \( C_\lambda = \{z \in \mathbb{T}^A : f_\lambda(z) = 0\} \) denotes the set of zeros of \( f_\lambda \) then

\[
\mu_{x+\lambda y}(C_\lambda) = \int_{C_\lambda} f_\lambda \, d\mu = 0.
\]

To obtain the equivalence \( \mu \sim \mu_{x+\lambda y} \) for some \( \lambda \in \mathbb{C} \), it is sufficient to prove that \( \mu(C_\lambda) = 0 \). If \( C_\lambda, C_\kappa \) are the corresponding zero sets for two different complex numbers \( \lambda \neq \kappa \), for \( C = C_\lambda \cap C_\kappa \) it holds

\[
0 \leq \mu_{x+\lambda y}(C) \leq \mu_{x+\lambda y}(C_\lambda) = 0,
\]

and analogously \( \mu_{x+\kappa y}(C) = 0 \), which implies that

\[
E(C)(x + \lambda y) = E(C)(x + \kappa y) = 0,
\]

\[ (\lambda - \kappa)E(C)y = 0, \]

and as \( \lambda - \kappa \neq 0 \) we get \( E(C)y = E(C)x = 0 \), i.e. \( \mu_x(C) = \mu_y(C) = 0 \), hence \( \mu(C) = 0 \). Summing up these equalities we obtain that

\[
\mu(C_\lambda \cup C_\kappa) = \mu(C_\lambda) + \mu(C_\kappa).
\]

The last equality implies that there could be only countable number of those \( \lambda \in \mathbb{C} \) for which \( \mu(C_\lambda) \neq 0 \), which proves the existence of the desired complex \( \lambda \).

For any \( x \in \mathcal{H} \) the restriction \( \widetilde{T}|_{\mathcal{Z}(x)} \) is unitarily equivalent to the multiplication \( M_{t_x} \) on the space \( L^2(\mu_x) \) by some function \( t_x \in L^\infty(\mu_x) \). The equivalence is given by the unitary operator

\[
\Phi_x : \mathcal{Z}(x) \to L^2(\mu_x),
\]

with

\[
\Phi_x V_\alpha = M_{z_\alpha} \Phi_x, \quad \Phi_x \widetilde{T} = M_{t_x} \Phi_x, \quad \Phi_x x = 1.
\]

Hence, the operator \( \widetilde{T} \in B(\mathcal{K}) \) can be viewed as \( t_x(\widetilde{V}) \) on \( \mathcal{Z}(x) \). Moreover, we may suppose that

\[
\|t_x\|_\infty = \sup_{z \in \mathbb{T}^A} |t_x(z)| \leq \|\tilde{T}\| = \|T\|.
\]

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Lemma 5. Let \( x, y \in \mathcal{H} \) be any vectors in a Hilbert space \( \mathcal{H} \) such that the corresponding spectral measures satisfy \( \mu_x < \mu_y \). Then

\[
t_x = t_y \quad \mu_x\text{-a.e.}
\]

Proof. Let \( \varepsilon > 0 \) and let \( f \in L^1(\mu_y) \) satisfy \( d\mu_x = f\,d\mu_y \). For given \( \delta > 0 \) let us denote \( N_\delta = \{ z : |f(z)| \geq \delta \} \). From the inclusion \( T(x + y) \in Z_+(x + y) = \{ p(\tilde{V})(x + y) \} \) we can deduce that there exists a polynom \( p \) such that

\[
\|(T - p(\tilde{V}))(x + y)\| < \frac{\varepsilon^3 \sqrt{\delta}}{4\|T\|}.
\]

Denoting by \( z = (T - p(\tilde{V}))(x + y) = (t_y - p)(\tilde{V})y + (t_x - p)(\tilde{V})x \) we obtain

\[
\begin{align*}
\|Tz\| &= \|t_y(t_y - p)(\tilde{V})y + t_x(t_x - p)(\tilde{V})x\| < \frac{\varepsilon^3 \sqrt{\delta}}{4}, \\
\|t_y(\tilde{V})z\| &= \|t_y(t_y - p)(\tilde{V})y + t_y(t_x - p)(\tilde{V})x\| < \sup_{z \in T^A} |t_y(z)| \frac{\varepsilon^3}{4\sqrt{\delta}\|T\|} \\
&\leq \frac{\varepsilon^3 \sqrt{\delta}}{4}, \\
\|t_x(\tilde{V})z\| &= \|t_x(t_y - p)(\tilde{V})y + t_x(t_x - p)(\tilde{V})x\| < \frac{\varepsilon^3 \sqrt{\delta}}{4}.
\end{align*}
\]

(1) \( \|Tz\| < \frac{\varepsilon^3 \sqrt{\delta}}{4} \),

(2) \( \|t_x(\tilde{V})z\| < \frac{\varepsilon^3 \sqrt{\delta}}{4} \),

(3) \( \|t_y(\tilde{V})z\| < \frac{\varepsilon^3 \sqrt{\delta}}{4} \).

