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LOCAL PROPERTIES AND UPPER EMBEDDABILITY OF CONNECTED MULTIGRAPHS

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This note has been motivated by the following three theorems:

Theorem A (Glukhov [3]). *If G is a 2-connected multigraph with the property that each edge of G belongs to a cycle of length 2 or 3, then G is upper embeddable.*

Theorem B (Nebeský [8]). *If G is a connected, locally quasiconnected graph, then G is upper embeddable.*

Theorem C (Nebeský [9]). *If G is a connected, N_2-locally connected graph, then G is upper embeddable.*

In this note we will give a common generalization of Theorems A, B, and C.

Let G be a multigraph (in the sense of [1], for example) with a vertex set $V(G)$ and an edge set $E(G)$. We say that a multigraph F is a submultigraph of G if $V(F) \subseteq V(G)$, $E(F) \subseteq E(G)$ and the implication

if a vertex u and an edge e are incident in F, then they are incident in G

holds for any $u \in V(F)$ and $e \in E(F)$. If $W \subseteq V(G)$, $W \neq \emptyset$, then we say that H is the submultigraph of G induced by W if H is a submultigraph of G, $V(H) = W$, and each edge of G incident only with vertices in W belongs to H. If $A \subseteq E(G)$, then we say that H' is the submultigraph of G induced by A if H' is a submultigraph of G, $E(H') = A$, and a vertex of G belongs to H' if and only if it is incident with an edge in A. Let $u \in V(G)$; we denote by $V(u, G)$ the set of all vertices adjacent to u in G; moreover, we denote by $E(u, G)$ the set of all edges e in G with the properties that e is not incident with u but e is incident with a vertex adjacent to u in G; if $V(u, G) \neq \emptyset$, then we denote by $N(u, G)$ the submultigraph of G induced by

$V(u, G)$; finally, if $E(u, G) \neq \emptyset$, then we denote by $N_2(u, G)$ the submultigraph of G induced by $E(u, G)$. We say that G is locally connected if $V(v, G) \neq \emptyset$ and $N(v, G)$ is connected for each $v \in V(G)$. We say that G is locally quasiconnected if at least one of the multigraphs $N(v_1, G)$ and $N(v_2, G)$ is connected, for each pair of adjacent vertices v_1 and v_2 of G. Finally, we say that G is N_2-locally connected if $E(v, G) \neq \emptyset$ and $N_2(v, G)$ is connected, for each $v \in V(G)$. Clearly, if G is locally connected and no component of G has less than 3 vertices, then G is both locally quasiconnected and N_2-locally connected.

For locally connected graphs, locally quasiconnected graphs, or N_2-locally connected graphs, see [2], [8], or [11], respectively. Recall that a multigraph is a graph if and only if it has no parallel edges.

Figure 1 shows three examples of connected graphs: G_1 is 2-connected, each edge of G_1 belongs to a triangle; as we can see, G_1 is neither locally quasiconnected nor N_2-locally connected. G_2 is locally quasiconnected; it is neither 2-connected nor N_2-locally connected. G_3 is N_2-locally connected; it is not locally quasiconnected (and, of course, it contains no triangle).

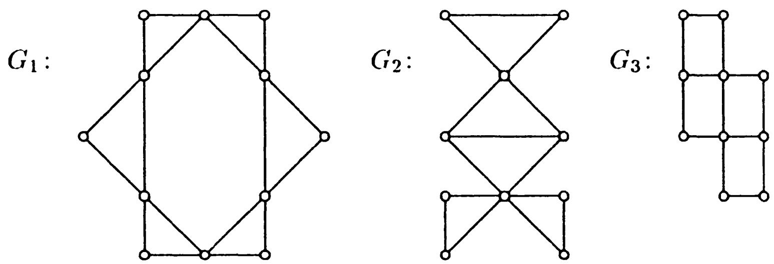

Fig. 1

If F is a multigraph with $|V(F)| \geqslant 2$ and $u \in V(F)$, then we say that u is a cut-vertex of F if the multigraph $F - u$ has more components than F has. We shall introduce the main notion of the present note. We say that a multigraph G is *interlaced* if the following two conditions hold:

(i) if no cycle of length 2 or 3 is passing through an edge incident both with u and with v, then both $N_2(u, G)$ and $N_2(v, G)$ are connected, for each pair of adjacent vertices u and v of G;

(ii) $N(w, G)$ is connected, for each vertex w which is adjacent to a cut-vertex of G.

