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NOTE ON TURÁN'S GRAPH

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Graphs considered in the paper are finite, undirected and simple (without loops or multiple edges), [1, 2] being followed for terminology and notation. We denote by $S(p, q)$ the Stirling number of the second kind, that is, the number of partitions of a $p$-set into $q$ classes.

A $k$-partite complete graph is a graph consisting of $k$ independent sets, such that two vertices are adjacent if and only if they belong to different independent sets.

Turán's graph, denoted by $T(n, k)$, is a $k$-partite complete graph with $n$ vertices, for which $m$ parts contain $t + 1$ vertices and $k - m$ parts contain $t$ vertices, where $n = kt + m$ and $0 \leq m \leq k - 1$. According to [3], $T(n, k)$ is the unique (up to an isomorphism) graph with $n$ vertices which does not contain $(k + 1)$-cliques and has the chromatic number equal to $k$, its number of edges being maximal in the class of graphs with these properties.

A $(k+r)$-colouring of a graph with $n$ vertices and the chromatic number equal to $k$ is a partition of its vertex set into $k+r$ classes $(0 \leq r \leq n-k)$ such that two vertices belonging to the same class are not adjacent, the order of class being indifferent.

**Theorem 1.** The number $C(n, k, r)$ of $(k+r)$-colourings of $T(n, k)$ is given by

$$C(n, k, r) = \sum_{n_1, \ldots, n_k \geq 1 \atop n_1 + \ldots + n_k = k+r} \left( \prod_{i=1}^{m} S(t+1, n_i) \right) \cdot \left( \prod_{i=m+1}^{k} S(t, n_i) \right).$$

**Proof.** By $n_i$ for $i = 1, \ldots, k$ let us denote the number of classes of the partition of the $i$-th part of $T(n, k)$ induced by a $(k+r)$-colouring of $T(n, k)$. Then

$$n_1 + \ldots + n_k = k + r$$
and

\[ n_i \geq 1 \quad \text{for } i = 1, \ldots, k. \]

One can observe that all colourings with \( k + r \) classes of \( T(n, k) \) are obtained without repetitions from the divisions of \( k + r \) into \( k \) parts, two divisions being considered different if they differ only by the order of terms.

Obviously, \( C(n, k, r) = 1 \) for \( r = 0 \) and \( r = n - k \), and \( C(n, k, r) = 0 \) for \( r > n - k \).

\[ \square \]

**Theorem 2.** If we denote \( [\lambda]_k = \lambda(\lambda - 1) \ldots (\lambda - k + 1) \), then the chromatic polynomial of \( T(n, k) \) is equal to

\[
P(T(n, k); \lambda) = \sum_{p_1 + \ldots + p_{t+1} = m} \binom{m}{p_1, \ldots, p_{t+1}} \binom{k-m}{q_1, \ldots, q_t} 
\times \prod_{i=2}^{t} (S(t+1, i))^{p_i} \cdot \prod_{j=2}^{t-1} (S(t, j))^{q_j}[\lambda]_{p \oplus q},
\]

where

\[ p \oplus q = p_1 + 2p_2 + \ldots + (t + 1)p_{t+1} + q_1 + 2q_2 + \ldots + tq_t. \]

**Proof.** Obviously, the chromatic polynomial of a graph consisting of \( p \) isolated vertices is equal to

\[
\lambda^p = \sum_{k=1}^{p} S(p, k)[\lambda]_k.
\]

Thus, having in view the method of Read [4], we obtain

\[
P(T(n, k); \lambda) = \left( \sum_{p=1}^{t+1} S(t+1, p)[\lambda]_p \right)^m \left( \sum_{q=1}^{t} S(t, q)[\lambda]_q \right)^{k-m},
\]

where, by definition,

\[ [\lambda]_p[\lambda]_q = [\lambda]_{p+q} \quad \text{for all } p \text{ and } q. \]

Using the multinomial formula we obtain the result. \[ \square \]

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References


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