Amiran Gogatishvili; Luboš Pick
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WEAK AND EXTRA-WEAK TYPE INEQUALITIES FOR THE MAXIMAL OPERATOR AND THE HILBERT TRANSFORM

AMIRAN GOGATISHVILI, Tbilisi, and LUBOŠ PICK, Praha

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1. INTRODUCTION

Let $\Phi$ be a nondecreasing finite function on $[0, \infty)$, not vanishing identically and satisfying $\Phi(0) = 0$, let $\sigma, \varepsilon$ be appropriate measures in $\mathbb{R}^n$, and let $T$ be a homogeneous operator. The usual two-weight weak type inequality in $L^p$,

$$\varepsilon\{|Tf| > \lambda\} \leq C \lambda^{-p} \int |f(x)|^p \, d\sigma,$$

where $C$ is independent of $f$ and $\lambda > 0$, and $\{|Tf| > \lambda\}$ stands for $\{x \in \mathbb{R}^n; |Tf(x)| > \lambda\}$, has at least two different analogues when replacing $t^p$ by $\Phi(t)$:

(1) $$\varepsilon\{|Tf| > \lambda\} \cdot \Phi(\lambda) \leq C \int \Phi(C|f(x)|) \, d\sigma,$$

"weak type inequality", and

(2) $$\varepsilon\{|Tf| > \lambda\} \leq C \int \Phi(C|f(x)|/\lambda) \, d\sigma,$$

"extra-weak type inequality" (this terminology goes back to [18], for justification see Remark 1 and Remark 2).

We start with proving some simple preliminary results concerning $\Phi$ and related functions (Section 2), and use them in Section 3 to give a characterization of the couples of measures $(\sigma, \varepsilon)$ for which (1) or (2) hold with $T = M_\mu$, where $M_\mu$ is the Hardy-Littlewood maximal operator related to a doubling measure $\mu$ (cf. [6], [1], [2], [15], [17] and [18]). This characterization is slightly more general than that in [18],
where $\Phi$ is assumed to be a Young function. We also give a new direct proof of necessity of the condition for the extra-weak type inequality.

As a consequence we obtain in Section 4 a new general characterization for the $A_\infty$ condition, of independent interest, which sheds light onto the relationship between two conditions proved earlier by Hruščev [11] and Fujii [5].

The main results are the theorems in Section 5, which give necessary and sufficient conditions on a weight $w$ for the inequalities

$$w(\{H^* f > \lambda\}) \cdot \Phi(\lambda) \leq C \int_{-\infty}^{\infty} \Phi(C|f|)w,$$

and

$$w(\{H^* f > \lambda\}) \leq C \int_{-\infty}^{\infty} \Phi(C|f|\lambda^{-1})w$$

to hold, where $H^*$ is the maximal Hilbert transform. In the latter case $\Phi$ is assumed to satisfy the $A_2$ condition near zero.

Positive constants independent of the appropriate quantities are always denoted with $C$ and need not keep their value from line to line. Throughout we take $0 \cdot \infty$ to be zero.

2. THE FUNCTIONS $\Phi$, $\hat{\Phi}$, $R_\Phi$ AND $S_\Phi$

We define the complementary function to $\Phi$ by

$$\hat{\Phi}(t) = \sup_{s \geq 0} (st - \Phi(s)).$$

Clearly, $\hat{\Phi}(0) = 0$ and $\hat{\Phi}$ is nondecreasing. The subadditivity of supremum easily implies that $\hat{\Phi}$ is always convex. For any $\Phi$ we have $(\hat{\Phi})^{-} \leq \Phi$, equality holds if $\Phi$ itself is convex. If $\Phi_1 \leq \Phi_2$, then $\hat{\Phi}_2 \leq \hat{\Phi}_1$, and if $\Phi_1(t) = a\Phi(bt)$, $a, b > 0$, then

$$\hat{\Phi}_1(t) = a\hat{\Phi}(t/ab).$$

Moreover, the Young inequality $st \leq \Phi(s) + \hat{\Phi}(t)$ holds.

We say that $\Phi \in \Delta_2$ if $\Phi(2t) \leq C\Phi(t)$ for $t \geq 0$.

It is also worth to notice that unlike $\Phi$, the function $\hat{\Phi}$ may jump to infinity at some point $t > 0$. For example, if $\Phi(t) = t$, then $\hat{\Phi}(t) = \infty \cdot \chi_{(1,\infty)}(t)$. It can even be $\hat{\Phi} \equiv \infty$ everywhere on $(0, \infty)$ (put e.g. $\Phi(t) = \sqrt{t}$). We say that $\Phi$ is reasonable if there exists $t > 0$ such that $\hat{\Phi}(t) < \infty$. 

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We put
\[ R_\Phi(t) = \frac{\Phi(t)}{t} \quad \text{and} \quad S_\Phi(t) = \frac{\tilde{\Phi}(t)}{t}, \quad t \geq 0. \]

**Lemma 1.** The following statements are equivalent.

(i) The function \( \Phi \) is reasonable;

(ii) there exists \( \varepsilon > 0 \) such that \( S_\Phi \) is bounded on \([0, \varepsilon)\);

(iii) there exist \( C, T > 0 \) such that \( R_\Phi(t) \geq C \) for \( t \geq T \).

**Proof.** (i)⇒(iii). Suppose that (iii) is not true, i.e., there is a sequence \( t_n \to \infty \) such that \( R_\Phi(t_n) < 1/n \). Then for any \( t > 0 \)
\[ \hat{\Phi}(t) \geq \sup_{n \in \mathbb{N}} t_n(t - R_\Phi(t_n)) \geq \sup_{n \in \mathbb{N}} t_n(t - \frac{1}{n}) = \infty, \]
whence \( \Phi \) is not reasonable.

(iii)⇒(ii). Assume that (ii) is not valid; then there is a sequence \( t_n \to 0_+ \) such that \( S_\Phi(t_n) > n, n \in \mathbb{N} \). So, there exists another sequence, \( s_n \), such that \( nt_n < s_n(t_n - R_\Phi(s_n)) \). Obviously it must be \( s_n > n \) and \( R_\Phi(s_n) < t_n \), which contradicts (iii). The remaining implication is obvious.

The equivalence of (i) and (ii) says that once \( \hat{\Phi} \) is finite near zero, it is bounded by a linear function near zero, which might seem to be somewhat surprising. But it naturally corresponds to the fact that \( \hat{\Phi}(0) = 0 \) and \( \hat{\Phi} \) is convex.