Let G be a connected multigraph with $|V(G)| \geqslant 3$. It is easy to see that if G is locally quasiconnected, then each edge of G belongs to a triangle. Thus, it is clear that if either (a) G has no cut-vertex and each edge of G belongs to a cycle of length 2 or 3, or (b) G is locally quasiconnected, or (c) G is N_2-locally connected with no

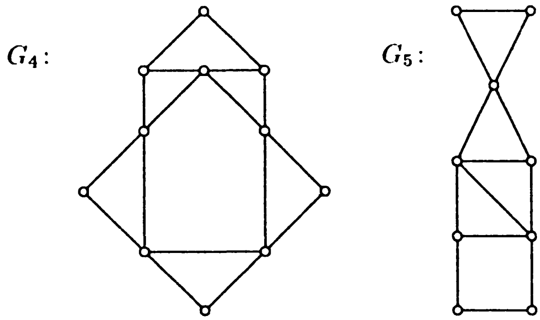

Fig. 2

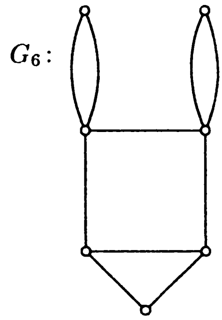

Fig. 3

cut-vertex, then G is interlaced. Fig. 2 shows two examples of connected interlaced graphs fulfilling none of the conditions (a), (b), (c).

In the present note we shall prove that every connected interlaced multigraph is upper embeddable. As a step to this result we shall prove a theorem on a certain global property of connected interlaced multigraphs.

Let G be a multigraph. Consider a partition $\mathscr{P}$ of $V(G)$. Let $\mathscr{R} \subseteq \mathscr{P}$; we denote by $E_{\mathscr{R}}(G)$ the set of all $e \in E(G)$ with the property that the vertices incident with e in G belong to two distinct elements of $\mathscr{R}$; the submultigraph of G induced by

$$\bigcup_{R \in \mathscr{R}} R$$

will be denoted by $G(\mathscr{R})$. If

$$|P| \geqslant 2 \quad \text{and} \quad G(\{P\}) \text{ is connected for each } P \in \mathscr{P},$$

then $\mathscr{P}$ will be referred to as a C-partition of G.

The following theorem is a generalization of Theorem 1 in [9].

Theorem 1. *Let G be a connected interlaced multigraph. Then*

$$|E_{\mathscr{P}}(G)| \geqslant 2(|\mathscr{P}| - 1) \tag{1}$$

for every C-partition of G.

Proof. If $|V(G)| = 1$, then the statement of the theorem holds trivially. Let $|V(G)| \geqslant 2$. Then there exists a C-partition of G. Consider a C-partition $\mathscr{P}$ of G. We proceed by induction on $|\mathscr{P}|$. If $|\mathscr{P}| = 1$, then $E_{\mathscr{P}}(G) = \emptyset$, and thus (1) holds. Assume that $|\mathscr{P}| \geqslant 2$. We distinguish two cases.

1. Assume that there exist distinct $P_1, P_2 \in \mathscr{P}$ such that

$$|E_{\{P_1,P_2\}}(G)| \geqslant 2.$$

Denote $P' = P_1 \cup P_2$ and $\mathscr{P}' = (\mathscr{P} - \{P_1, P_2\}) \cup \{P'\}$. Obviously, $\mathscr{P}'$ is a C-partition of G. Since $|\mathscr{P}'| = |\mathscr{P}| - 1$, it follows from the induction hypothesis that

$$|E_{\mathscr{P}'}(G)| \geqslant 2(|\mathscr{P}'| - 1) = 2(|\mathscr{P}| - 1) - 2.$$

Since $|E_{\mathscr{P}'}(G)| \leqslant |E_{\mathscr{P}}(G)| - 2$, we get (1).