By substracting (2) – (1) and (1) – (3) we have

\[
\begin{align*}
\|(t_y - t_x)(t_y - p)(\tilde{V})x\| &< \frac{\varepsilon^3 \sqrt{\delta}}{2}, \\
\|(t_y - t_x)(t_y - p)(\tilde{V})y\| &< \frac{\varepsilon^3 \sqrt{\delta}}{2},
\end{align*}
\]

which means that

\[
\begin{align*}
\|(t_y - t_x)(t_x - p)\|_{L^2(\mu_x)} &= \|(t_y - t_x)(t_x - p)|f|^{1/2}\|_{L^2(\mu_y)} < \frac{\varepsilon^3 \sqrt{\delta}}{2}, \\
\|(t_y - t_x)(t_y - p)\|_{L^2(\mu_y)} < \frac{\varepsilon^3 \sqrt{\delta}}{2} \leq \frac{\varepsilon^3}{2}
\end{align*}
\]

hence

\[
\|(t_y - t_x)(t_x - p)|N_\delta\|_{L^2(\mu_y)} \leq \|(t_y - t_x)(t_x - p)|f|^{1/2}|N_\delta\|_{L^2(\mu_y)} \cdot \delta^{-1/2} < \frac{\varepsilon^3}{2}
\]

\[
\|(t_y - t_x)(t_x - p)|N_\delta\|_{L^2(\mu_y)} < \frac{\varepsilon^3}{2},
\]

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which by subtracting results into

\[\|(t_y - t_x)^2\|_{L^2(\mu_y)} < \varepsilon^3\]

and

\[\mu_y\left\{z \in N_\delta : |(t_y - t_x)(z)| \geq \varepsilon\right\} \leq \varepsilon.\]

As \(\varepsilon\) was arbitrary we have obtained

\[\mu_y\left\{z \in N_\delta : t_y(z) \neq t_x(z)\right\} = 0,

and

\[\mu_x\left\{z \in T^A : t_y(z) \neq t_x(z)\right\} \leq \sum_{n=1}^\infty \mu_x\left\{z \in N_{1/n} : t_y(z) \neq t_x(z)\right\} = 0.\]

\[\square\]

**Theorem 2.** Let \(V = (V_\alpha)_{\alpha \in A}\) be a commuting system of isometries on a Hilbert space \(\mathcal{H}\). Then \(\text{AlgLat } V \subset V''\).

**Proof.** Let \(T \in \text{AlgLat } V\) and let \(S \in B(\mathcal{H})\) be any operator in commutant of \((V_\alpha)_{\alpha \in A}\), \(SV_\alpha = V_\alpha S\) (\(\alpha \in A\)). Using Corollary of Lemma 3 we can construct Hilbert space \(\mathcal{K} \supset \mathcal{H}\) and operators \(\tilde{V}_\alpha, \tilde{S}\) and \(\tilde{T}\) such that \(\tilde{V}_\alpha \tilde{S} = \tilde{S} \tilde{V}_\alpha, \tilde{V}_\alpha |\mathcal{H} = V_\alpha, \tilde{S}|\mathcal{H} = S, \tilde{T}|\mathcal{H} = T\).

Let \(x \in \mathcal{H}\) be arbitrary, and \(y\) be a linear combination (which exists by Lemma 4) of vectors \(x\) and \(Sx\) such that \(\mu_x < \mu_y, \mu_{Sx} < \mu_y\). Then \(\tilde{T}|\mathcal{Z}(x) = t_x(\tilde{V})|\mathcal{Z}(x) = t_y(\tilde{V})|\mathcal{Z}(x)\) and \(\tilde{T}|\mathcal{Z}(Sx) = t_{Sx}(\tilde{V})|\mathcal{Z}(Sx) = t_y(\tilde{V})|\mathcal{Z}(Sx)\) by the previous result.