We say that \( \Phi \) is quasiconvex if there exists a convex function \( \Phi_0 \) such that \( \Phi(t) \leq \Phi_0(t) \leq C\Phi(Ct), t \geq 0. \)

**Lemma 2.** ([10]) The following statements are equivalent.

(i) \( \Phi \) is quasiconvex;

(ii) there exists \( C > 0 \) such that for \( s \leq t \)
\[ \frac{\Phi(s)}{s} \leq C \frac{\Phi(Ct)}{t}; \]

(iii) there exists \( C > 0 \) such that for any cube \( Q \) and function \( f \)
\[ \Phi \left( \frac{1}{\mu(Q)} \int_Q |f(x)| \, d\mu(x) \right) \leq C \frac{1}{\mu(Q)} \int_Q \Phi(C|f(x)|) \, d\mu(x); \]

(iv) there exists \( C > 0 \) such that for all \( s, t > 0 \) and \( \alpha \in (0, 1) \) we have
\[ \Phi(\alpha s + (1 - \alpha)t) \leq C[\alpha \Phi(Cs) + (1 - \alpha)\Phi(Ct)]. \]
Let us recall that if \( \Phi \) itself is convex, then all the statements of Lemma 2 hold with \( C = 1 \). In particular, \( S_\Phi \) is always nondecreasing.

Analogously to quasiconvexity we can define quasiconcavity; then the corresponding counterpart lemma holds. We omit the details.

**Corollary 1.** A quasiconvex function is reasonable.

**Proof.** Let \( \Phi \) be quasiconvex. Then it follows from Lemma 2, (ii), that

\[
R_\Phi(t) \geq C^{-1}T^{-1}\Phi(C^{-1}T)
\]

for any \( 0 \leq T \leq t \). Taking a \( T \) so that \( \Phi(C^{-1}T) > 0 \), we get from Lemma 1 that \( \Phi \) is reasonable.

Let \( \Phi \) be quasiconvex. We say that \( \Phi \) is a Young’s function if \( \lim_{t \to 0^+} R_\Phi(t) = 0 \) and \( \lim_{t \to -\infty} R_\Phi(t) = \infty \). If \( R_\Phi(t) \leq C, \ t \geq 0 \), we say that \( \Phi \) is of bounded type near \( \infty \) (\( \Phi \in B_\infty \)). If \( R_\Phi(t) \geq C^{-1}, \ t > 0 \), we say that \( \Phi \) is of bounded type near \( 0 \) (\( \Phi \in B_0 \)).

**Lemma 3.** Let \( \Phi \) be convex. Then \( \Phi \in B_0 \) if, and only if, \( \tilde{\Phi} \equiv 0 \) near \( 0 \), and \( \Phi \in B_\infty \) if, and only if, \( \tilde{\Phi} \equiv \infty \) near \( \infty \).

**Proof.** Assume that \( \Phi \in B_\infty \), that is, \( R_\Phi(t) \leq C \). Then, clearly, for \( t > C \),

\[
\tilde{\Phi}(t) = \sup_{s > 0} s(t - R_\Phi(s)) = \infty.
\]

If \( \Phi \in B_0 \), that is, \( R_\Phi \geq C^{-1} \), then for \( t \leq C^{-1} \)

\[
\tilde{\Phi}(t) = \sup_{s > 0} s(t - R_\Phi(s)) = 0,
\]

since the expression in the brackets is negative.

Conversely, let \( \tilde{\Phi} \equiv 0 \) on \([0, \epsilon]\). Note that as \( \Phi \) is convex, we have \((\tilde{\Phi})^- = \Phi\).

Therefore,

\[
\Phi(t) = \max \{ \sup_{s \leq \epsilon} ts; \sup_{s > \epsilon} (ts - \tilde{\Phi}(s)) \} \geq \epsilon t, \quad t \geq 0.
\]

If \( \tilde{\Phi} \equiv \infty \) on \([T, \infty)\), then

\[
\Phi(t) = \sup_{s > 0} (st - \tilde{\Phi}(s)) = \sup_{s \leq T} (st - \tilde{\Phi}(s)) \leq Tt, \quad t \geq 0.
\]

\( \square \)
Lemma 4. If $\Phi$ is convex, then

\begin{equation}
\Phi(\lambda S_{\Phi}(t)) \leq C \lambda \hat{\Phi}(t), \quad t \geq 0, \ \lambda \in [0,1].
\end{equation}

Proof. Since $\Phi(0) = 0$ and $\Phi$ is convex, it will suffice to prove

\begin{equation}
\Phi(S_{\Phi}(t)) \leq C \hat{\Phi}(t), \quad t \geq 0.
\end{equation}

First, if $\Phi$ is a Young function, then (4) holds with $C = 1$ (see [18]). In this case the Young inequality implies $t \leq \Phi^{-1}(t)\hat{\Phi}^{-1}(t)$, and it thus suffices to substitute $t \to \hat{\Phi}(t)$.

Next, keeping in mind that $\Phi$ and $\hat{\Phi}$ are convex, we can observe using Lemma 2, (ii), that for $t \in [\varepsilon, T], \varepsilon, T > 0$, it is

$$
\Phi(S_{\Phi}(t)) = R_{\Phi}(S_{\Phi}(t))S_{\Phi}(t) \leq \varepsilon^{-1}R_{\Phi}(S_{\Phi}(T))\hat{\Phi}(T).
$$

Hence, it will suffice to prove that (4) holds near 0 and near $\infty$.

Let $\Phi \in B_0 \cap B_\infty$. Then by Lemma 3, (4) holds trivially for $t \in [0,\varepsilon] \cup [T,\infty]$.

If $\Phi \in B_\infty \setminus B_0$, then (4) holds trivially for $t \in [T,\infty)$. Moreover, there exists a Young function $\Psi$ such that $\Psi(t) = \Phi(t)$ for $t \in [0,\varepsilon]$. Let $t \in (0, R_{\Phi}(\varepsilon))$, and $\tau = R_{\Psi}^{-1}(t)$. Then

$$
\tilde{\Psi}(t) = \sup_{0 < s < \tau} s(t - R_{\Psi}(s)) = \sup_{0 < s < \tau} s(t - R_{\Phi}(s)) = \hat{\Phi}(t),
$$

that is, $\hat{\Phi}$ near zero is determined only by the behaviour of $\Phi$ near zero. As $\Psi$ is Young's, (4) holds for $\Psi$, and hence also for $\Phi$ and small values of $t$.