2. Assume that

$$\text{(2)} \qquad |E_{\{P^*,P^{**}\}}(G)| \leqslant 1 \quad \text{for any distinct} \quad P^*, P^{**} \in \mathscr{P}.$$

As follows from (2), no edge in $E_{\mathscr{P}}(G)$ belongs to a cycle of length 2 (i.e. no edges in $E_{\mathscr{P}}(G)$ are parallel). If $e \in E_{\mathscr{P}}(G)$ and u and v are the vertices incident with e, then for the sake of simplicity we will write $e = uv$.

We first assume that there exists a cut-vertex u of G incident with an edge in $E_{\mathscr{P}}(G)$. Then there exists $v \in V(G)$ such that $v \neq u$ and $uv \in E_{\mathscr{P}}(G)$. As follows from definition of an interlaced multigraph, $N(v, G)$ is connected. Let P_v denote the element of $\mathscr{P}$ containing v. Since v is incident with an edge in $E_{\mathscr{P}}(G)$ and $|P_v| \geqslant 2$, we can see that there exist $w_1, w_2 \in N(v, G)$ such that $w_1 \in P_v$, $w_2 \notin P_v$, and $w_1 w_2 \in E_{\mathscr{P}}(G)$. Since $w_2 \in N(v, G)$, we have that w_2 is adjacent to v in G. Let P' denote the element of $\mathscr{P}$ containing w_2. Since $v \neq w_1$, we get that $|E_{\{P_v,P'\}}(G)| \geqslant 2$, which is a contradiction with (2). Thus, no cut-vertex of G is incident with an edge in $E_{\mathscr{P}}(G)$.

Consider an arbitrary $P \in \mathscr{P}$ and an arbitrary $u \in P$ such that u is incident with an edge e in $E_{\mathscr{P}}(G)$. It follows from the definition of an interlaced multigraph that either e belongs to a triangle or $N_2(u, G)$ is connected. Clearly, $|P| \geqslant 2$ and u is not a cut-vertex of G. Thus, we can derive from (2) that there exist distinct P_1, $P_2 \in \mathscr{P} - \{P\}$ and u_1, u_2, $v \in V(G)$ such that $v \in P$, $u_1 \in P_1$, $u_2 \in P_2$ and uu_1, u_1u_2, $u_2v \in E_{\mathscr{P}}(G)$ (note that the case when $v = u$ is possible). This observation can be summarized as follows:

(3) for every $P \in \mathscr{P}$ and every $u \in P$ such that u is incident with an edge in $E_{\mathscr{P}}(G)$ there exist distinct $P_1, P_2 \in \mathscr{P} - \{P\}$ and $u_1, u_2, v \in V(G)$ such that $v \in P$, $u_1 \in P_1$, $u_2 \in P_2$ and $uu_1, u_1u_2, u_2v \in E_{\mathscr{P}}(G)$.

We will construct sets $\mathscr{R}^m$, X^m and Y^m for every integer $m \geqslant 1$. We will proceed by induction on m.

Consider an arbitrary $P^1 \in \mathscr{P}$. Since $|\mathscr{P}| \geqslant 2$ and G is connected, there exists $u^1 \in P^1$ such that u^1 is incident with an edge in $E_{\mathscr{P}}(G)$. According to (3), we can find distinct $P_1^1, P_2^1 \in \mathscr{P} - \{P^1\}$ and vertices $u_1^1 \in P_1^1$, $u_2^1 \in P_2^1$, $v^1 \in P^1$ such that $u^1u_1^1, u_1^1u_2^1, u_2^1v^1 \in E_{\mathscr{P}}(G)$. Denote $\mathscr{R}^1 = \{P^1, P_1^1, P_2^1\}$, $X^1 = \{u_1^1u_2^1, u_2^1v^1\}$ and $Y^1 = \{u^1u_1^1\}$.