It follows that

\[STx = \tilde{S}T\tilde{x} = \tilde{S}t_y(\tilde{V})x = t_y(\tilde{V})\tilde{S}x = t_y(\tilde{V})Sx = \tilde{T}Sx = TSx,\]

therefore

\[ST = TS, \quad T \in (V_\alpha)''.\]

\[\square\]

**Theorem 3.** Let \((V_\alpha)_{\alpha \in A}\) be a commuting system of isometries on a Hilbert space \(\mathcal{H}\) and suppose that \(T \in \text{AlgLat } V\). Then \(\tilde{T}\), the extension of \(T\) defined in Corollary of Lemma 3, belongs to the double commutant of \(\tilde{V} = (\tilde{V}_\alpha)_{\alpha \in A}\), the minimal extension of \(V\).
Proof. Let $\tilde{S} \in B(\mathcal{K})$ be any operator in commutant of $(\tilde{V}_\alpha)_{\alpha \in A}$, $\tilde{S} \tilde{V}_\alpha = \tilde{V}_\alpha \tilde{S}$ ($\alpha \in A$), and let $u \in \mathcal{K}$ be any vector of $\mathcal{K}$. Let $\varepsilon > 0$ be given. As $\mathcal{K} = \bigvee_{x \in \mathcal{H}} \mathcal{Z}(x)$, one can find vectors $x_1, \ldots, x_n \in \mathcal{H}$, $x'_1, \ldots, x'_m \in \mathcal{H}$, and vectors $u_1, \ldots, u_n \in \mathcal{K}$, $u'_1, \ldots, u'_m \in \mathcal{K}$ such that

$$
\left\| u - \sum_{i=1}^n u_i \right\| < \varepsilon, \quad \left\| \tilde{S}u - \sum_{i=1}^m u'_i \right\| < \varepsilon,
$$

$$
u_i \in \mathcal{Z}(x_i) \ (1 \leq i \leq n), \quad u'_i \in \mathcal{Z}(x'_i) \ (1 \leq i \leq m).
$$

By Lemma 4 and 5 there exists a function $f \in L^\infty(\mu)$, where $\mu = \bigvee_{i=1}^n \mu_{x_i} \bigvee_{i=1}^m \mu_{x'_i}$ such that $\tilde{T}|\mathcal{Z}_{x_i} = f(\tilde{V})|\mathcal{Z}_{x_i} \ (i = 1, \ldots, n)$ and $\tilde{T}|\mathcal{Z}_{x'_i} = f(\tilde{V})|\mathcal{Z}_{x'_i} \ (i = 1, \ldots, m)$. Then

$$
\| \tilde{S}\tilde{T}u - \tilde{T}\tilde{S}u \| \leq \| \tilde{S}\tilde{T}u - \tilde{S}\tilde{T} \sum_{i=1}^n u_i \| + \| \tilde{S}\tilde{T} \sum_{i=1}^n u_i - \tilde{T} \sum_{i=1}^m u'_i \| + \| \tilde{T} \sum_{i=1}^m u'_i - \tilde{T}\tilde{S}u \|
\leq \| \tilde{S} \| \| \tilde{T} \| \left\| u - \sum_{i=1}^n u_i \right\| + \| \tilde{S}f(\tilde{V}) \sum_{i=1}^n u_i - f(\tilde{V}) \sum_{i=1}^m u'_i \| + \| \tilde{T} \| \| \sum_{i=1}^m u'_i - \tilde{S}u \|
\leq \| \tilde{S} \| \| \tilde{T} \| \varepsilon + \| f(\tilde{V}) \| \left( \| \tilde{S} \sum_{i=1}^n u_i - \tilde{S}u \| + \| \tilde{S}u - \sum_{i=1}^m u'_i \| \right) + \| \tilde{T} \| \varepsilon
\leq 2\| \tilde{S} \| \| \tilde{T} \| \varepsilon + 2\| \tilde{T} \| \varepsilon.
$$

As $\varepsilon$ was arbitrary we have $\tilde{S}\tilde{T}u = \tilde{T}\tilde{S}u$, i.e. $\tilde{S}\tilde{T} = \tilde{T}\tilde{S}$ and $\tilde{T} \in (\tilde{V}_\alpha)''$. \qed

References


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