Finally, if $\Phi \in B_0 \setminus B_\infty$, then (4) holds trivially for $t \in [0,\varepsilon]$, and there exists a Young function $\Psi$ such that $\Psi(t) = \Phi(t)$ for $t \geq T$. It is not hard to verify that $\hat{\Phi}$ and $\tilde{\Psi}$ coincide for large values of $t$ (cf. [14], Theorem I.2.1). As $\Psi$ is Young's, (4) holds for $\Psi$, and hence also for $\Phi$ and large values of $t$. \hfill \Box

Corollary 2. (cf. [18]). If $\Phi$ is convex, then for all $t \geq 0$

\begin{equation}
R_{\Phi}(S_{\Phi}(t)) \leq C t.
\end{equation}

Proof. Multiply (4) by $1/S_{\Phi}(t)$. \hfill \Box

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3. THE HARDY-LITTLEWOOD MAXIMAL OPERATOR

Let \( \mu \) be a complete \( \sigma \)-finite Borel measure, satisfying the doubling condition 
\( \mu(2Q) \leq C \mu(Q) \), where \( 2Q \) is the cube concentric with \( Q \) and with sides twice as long. Let \( \varrho \) and \( \sigma \) be measures absolutely continuous with respect to \( \mu \) and vice versa, that is, there exist measurable functions \( \frac{d \varrho}{d \mu}, \frac{d \sigma}{d \mu}, \) and \( \frac{d \varrho}{d \mu} \).

For a \( \mu \)-measurable function \( h \) and a \( \mu \)-measurable set \( E \) we shall write 
\( h(E) = \int_E h \, d\mu \) and \( h_E = (\mu(E)^{-1})h(E) \).

In this section we shall be concerned with the inequalities

\[
\varrho(\{M_\mu f > \lambda\}) \cdot \Phi(\lambda) \leq C \int \Phi(C |f(x)|) \, d\sigma, \tag{6}
\]

and

\[
\varrho(\{M_\mu f > \lambda\}) \leq C \int \Phi(C |f(x)|/\lambda) \, d\sigma, \tag{7}
\]

where the Hardy-Littlewood maximal operator related to \( \mu \) is given by

\[
M_\mu f(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(y)| \, d\mu(y).
\]

Lemma 5. (i) Let the weak type inequality (6) hold. Then \( \Phi \) is quasiconvex.
(ii) Let the extra-weak type inequality (7) hold. Then \( \Phi \) is reasonable.

Proof. (i) Take \( K \) such that the set \( E = \{ \frac{d \varrho}{d \mu}(x) \geq K^{-1} ; \frac{d \sigma}{d \mu}(x) \leq K \} \) has positive measure and let \( Q \) be a cube such that \( \mu(Q \cap E) > \mu(Q)/2 \). By (6),

\[
\Phi \left( 2^{-1} |f|_{Q \cap E} \right) \leq C K^2 \Phi(C |f|)_{Q \cap E}. \tag{8}
\]

Let \( s, t > 0 \) and \( \alpha \in (0, 1) \). Write \( Q \cap E \) as \( F \cup F' \), where \( \mu(F) = \alpha \cdot \mu(Q \cap E) \), and define \( f(x) = s \cdot \chi_F(x) + t \cdot \chi_{F'}(x) \). Then (8) turns to

\[
\Phi(2^{-1}(\alpha s + (1 - \alpha)t)) \leq C K^2 (\alpha \Phi(Cs) + (1 - \alpha)\Phi(Ct)),
\]

which is by Lemma 2 equivalent to the quasiconvexity of \( \Phi \).

(ii) Assume that \( \Phi \) is not reasonable. Then, by Lemma 1, there is a sequence \( \{t_n\}, t_n \not\to \infty \), such that \( \Phi(t_n) < n^{-1}t_n \). Taking arbitrary cube \( Q \) and its subsets
$E_n$ in order that $\mu(Q) = C^{-1} t_n \mu(E_n)$ where $C$ is from (7), and putting $f = \chi_{E_n}$ and $\lambda = \frac{\mu(E_n)}{\mu(Q)}$ in (7) we get
\[
\varrho(Q) \leq C \Phi \left( C \cdot \frac{\mu(Q)}{\mu(E_n)} \right) \sigma(E_n) < C \cdot \frac{\mu(Q)}{\mu(E_n)} \cdot \frac{\sigma(E_n)}{n},
\]
which yields $\varrho Q \leq \frac{C}{n} \cdot \sigma_{E_n}$. Letting shrink $E_n$ to a density point of $\{0 < \sigma(x) < \infty\}$, we get $\varrho = 0$ almost everywhere on the set where $\sigma$ is finite. However, this contradicts the mutual absolute continuity of the measures $\varrho$ and $\sigma$.

We have seen that the weak type inequalities turn out to be strong enough to guarantee quasiconvexity of $\Phi$, while the extra-weak type ones imply merely reasonability of $\Phi$. This is caused by the fact that (7), unlike (6), provides some control of the growth of $\Phi$ only from one side.

**From now on we shall assume for simplicity sake that $\Phi$ itself is convex.**

The pair $(\sigma, \varrho)$ is said to satisfy the $A_\Phi(\mu)$ condition $((\sigma, \varrho) \in A_\Phi(\mu))$ if either $\Phi$ is Young’s and there exist $C, \varepsilon$ such that
\[
\sup_{\alpha > 0} \sup_{Q} \frac{\varrho(Q)}{\mu(Q)} R_\Phi \left( \frac{\varepsilon}{\mu(Q)} \int_{Q} S_\Phi \left( \alpha^{-1} \frac{d\mu}{d\sigma} \right) d\mu \right) \leq C,
\]
or $\Phi \in B_0 \cup B_\infty$ and there is $C$ such that for all $Q$ and almost every $x \in Q$
\[
\frac{\varrho(Q)}{\mu(Q)} \leq C \frac{d\sigma}{d\mu}(x).
\]

The pair $(\sigma, \varrho)$ is said to satisfy the $E_\Phi(\mu)$ condition if there are $C, \varepsilon > 0$ such that
\[
\sup_{Q} \frac{1}{\mu(Q)} \int_{Q} S_\Phi \left( \varepsilon \cdot \frac{d\mu}{d\sigma}(x) \cdot \frac{\varrho(Q)}{\mu(Q)} \right) d\mu \leq C.
\]

We shall prove that the pairs $(\sigma, \varrho)$ satisfying $A_\Phi(\mu)$, or $E_\Phi(\mu)$, are good for weak, or extra-weak, resp., type inequalities involving the operator $M_\mu$.