Let $m \geqslant 2$. Assume that the sets $\mathscr{R}^{m-1}$, X^{m-1}, and Y^{m-1} have been constructed. We first assume that there exists $P^m \in \mathscr{R}^{m-1} - \{P^1\}$ such that exactly one vertex in P^m, say a vertex w^m, is incident with an edge in $X^{m-1} \cup Y^{m-1}$. Since w^m is not a cut-vertex of G, there exists $u^m \in P^m - \{w^m\}$ such that u^m is incident with an edge in $E_{\mathscr{P}}(G)$. According to (3), there exist distinct $P_1^m, P_2^m \in \mathscr{P} - \{P^m\}$ and vertices $u_1^m \in P_1^m$, $u_2^m \in P_2^m$, $v^m \in P^m$ such that $u^mu_1^m, u_1^mu_2^m, u_2^mv^m \in E_{\mathscr{P}}(G)$. We put $\mathscr{R}^m = \mathscr{R}^{m-1} \cup \{P_1^m, P_2^m\}$. We denote by $\mathscr{S}^m$ the set of all $P \in \mathscr{R}^{m-1} - \{P^1\}$ such that exactly one vertex of P is incident with an edge in $X^{m-1} \cup Y^{m-1}$. If $P \in \mathscr{S}^m$, then we denote by $w^m(P)$ the vertex of P which is incident with an edge in $E_{\mathscr{P}}(G)$. Obviously, $P^m \in \mathscr{S}^m$ and $w^m = w^m(P^m)$. We put $X^m = X^{m-1} \cup X$ and $Y^m = Y^{m-1} \cup \{u^mu_1^m\} \cup Y$, where

$$\begin{aligned}
&X \subseteq \{u_1^mu_2^m, u_2^mv^m\}, \quad Y \subseteq \{u_1^mu_2^m, u_2^mv^m\} - X,\\
&u_1^mu_2^m \in X \quad \text{if and only if} \quad P_1^m \notin \mathscr{R}^{m-1},\\
&u_2^mv^m \in X \quad \text{if and only if} \quad P_2^m \notin \mathscr{R}^{m-1},\\
&u_1^mu_2^m \in Y \quad \text{if and only if} \quad P_1^m \in \mathscr{S}^m \quad \text{and} \quad u_1^m \neq w^m(P_1^m), \quad \text{and}\\
&u_2^mv^m \in Y \quad \text{if and only if} \quad P_2^m \in \mathscr{S}^m \quad \text{and} \quad u_2^m \neq w^m(P_2^m).
\end{aligned}$$

Clearly, $Y^m - Y^{m-1} \neq \emptyset$.

We shall now assume that there exists no $P \in \mathscr{R}^{m-1} - \{P^1\}$ such that exactly one vertex in P is incident with an edge in $X^{m-1} \cup Y^{m-1}$. We put $\mathscr{R}^m = \mathscr{R}^{m-1}$, $X^m = X^{m-1}$ and $Y^m = Y^{m-1}$. Moreover, we denote $\mathscr{S}^m = \emptyset$.

Since $E_{\mathscr{P}}(G)$ is finite, we see that there exists an integer $n > 1$ such that $Y^n - Y^{n-1} \neq \emptyset$ and $Y^{n+1} = Y^n$. Obviously, $Y^{n+j} = Y^n$ for every integer $j \geqslant 1$. We put $\mathscr{R} = \mathscr{R}^n$. By the construction we get

$$\begin{aligned}
&X^k \cap Y^k = \emptyset, \quad |X^k| = |\mathscr{R}^k| - 1, \quad |Y^k| = |\mathscr{R}^k| - |\mathscr{S}^{k+1}|,\\
&X^k \cup Y^k \subseteq E_{\mathscr{R}^k}(G), \quad \text{and} \quad G(\mathscr{R}^k) \text{ is connected}
\end{aligned}$$

for every integer k, $1 \leqslant k \leqslant n$. Hence, $X^n \cap Y^n = \emptyset$, $|X^n| = |\mathscr{R}| - 1$ and $|Y^n| = |\mathscr{R}|$. Thus, we have obtained that

$$|E_{\mathscr{R}}(G)| \geqslant 2|\mathscr{R}| - 1. \tag{4}$$

Denote

$$P_0 = \bigcup_{P \in \mathscr{R}} P$$

and $\mathscr{P}_0 = (\mathscr{P} - \mathscr{R}) \cup \{P_0\}$. It is obvious that $\mathscr{P}_0$ is a C-partition of G. Since $|\mathscr{P}_0| < |\mathscr{P}|$, it follows from the induction hypothesis that

$$(5) \qquad |E_{\mathscr{P}_0}(G)| \geqslant 2(|\mathscr{P}_0| - 1).$$

Clearly, $E_{\mathscr{P}}(G) = E_{\mathscr{P}_0}(G) \cup E_{\mathscr{R}}(G)$, $E_{\mathscr{P}_0}(G) \cap E_{\mathscr{R}}(G) = \emptyset$ and $|\mathscr{P}| = |\mathscr{P}_0| - 1 + |\mathscr{R}|$. Combining (4) and (5), we get that $|E_{\mathscr{P}}(G)| \geqslant 2|\mathscr{P}| - 1$. Thus, (2) holds. The proof is complete. □