The conditions (9) and (11) take their origin in the well-known Muckenhoupt’s $A_p$ condition for couples of weights $(w, u)$ (see [16])
\[
\sup_{Q} \left( \frac{1}{|Q|} \int_{Q} u(x) \, dx \right) \left( \frac{1}{|Q|} \int_{Q} w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C,
\]
and its simple reformulation
\[
\sup_{Q} \frac{1}{|Q|} \int_{Q} \left( \frac{u_Q}{w(x)} \right)^{p'-1} \, dx \leq C,
\]
respectively, where \(|Q| = \int_Q dx\), \(u_Q = |Q|^{-1} \int_Q u\), and \(p' = p/(p-1)\). The inequality (10) is known as the \(A_1\) condition ([16]). The \(A_\Phi(\mu)\) condition in the form similar to (9) was introduced in [18], but the key discovery is due to Kerman and Torchinsky [12], see also [6]. Clearly, if \(\Phi(t) = t^p\), then \(A_\Phi(\mu) = E_\Phi(\mu) = A_p(\mu)\).

**Theorem 1.** The following statements are equivalent.

(i) There exists \(C > 0\) such that for all \(f\) and \(\lambda\) the inequality (6) holds;
(ii) there exists \(C > 0\) such that for all \(f\) and \(Q\),

\[
\varrho(Q) \cdot \Phi(|f|_Q) \leq C \int_Q \Phi(C |f(x)|) \, d\sigma;
\]

(iii) \((\sigma, \varrho) \in A_\Phi(\mu)\).

**Theorem 2.** The following statements are equivalent.

(i) There exists \(C > 0\) such that for all \(f\) and \(\lambda > 0\) the inequality (7) holds;
(ii) there exists \(C > 0\) such that for all \(f\) and \(Q\),

\[
\varrho(Q) \leq C \int_Q \Phi(C |f(x)|/|f|_Q) \, d\sigma;
\]

(iii) \((\sigma, \varrho) \in E_\Phi(\mu)\).

The next remark sheds light on the connection between the statements of both theorems and justifies our terminology “weak” and “extra-weak”.

**Remark 1.** Each statement of Theorem 1 implies its counterpart in Theorem 2.

Indeed, inserting \(\lambda = 1\) in (6) we get

\[
\varrho(\{M_\mu f > 1\}) \leq C \int_{\mathbb{R}^n} \Phi(C |f(x)|) \, d\sigma,
\]

which is by homogeneity of \(M_\mu\) equivalent to (7). Similarly, taking \((|f|_Q)^{-1} \cdot f\) instead of \(f\) in (12) we get (13). Lastly, to see that \(A_\Phi(\mu) \subset E_\Phi(\mu)\), simply put \(\alpha = \frac{\mu(Q)}{\varrho(Q)}\) in (9) in case \(\Phi\) is Young’s, or use (10) in case \(\Phi \in B_0 \cup B_\infty\).

**Lemma 6.** Assume that \((\sigma, \varrho) \in A_1(\mu)\) (that is, (10) holds). Then the weak-type inequality (6) holds for any \(\Phi\).
Proof. As $\mu$ is doubling, standard covering argument yields

$$\lambda \cdot \varrho(\{M_\mu f > \lambda\}) \leq C \int |f(x)| \, d\sigma.$$ 

Moreover, the convexity of $\Phi$ gives via Lemma 2, (iii), that $\Phi(M_\mu f) \leq M_\mu(\Phi(f))$. Hence

$$\varrho(\{M_\mu f > \lambda\}) \cdot \Phi(\lambda) = \varrho(\{\Phi(M_\mu f) > \Phi(\lambda)\}) \cdot \Phi(\lambda) \leq C \int \Phi(|f(x)|) \, d\sigma.$$ 

Lemma 7. If $\Phi \in B_0 \cup B_\infty$ and the estimate (12) holds, then $(\sigma, \varrho) \in A_1(\mu)$.

Proof. Let $\Phi \in B_0$. Then inserting $f = \chi_E, E \subset Q$, in (12), we get

$$\frac{\mu(E)}{\mu(Q)} \leq C \Phi \left( \frac{\mu(E)}{\mu(Q)} \right) \leq C \Phi(C) \frac{\sigma(E)}{\varrho(Q)},$$

which yields $(\sigma, \varrho) \in A_1(\mu)$. Now let $\Phi \in B_\infty$. As already observed (Remark 1), (12) suffices for (13). Putting $f = \chi_E$ this time in (13) we obtain

$$\varrho(Q) \leq C \sigma(E) \Phi \left( C \frac{\mu(Q)}{\mu(E)} \right) \leq C \sigma(E) \frac{\mu(Q)}{\mu(E)},$$

which is $A_1(\mu)$, again.

Proof of Theorem 1. If $\Phi$ is a Young function, the proof can be done as in [18] with trivial changes. Assume that $\Phi \in B_0 \cup B_\infty$; then the implications (ii)$\Rightarrow$(iii)$\Rightarrow$(i) follow from Lemmas 7 and 6, and the implication (i)$\Rightarrow$(ii) is a consequence of the obvious inclusion $Q \subset \{M_\mu f > |f|_{Q/2}\}$.

Proof of Theorem 2. That (i) implies (ii) follows again from the inclusion $Q \subset \{M_\mu f > |f|_{Q/2}\}$.

The implication (iii)$\Rightarrow$(i) can be proved following the lines of the proof in [18].

The proof of (ii)$\Rightarrow$(iii) in [18] requires somewhat complicated theory of norms in Orlicz spaces and saturation of the Hölder inequality. We give here a much simpler direct proof, applicable to a general $\Phi$.

Let $Q$ be a fixed cube. If $\varrho(Q) = 0$, there is nothing to prove. Let $0 < \varrho(Q) < \infty$. 555
Assume first that $\Phi \notin B_\infty$. Then $\hat{\Phi}$, and hence also $S_\Phi$, is finite on $(0, \infty)$. Given $k \in \mathbb{N}$, put $Q_k = \{x \in \mathbb{Q}; \frac{g(x)}{\mu(x)} > 1/k\}$ and

$$g(x) = g_k(x) = S_\Phi \left( \frac{\rho(Q)}{\mu(Q)} \frac{d\mu}{d\sigma}(x) \right) \chi_{Q_k}(x)$$

with $\varepsilon$ to be specified later. It follows from (ii) that

$$\int_{Q_k} \hat{\Phi} \left( \frac{\rho(Q)}{\mu(Q)} \frac{d\mu}{d\sigma}(x) \right) d\sigma = \varepsilon g_Q \rho(Q) \leq C\varepsilon \rho(Q) + I_Q,$$

where $I_Q$ is defined as follows: $I_Q = 0$ if $g_Q \leq C$ ($C$ is the bigger of the constants from (13) and (3)), and

$$I_Q = C\varepsilon g_Q \int_Q \Phi \left( \frac{g(Q)}{g_Q} g(x) \right) d\sigma \quad \text{if} \quad g_Q > C.$$

Hence, using (3) with $\lambda = C/g_Q$,

$$I_Q = C\varepsilon g_Q \int_{Q_k} \Phi \left( \frac{C}{g_Q} S_\Phi \left( \frac{\rho(Q)}{\mu(Q)} \frac{d\mu}{d\sigma}(x) \right) \right) d\sigma \leq C^3 \varepsilon \int_{Q_k} \hat{\Phi} \left( \frac{\rho(Q)}{\mu(Q)} \frac{d\mu}{d\sigma}(x) \right) d\sigma,$$

which yields

$$\int_{Q_k} \hat{\Phi} \left( \varepsilon \frac{\rho(Q)}{\mu(Q)} \frac{d\mu}{d\sigma}(x) \right) d\sigma \leq C\varepsilon \rho(Q) + C^3 \varepsilon \int_{Q_k} \hat{\Phi} \left( \varepsilon \frac{\rho(Q)}{\mu(Q)} \frac{d\mu}{d\sigma}(x) \right) d\sigma. \quad (14)$$

Now (remember that $S_\Phi$ is nondecreasing),

$$\int_{Q_k} \hat{\Phi} \left( \frac{\rho(Q)}{\mu(Q)} \frac{d\mu}{d\sigma}(x) \right) d\sigma = \varepsilon \rho(Q) \int_{Q_k} \Phi \left( \varepsilon \frac{\rho(Q)}{\mu(Q)} \frac{d\mu}{d\sigma}(x) \right) d\mu \leq \varepsilon \rho(Q) \cdot S_\Phi \left( k \varepsilon \frac{\rho(Q)}{\mu(Q)} \right) < \infty,$$

whence we can take $\varepsilon$ sufficiently small ($\varepsilon < C^{-3}$) and subtract in (14) to get thereby

$$\int_{Q_k} \hat{\Phi} \left( \varepsilon \frac{\rho(Q)}{\mu(Q)} \frac{d\mu}{d\sigma}(x) \right) d\sigma \leq \frac{C\varepsilon}{1 - C^3 \varepsilon} \rho(Q).$$
Since $\mu(Q \setminus \bigcup Q_k) = 0$ and the constant at the right does not depend on $k$, (iii) follows.

The situation is much simpler if $\Phi \in B_\infty$, since then $R_\Phi(t) \leq C$, and inserting $f = \chi_E, E \subset Q$, into (ii) gives

$$g(Q) \leq C \frac{\mu(Q)}{\mu(E)} \sigma(E).$$

So, $(\sigma, g)$ belongs to $A_1(\mu)$. It follows easily from (5) that (10) always implies (9), and therefore $A_1(\mu) \subset A_\Phi(\mu)$ for every $\Phi$. As $A_\Phi(\mu) \subset E_\Phi(\mu)$ for every $\Phi$ (Remark 1), we are done.

\textbf{Corollary 3.} If $\Phi \in B_\infty$, then $A_\Phi(\mu) = E_\Phi(\mu) = A_1(\mu)$.

\textbf{Proof.} The proof of Lemma 7 shows that if $\Phi \in B_\infty$, then $E_\Phi(\mu) \subset A_1(\mu)$. The remaining inclusions have been already established.

\section{The condition $A_\infty$}

In this section we assume that $\sigma \equiv g$. Recall that $\Phi$ is convex.

We say that $g \in A_\infty(\mu)$ if there exist $\delta, \varepsilon \in (0, 1)$ such that $E \subset Q$ and $\mu(E) < \delta \mu(Q)$ imply $g(E) < \varepsilon g(Q)$.

Both the endpoints of the $A_p$ scale, the classes $A_1$ and $A_\infty$, are of exceptional meaning. Between $A_1$ and all other $A_p$'s there is a significant gap. For example, putting $\Phi(t) = t(1 + \log^+ t)^K$, we get $A_1(\mu) \subset E_\Phi(\mu) \subset \bigcap_{p>1} A_p(\mu)$, where both the inclusions are proper (see [2], [15], [17]). A different situation can be found near $A_\infty$; it is known (e.g. [4]) that $A_\infty = \bigcup_{p>1} A_p$. This fact will allow us to obtain new characterizations of $A_\infty$.

The idea is simple: First, it is easy to prove that $E_\Phi(\mu) \subset A_\infty(\mu)$ in any case of $\Phi$. Further, we know that $A_\Phi(\mu) \subset E_\Phi(\mu)$ (Remark 1). Therefore, it will suffice to take $\Phi$ with sufficiently rapid growth so that $A_p(\mu) \subset A_\Phi(\mu)$ for all $p$, and then it must be $A_\Phi(\mu) = E_\Phi(\mu) = A_\infty(\mu)$.

The condition $A_\infty$ has been intensively studied and a lot of equivalent statements have been proved ([4], [7], [11], [5] etc.). In the particular (weighted) case $d\mu = dx$ and $dg = w(x)dx$, Hruščev ([11]) proved that $w \in A_\infty$ (we write $w \in A_\infty$ instead of $g \in A_\infty(\mu)$), if, and only if,

\begin{equation}
\sup_Q \left( \frac{1}{|Q|} \int_Q w(x)\, dx \right) \exp \left( \frac{1}{|Q|} \int_Q \log \frac{1}{w(x)}\, dx \right) \leq C.
\end{equation}
(An independent proof of this result was given by García-Cuerva and Rubio de Francia in [7]). By a different argument, Fujii ([5]) obtained (among others) another characterization of $A_\infty$,

$$\sup_Q \int_Q \log^+ \left( \frac{w(x)}{w_Q} \right) w(x) \, dx \leq C \, w(Q).$$

We shall prove a new general characterization of $A_\infty$ expressed in terms of $E_\Phi(\mu)$ conditions, which covers (15) and (16) as particular cases and clarifies their mutual relationship.

**Theorem 3.** Let $\Phi$ be such that $S_\Phi(t^\alpha)$ is quasiconcave on $(0, \infty)$ for any $\alpha \geq \alpha_0$ and some $\alpha_0$. Then $A_\Phi(\mu) = E_\Phi(\mu) = A_\infty(\mu)$.

**Proof.** First, let $\varrho \in E_\Phi(\mu)$. Then, inserting $\varrho = \sigma$ and $f = \chi_E$, $E \subset Q$, in Theorem 2, (ii), we get

$$\frac{\varrho(Q)}{\varrho(E)} \leq C \Phi \left( \frac{\mu(Q)}{\mu(E)} \right).$$

Therefore, if $E' = Q \setminus E$ and $\mu(E') < \delta \mu(Q)$, we have $\varrho(E') < \varepsilon \varrho(Q)$, where $(1 - \varepsilon)^{-1} = C \Phi(C/(1 - \delta))$. In other words, $\varrho \in A_\infty(\mu)$. Note that this inclusion, $E_\Phi(\mu) \subset A_\infty(\mu)$, holds for any $\Phi$.