The upper embeddability belongs to central notions in the study of the maximum genus of a pseudograph; cf. [12] or Chapter 5 in [1]. (Note that a pseudograph is a multigraph if and only if it is loopless). Let G be a connected pseudograph. If there exists a 2-cell embedding of G into the closed orientable surface of genus

$$\left[\tfrac{1}{2}(|E(G)| - |V(G)| + 1)\right],$$

then G is called upper embeddable.

Let H be a pseudograph. We denote by $b(H)$ the number of components F of H such that $|E(H)| - |V(H)|$ is even. Moreover, we denote by $c(H)$ the number of all components of H.

We shall need the following theorem:

Theorem D. *If G is a connected pseudograph, then the statements* (6), (7) *and* (8) *are equivalent:*

(6) *G is upper embeddable;*

(7) *there exists a spanning tree T of G such that at most one component of $G - E(T)$ has an odd number of edges;*

(8) $|A| \geqslant b(G - A) + c(G - A) - 2$ *for every* $A \subseteq E(G)$.

The equivalence (6) $\Leftrightarrow$ (7) was proved independently in [4], [6] and [13]; the equivalence (7) $\Leftrightarrow$ (8) was proved independently in [3] and [7]. (However, the results in [3] and [4] were formulated rather differently.)

The following theorem can be proved in the same way as Theorem 2 in [9].

Theorem 2. *Let G be a connected interlaced multigraph. Then G is upper embeddable.*

P r o o f (outlined). There exists $A \subseteq E(G)$ such that

$$b(G - A) + c(G - A) - 2 - |A| \geqslant b(G - A') + c(G - A') - 2 - |A'|$$

for every $A' \subseteq E(G)$ and

$$b(G-A)+c(G-A)-2-|A| > b(G-A'')+c(G-A'')-2-|A''|$$

for every $A'' \subseteq E(G)$ such that $|A''| < |A|$. It is not difficult to show that there exists a C-partition $\mathscr{P}$ of G such that $A = E_{\mathscr{P}}(G)$. As follows from Theorem 1, $|A| \geqslant 2(c(G-A)-1)$. Clearly, $2(c(G-A)-1) \geqslant b(G-A)+c(G-A)-2$. The result of the theorem can be derived from the implication (8) $\Rightarrow$ (6). □

It is clear that Theorems A and B are consequences of Theorem 2. The following corollary of Theorem 2 is a common generalization of Theorems A, B and C. The corollary can be easily derived from Theorem 2 by the equivalence (6) $\Leftrightarrow$ (7).

Corollary 1. *Let G be a connected multigraph, and let W be a nonempty subset of $V(G)$. Assume that the submultigraph of G induced by W is connected and interlaced, and that either $W = V(G)$ or $G - W$ is a forest. Then G is upper embeddable.*

If G is connected, N_2-locally connected graph, then–as was shown in [11]–at most one block of G contains a cycle. This is not true for multigraphs; the multigraph G_6 in Fig. 3 is an example of a connected N_2-locally connected multigraph with three blocks, each of them containing a cycle. It is not difficult to show that if G is a connected, N_2-locally connected multigraph such that exactly one block H of G contains a cycle, then H is interlaced. Combining this observation with Corollary 1, we get the following result:

Corollary 2. *Let G be a connected, N_2-locally connected multigraph. If at most one block of G contains a cycle, then G is upper embeddable.*

Note that multigraph G_6 in Fig. 3 is not upper embeddable.

Remark. The subject of the present note is not too far from the subject of the paper [10]. Nedela and Škoviera [10] proved that if G is a connected multigraph such that there exists a 2-cell embedding of G in a closed surface with the property that the length of no face is greater than 4, then G is upper embeddable. In June 1991, the author of the present note was informed by Nedela and Škoviera that if G is a graph (i.e. a multigraph with no parallel edges), then the result of [10] can be derived from Theorem C.

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