Now, let $\varrho \in A_\infty(\mu)$. Then there is $p > \alpha_0 + 1$ such that $\varrho \in A_p(\mu)$ (see e.g. [4]). By our assumption, the function $F(t) = S_\Phi(t^{p-1})$ is quasiconcave. Taking $\varepsilon$ small enough ($\varepsilon C \leq 1$) and using Jensen's inequality and (5), we get

$$\frac{\alpha \varrho(Q)}{\mu(Q)} \cdot R_\Phi \left( \frac{\varepsilon}{\mu(Q)} \int_Q S_\Phi \left( \frac{1}{\alpha} \cdot \frac{d\mu}{d\varrho}(x) \right) \, d\mu \right)$$

$$= \frac{\alpha \varrho(Q)}{\mu(Q)} \cdot R_\Phi \left( \frac{\varepsilon}{\mu(Q)} \int_Q F \left( \left( \frac{1}{\alpha} \cdot \frac{d\mu}{d\varrho}(x) \right)^{p'-1} \right) \, d\mu \right)$$

$$\leq \frac{\alpha \varrho(Q)}{\mu(Q)} \cdot R_\Phi \left( C \varepsilon F \left( \frac{C}{\mu(Q)} \int_Q \left( \frac{1}{\alpha} \cdot \frac{d\mu}{d\varrho}(x) \right)^{p'-1} \, d\mu \right) \right)$$

$$\leq C \cdot \frac{\alpha \varrho(Q)}{\mu(Q)} \cdot \left( \frac{1}{\mu(Q)} \int_Q \left( \frac{1}{\alpha} \cdot \frac{d\mu}{d\varrho}(x) \right)^{p'-1} \, d\mu \right)^{p-1}$$

Hence, $A_\infty(\mu) \subset A_\Phi(\mu)$. Since $A_\Phi(\mu) \subset E_\Phi(\mu)$ always, the proof is complete. \qed

**Remark 2.** As we know, if $\Phi(t) = t^p$ or $\Phi \in B_\infty$, then $A_\Phi = E_\Phi$. Now, Theorem 3 describes another class of functions $\Phi$ with this property. However, the inclusion $A_\Phi \subset E_\Phi$ is proper in general. The following two examples are essentially due to Bagby [1]:

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If \( \Phi(t) = t^p \) for \( t \in [0, 1) \) and \( \Phi(t) = t^q \) for \( t \in [1, \infty) \), where \( p < q \), then \( A_\Phi = A_p \) but \( E_\Phi = A_q \).

If \( \Phi(t) = t^p(\log^+ t + 1)^{-q} \), \( p > 1 \), \( q > 0 \), \( \mu \) is Lebesgue measure, \( d\varrho = d\sigma = x^{p-1}dx \), then \((\sigma, \varrho) \in E_\Phi \), but \((\sigma, \varrho) \notin A_\Phi = A_p \).

**Theorem 4.** The following statements are equivalent.

(i) \( \varrho \in A_\infty(\mu) \);

(ii) there is \( C \) such that for every \( Q \)

\[
\sup_Q \frac{1}{\mu(Q)} \int_Q \log \left( \frac{d\mu}{d\varrho}(x) \cdot \frac{\varrho(Q)}{\mu(Q)} \right) d\mu \leq C;
\]

(iii) there is \( C \) such that for every \( Q \)

\[
\sup_Q \frac{1}{\varrho(Q)} \int_Q \log \left( \frac{d\varrho}{d\mu}(x) \cdot \frac{\mu(Q)}{\varrho(Q)} \right) d\varrho \leq C.
\]

**Proof.** To prove that (i) \( \Leftrightarrow \) (ii), put \( \tilde{\Phi}(t) = t(1 + \log^+ t) \). Then \( \tilde{\Phi} \) is convex and so it is indeed a complementary function (e.g. to the function \( (\tilde{\Phi})^* \)). On the other hand, \( S_\tilde{\Phi}(t^\alpha) = 1 + \alpha \log^+ t \) is evidently quasiconcave for any \( \alpha > 0 \). Theorem 3 therefore implies that \( \varrho \in A_\infty(\mu) \) if, and only if, \( \varrho \in E_\Phi(\mu) \), or

\[
\sup_Q \frac{1}{\mu(Q)} \int_Q \log^+ \left( \frac{d\mu}{d\varrho}(x) \cdot \frac{\varrho(Q)}{\mu(Q)} \right) d\mu \leq C.
\]

This inequality obviously implies (17), but in fact they are equivalent. This will be seen once we prove that for any \( Q \)

\[
\frac{1}{\mu(Q)} \int_Q \log^+ \left( \frac{d\mu}{d\varrho}(x) \cdot \frac{\varrho(Q)}{\mu(Q)} \right) d\mu \leq \frac{1}{\mu(Q)} \int_Q \log \left( \frac{d\mu}{d\varrho}(x) \cdot \frac{\varrho(Q)}{\mu(Q)} \right) d\mu + \frac{1}{e},
\]

cf. [11], Lemma 1.
To prove (19), put \( E = \{ x \in Q; \frac{\varphi(Q)}{\mu(Q)} \leq \frac{d\mu}{d\nu}(x) \} \). Then, by the Jensen inequality, applied to the convex function \(-\log\),

\[
\frac{1}{\mu(Q)} \int_Q \log \left( \frac{d\mu}{d\varphi}(x) \cdot \frac{\varphi(Q)}{\mu(Q)} \right) d\mu - \frac{1}{\mu(Q)} \int_Q \log^+ \left( \frac{d\mu}{d\varphi}(x) \cdot \frac{\varphi(Q)}{\mu(Q)} \right) d\mu \\
= \frac{\mu(E)}{\mu(Q)} \cdot \frac{1}{\mu(E)} \int_E \left( -\log \frac{d\varphi}{d\mu}(x) \right) d\mu + \frac{1}{\mu(Q)} \int_E \log \frac{\varphi(Q)}{\mu(Q)} d\mu \\
\geq - \frac{\mu(E)}{\mu(Q)} \log \left( \frac{1}{\mu(E)} \int_E \frac{d\varphi}{d\mu}(x) d\mu \right) + \frac{\mu(E)}{\mu(Q)} \cdot \log \frac{\varphi(Q)}{\mu(Q)} \\
= \frac{\mu(E)}{\mu(Q)} \cdot \log \left( \frac{\varphi(Q)}{\mu(Q)} \cdot \frac{\mu(E)}{\varphi(E)} \right) \geq \frac{\mu(E)}{\mu(Q)} \cdot \log \left( \frac{\mu(E)}{\varphi(E)} \right) \geq - \frac{1}{e},
\]

since \( \min_{t \in (0,1)} t \log t = -1/e \).

The equivalence of (ii) and (iii) follows from the equivalence of \( \varphi \in A_\infty(\mu) \) and \( \mu \in A_\infty(\varphi) \), which was proved by Coifman and Fefferman [4] provided that both \( \mu \) and \( \varphi \) were doubling. In our case \( \mu \) is assumed to be doubling from the very beginning and \( \varphi \in A_\infty(\mu) \) easily yields that also \( \varphi \) is doubling. The proof is thus complete.

\( \square \)

To round off this section, put finally \( d\mu(x) = dx \) and \( d\varphi(x) = w(x)dx \). Then (17) turns to

\[
\sup_Q \frac{1}{|Q|} \int_Q \log \frac{w(Q)}{w(x)} dx \leq C,
\]

exactly what we obtain after taking log of the left hand side of (15). In view of this, (17) is equivalent to the Hrusčev condition (15). Similarly, (18) turns to

\[
\sup_Q \frac{1}{w(Q)} \int_Q \log \left( \frac{w(x)}{w(Q)} \right) w(x) dx \leq C,
\]

which is the Fujii condition (16) (even a slightly better one, as the “+” sign is removed).
5. THE HILBERT TRANSFORM

In the sequel we assume that \( n = 1 \). Recall that \( \Phi \) is still convex. The symbol \( I \) will always stand for an open interval on the real line and if \( I = (a, b) \), we denote \( I' = [b, 2b-a] \). We shall also restrict ourselves to the case \( d\mu(x) = dx \) (the Lebesgue measure), and \( d\varrho(x) = d\sigma(x) = w(x)dx \), where \( w \) is a positive measurable function (weight). Thus, \( w \in A_\Phi \) if either \( \Phi \) is Young’s and

\[
\sup_{a,b} \alpha w_I \cdot R_\Phi \left( \frac{\varepsilon}{|I|} \int_I S_\Phi \left( \frac{1}{\alpha w(x)} \right) dx \right) \leq C,
\]

or \( \Phi \in B_0 \cup B_\infty \) and \( w \in A_1 \), that is,

\[
w_I \leq C \cdot \text{ess inf} \{w(x); x \in I\}.
\]

Similarly, \( w \in E_\Phi \) if

\[
\sup_{I} \frac{1}{|I|} \int_I S_\Phi \left( \frac{\varepsilon w_I}{w(x)} \right) dx \leq C.
\]

The maximal operator \( M \) treated in this section is defined by

\[
Mf(x) = \sup \{|f|_I; I \ni x\}.
\]

The Hilbert transform is given for any function \( f \) satisfying

\[
\int_{-\infty}^{\infty} |f(x)| (1 + |x|)^{-1} \, dx < \infty
\]

by the Cauchy principal value integral

\[
Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R} \setminus (x-\varepsilon, x+\varepsilon)} \frac{f(y)}{x-y} \, dy.
\]

Similarly we define the maximal Hilbert transform

\[
H^*f(x) = \frac{1}{\pi} \sup_{\varepsilon > 0} \left| \int_{\mathbb{R} \setminus (x-\varepsilon, x+\varepsilon)} \frac{f(y)}{x-y} \, dy \right|.
\]

We shall prove the following theorems.
Theorem 5. The following statements are equivalent.

(i) There exists $C > 0$ such that for all $f$ for which $H^* f$ is defined and all $\lambda$

\[
\omega(\{H^* f > \lambda\}) \cdot \Phi(\lambda) \leq C \int_{-\infty}^{\infty} \Phi(C |f(x)|) \omega(x) \, dx;
\]

(ii) $\Phi \in \Delta_2$, and there exists $C > 0$ such that for all $f$ and $\lambda$

\[
\omega(\{M f > \lambda\}) \cdot \Phi(\lambda) \leq C \int_{-\infty}^{\infty} \Phi(C |f(x)|) \omega(x) \, dx;
\]

(iii) $\Phi \in \Delta_2$ and $\omega \in A_\Phi$.

Theorem 6. Let $\Phi \in \Delta_2^0$, that is, $\Phi(2t) \leq C\Phi(t)$ for $t \in (0,1)$. Then the following statements are equivalent.

(i) There exists $C > 0$ such that for all $f$ for which $H^* f$ is defined and all $\lambda > 0$

\[
\omega(\{H^* f > \lambda\}) \leq C \int_{-\infty}^{\infty} \Phi(C |f(x)|/\lambda) \omega(x) \, dx;
\]

(ii) there exists $C > 0$ such that for all $f$ and $\lambda > 0$

\[
\omega(\{M f > \lambda\}) \leq C \int_{-\infty}^{\infty} \Phi(C |f(x)|/\lambda) \omega(x) \, dx;
\]

(iii) $\omega \in E_\Phi$.

Remark 3. For any interval $I = (a, b)$, $f \geq 0$ and $x \in I$ we have

\[
H^*(\chi_I f)(x) \geq |H(\chi_I f)(x)| \geq (2\pi)^{-1} f_I.
\]

Similarly, for $x \in I'$ we have

\[
H^*(\chi_I f)(x) \geq |H(\chi_I f)(x)| \geq (2\pi)^{-1} f_I.
\]

Now, (24) and (20), applied to $f = \chi_I$ and $\lambda < (2\pi)^{-1}$, lead to

\[
\omega(I) \leq C \omega(I').
\]
Note that (22) together with (24) implies (26), too. Given $f \geq 0$ and $\lambda > 0$, put $\Omega = \{Mf > \lambda\}$ and let $F$ be any compact subset of $\Omega$. Then

$$F \subset \bigcup_{j=1}^{N} I_j, \quad \text{where} \quad f_{I_j} > \lambda.$$  

By [8], Lemma 4.4, Chap. I, §4, there is a disjoint subfamily $\{J_j\}$ of $\{I_j\}$ such that $w(\bigcup I_j) \leq 2 \sum w(J_j)$. Thus, by (26), (24), and (25)

$$w(F) \leq w(\bigcup I_j) \leq 2 \sum w(J_j) \leq C \sum w(J_j') \leq C \sum w(\{|H(f\chi_{J_j})| > (2\pi)^{-1}\})$$

As $F$ was arbitrary, this inequality shows that

$$w(\{Mf > \lambda\}) \leq C w(\{|Hf| > \lambda\}),$$

and therefore in both the above theorems the implication (i) $\Rightarrow$ (ii) holds. Moreover, it is clear that we can replace $H^*f$ by $|Hf|$ in Theorems 5 and 6.

**Proof of Theorem 5.** Coifman [3] proved that if $w \in A_\infty$ and $\Phi \in \Delta_2$, then

$$\sup_{\lambda} \Phi(\lambda) \cdot w(\{H^*f > \lambda\}) \leq C \sup_{\lambda} \Phi(\lambda) \cdot w(\{Mf > \lambda\}).$$

This proves (ii) $\Rightarrow$ (i). It remains to prove that (i) suffices for $\Phi \in \Delta_2$, the rest follows from Theorem 1. We shall use the idea from [9]. Given $\lambda > 0$ we put $f(x) = (2C)^{-1} \lambda \chi_{(0,1)}(x)$. Then, by (i),

$$\Phi(\lambda) \leq C \frac{w(0,1)}{w(\{H^*\chi_{(0,1)} > 2C\})} \cdot \Phi(\lambda/2),$$

and we are done. \hfill \Box

**Proof of Theorem 6.** The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow from Remark 3 and Theorem 2. We shall prove (iii) $\Rightarrow$ (i). Given a function $f$ and $\lambda > 0$, put $\Omega = \{Mf > \lambda\}$, $F = \mathbb{R} \setminus \Omega$. Then $\Omega = \bigcup I_j$, where $I_j$ are closed intervals with disjoint interiors such that $\text{dist}(F, I_j) = |I_j|$ (the Whitney decomposition—cf. [8]). Since $4I_j$ always meets $F$, it must be $|f|_{I_j} \leq 4\lambda$. As usual, we split $f$ into the "good" and the "bad" parts, namely,

$$g(x) = f(x)\chi_F(x) + \sum_{j} f_{I_j} \cdot \chi_{I_j}(x),$$

$$b(x) = f(x) - g(x) = \sum_{j} (f(x) - f_{I_j}) \chi_{I_j}(x) = \sum_{j} b_j(x).$$

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To estimate the "good" part is easy. Our assumption $\Phi \in \Delta^0_2$ guarantees that $\Phi(\lambda) \geq C\lambda^p$ for $\lambda \in (0, 1]$ and all $p$ bigger than some $p_0$. As observed in the proof of Theorem 3, (iii) implies that $w \in A_\infty$, hence $w \in A_p$ for $p$ bigger than some $p_1$. Therefore, for such $p$, $H^*$ is bounded on $L_p$, $w$ ([7], Chap. IV, Theorem 3.6), and we have for $p \geq \max(p_0, p_1)$ (recall that $|f| \leq \lambda$ almost everywhere on $F$)

\[
(27) \quad \text{w}(\{H^*g > \lambda\}) \leq C \int_{-\infty}^{\infty} \left( \frac{g(x)}{\lambda} \right)^p w(x) \, dx \\
\leq C \int_F \left( \frac{|f(x)|}{\lambda} \right)^p w(x) \, dx + C w(\Omega) \\
\leq C \int_F \Phi \left( \frac{|f(x)|}{\lambda} \right) w(x) \, dx + C w(\Omega).
\]

Now let us deal with the "bad" part. As known ([19], Chap. II, 4.6.2), for $x \in F$ we have

\[
H^*b(x) \leq C \sum_j \int_{I_j} \left| \frac{1}{|x-t_j|} - \frac{1}{|x-t|} \right| |b_j(t)| \, dt + C_0 Mb(x),
\]

where $t_j$ is the center of $I_j$. Note that $|x-t_j|$ is comparable to $|x-t|$ for every $t \in I_j$ and $x \in F$. Hence, making use of the definition of $b_j$, the estimate $|f|_{I_j} \leq 4\lambda$, and the estimate

\[
\frac{|I_j|}{|x-t_j|} \leq C M(\chi_{I_j})(x), \quad x \in F,
\]

we obtain

\[
(28) \quad H^*b(x) \leq C \sum_j \frac{|I_j|}{|x-t_j|^2} \int_{I_j} |b_j(t)| \, dt + C_0 Mb(x)
\]

\[
\leq C \sum_j \left( \frac{|I_j|}{|x-t_j|} \right)^2 |f|_{I_j} + C_0 Mb(x)
\]

\[
\leq C \lambda \sum_j M^2(\chi_{I_j})(x) + C_0 Mb(x).
\]

As already mentioned, $w \in A_p$ for some $p > 2$. Put $r = p/2$, then $r > 1$ and we can invoke the vector-valued weighted strong-type inequality ([13], Theorem 1, or
[7], Chap. 5, Theorem 6.4 and Remark 6.5 a) to obtain thereby

\[ w(\{x \in F; C \lambda \sum_{j} M^2(\chi_{I_j})(x) > \lambda \}) \leq C \int_{F} \left[ \sum_{j} M^2(\chi_{I_j})(x) \right] \tau w(x) \, dx \]

\[ \leq C \int_{\Omega} \sum_{j} \chi_{I_j}(x) w(x) \, dx \]

\[ \leq C \sum_{j} w(I_j) = C w(\Omega), \]

as \( I_j \)'s have disjoint interiors. Since \(|b(x)| \leq |f(x)| + 4\lambda\), it is

\[ \{x \in F; C_0 M b(x) > 5C_0 \lambda \} \subset \{x \in F; M f(x) > \lambda \} = \emptyset. \]

Now, (28), (29) and (30) give

\[ w(\{x \in F; H^* b(x) > (5C_0 + 1)\lambda \}) \leq C w(\Omega). \]

It follows from Theorem 2 that

\[ w(\Omega) \leq C \int_{-\infty}^{\infty} \Phi \left( C \frac{|f(x)|}{\lambda} \right) w(x) \, dx. \]

Combined with (27) and (31) this leads to

\[ w(\{H^* f > (5C_0 + 2)\lambda \}) \]

\[ \leq w(\{H^* g > \lambda \}) + w(\{x \in F; H^* b > (5C_0 + 1)\lambda \}) + w(\Omega) \]

\[ \leq C \int_{-\infty}^{\infty} \Phi \left( C \frac{|f(x)|}{\lambda} \right) w(x) \, dx, \]

which easily yields the desired estimate. \( \square \)

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Authors’ addresses: Amiran Gogatishvili, Mathematical Institute of the Georgian Academy of Sciences, Z. Ruchadze 1, Tbilisi, 380093 Georgia; Luboš Pick, Mathematical Institute of the Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail PICK@CSEARN.BITNET, Current address: University of Wales College of Cardiff, School of Mathematics, Senghennydd Road, Cardiff CF2 4AG, United Kingdom, e-mail PICKL@TAFF.CARDIFF.AC.UK.