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V-LATTICES OF VARIETIES OF ALGEBRAS OF DIFFERENT TYPES

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1. INTRODUCTION

In this paper we introduce the notion of a V-lattice which is more general than that of a lattice. We show that V-lattices can also be characterized (similarly as lattices) as certain relation systems.

The notion of a V-lattice is then applied for investigating systems of varieties of algebras of certain, possibly different, types. We prove that the set of all varieties of algebras of certain types equipped with suitable operations forms a V-lattice.

For example, one can introduce the V-lattice of all varieties of ortholattices, lattices and semilattices, or the V-lattice of all varieties of rings and Abelian groups.

It is well known that beginning with the fundamental Birkhoff's results (of [1], [2]) a rather immense literature on varieties of algebraic structures has grown up. On the other hand, the literature on systems of algebras which can be of different types is scarce.

2. V-LATTICES AND V-POSETS

Definition 1. A V-lattice is an algebra $(L, \wedge, \vee)$, where L is a nonempty set and $\wedge$, $\vee$ are binary operations such that $(L, \wedge, \vee)$ satisfies the identities

$$\begin{aligned}
&(1) && x \wedge x = x, && x \vee x = x,\\
&(2) && (x \wedge y) \wedge z = (x \wedge z) \wedge y, && z \vee (y \vee x) = y \vee (z \vee x),\\
&(3) && ((x \wedge y) \wedge z) \wedge ((x \wedge u) \wedge v) && (v \vee (u \vee x)) \vee (z \vee (y \vee x))\\
& && = ((x \wedge u) \wedge v) \wedge ((x \wedge y) \wedge z), && = (z \vee (y \vee x)) \vee (v \vee (u \vee x)),\\
&(4) && x \wedge (y \vee x) = x, && (x \wedge y) \vee x = x.
\end{aligned}$$

Lemma 1. *Given a V-lattice* $(L, \wedge, \vee)$, *the following identities hold:*

$$
\begin{array}{lll}
(5) & x \wedge (x \wedge y) = x \wedge y, & (y \vee x) \vee x = y \vee x, \\
(6) & (x \wedge y) \wedge y = x \wedge y, & y \vee (y \vee x) = y \vee x, \\
(7) & \big((x \wedge y) \wedge z\big) \wedge y = (x \wedge y) \wedge z, & y \vee \big(z \vee (y \vee x)\big) = z \vee (y \vee x), \\
(8) & (x \wedge y) \wedge (x \wedge z) = (x \wedge y) \wedge z, & (z \vee x) \vee (y \vee x) = z \vee (y \vee x).
\end{array}
$$

Proof. a) By (1), (3), (1), (2) and (1), respectively, we have

$$
\begin{aligned}
x \wedge (x \wedge y) &= \big((x \wedge x) \wedge x\big) \wedge \big((x \wedge x) \wedge y\big) \\
&= \big((x \wedge x) \wedge y\big) \wedge \big((x \wedge x) \wedge x\big) \\
&= (x \wedge y) \wedge x = (x \wedge x) \wedge y = x \wedge y.
\end{aligned}
$$

b) By (5), (2) and (1), respectively, we have

$$(x \wedge y) \wedge y = \big(x \wedge (x \wedge y)\big) \wedge y = (x \wedge y) \wedge (x \wedge y) = x \wedge y.$$

c) From (2) and (6) it follows that

$$\big((x \wedge y) \wedge z\big) \wedge y = \big((x \wedge y) \wedge y\big) \wedge z = (x \wedge y) \wedge z.$$

d) By (2), (5) and (2), respectively, we get

$$(x \wedge y) \wedge (x \wedge z) = \big(x \wedge (x \wedge z)\big) \wedge y = (x \wedge z) \wedge y = (x \wedge y) \wedge z.$$

Using duality between $\wedge$ and $\vee$ we obtain the identities with $\vee$. □

Let $(L, \wedge, \vee)$ be a V-lattice. Let $\leqslant$, $\leqq$, $\lessapprox$, $\lesseqqgtr$ be binary relations on L defined by

$$
\begin{array}{lrll}
(a) & a \leqslant b & \text{iff} & b \wedge a = a, \\
(b) & a \leqq b & \text{iff} & a \wedge b = a, \\
(c) & a \lessapprox b & \text{iff} & b \vee a = b, \\
(d) & a \lesseqqgtr b & \text{iff} & a \vee b = b
\end{array}
$$

for every a, b in L.

Lemma 2. *The relations defined by* (a)–(d) *satisfy the following conditions*

(e) $a \leqslant b \implies a \leqq b$ and $a \mathrel{\bar{\bar{<}}} b$,

$a \mathrel{\bar{<}} b \implies a \leqq b$ and $a \mathrel{\bar{\bar{<}}} b$;

(f) if $a \leqslant b$ and $b \leqq c$ then $a \leqq c$,

if $a \mathrel{\bar{\bar{<}}} b$ and $b \mathrel{\bar{<}} c$ then $a \mathrel{\bar{\bar{<}}} c$

(a weak transitivity holds);

(g) both $\leqslant$ and $\mathrel{\bar{<}}$ are partial orders on L;

(h) $a \leqq b,\ a \leqslant c,\ b \leqslant c \implies a \leqslant b$,

$a \mathrel{\bar{\bar{<}}} b,\ c \mathrel{\bar{<}} a,\ c \mathrel{\bar{<}} b \implies a \mathrel{\bar{<}} b$

(i.e. if the elements a, b have the same upper bound with respect to the partial order $\leqslant$, then $a \leqq b$ implies $a \leqslant b$ and analogously for $\mathrel{\bar{\bar{<}}}$ and $\mathrel{\bar{<}}$).

Proof. (e). By (a), (6), (1), (3) and (5), the relation $a \leqslant b$ implies $b \wedge a = a$, hence $a \wedge b = (b \wedge a) \wedge b = \big((b \wedge a) \wedge a\big) \wedge \big((b \wedge b) \wedge b\big) = \big((b \wedge b) \wedge b\big) \wedge \big((b \wedge a) \wedge a\big) = b \wedge (b \wedge a) = b \wedge a = a$ and thus $a \leqq b$ (by (b)).

By (a), (4) and (d), the relation $a \leqslant b$ yields $b \wedge a = a$, so $a \vee b = (b \wedge a) \vee b = b$, hence $a \mathrel{\bar{\bar{<}}} b$.

In the other parts of the proof we omit some details.

(f). If $a \leqslant b$ and $b \leqq c$, then $a \wedge c = (b \wedge a) \wedge c = (b \wedge c) \wedge a = b \wedge a = a$, hence $a \leqq c$.

(g). If $a \leqslant b$ and $b \leqslant a$, then $a \leqslant b$ and $b \leqq a$, so $b \wedge a = a$, $b \wedge a = b$, hence $a = b$.

If $a \leqslant b$ and $b \leqslant c$, then $c \wedge a = c \wedge (b \wedge a) = c \wedge \big((c \wedge b) \wedge a\big) = \big((c \wedge b) \wedge a\big) \wedge c = (b \wedge a) \wedge c = a \wedge c = a$, since the assumptions $a \leqslant b$ and $b \leqslant c$ combined with (f) and (e) imply $a \leqq c$. Hence $a \leqslant c$.

(h). If $a \leqq b$, $a \leqslant c$ and $b \leqslant c$, then $b \wedge a = (c \wedge b) \wedge a = (c \wedge a) \wedge b = a \wedge b = a$, hence $a \leqslant b$. □

Definition 2. A V-partially ordered set, or more briefly a V-poset, is a 5-tuple $(L, \leqslant, \leqq, \mathrel{\bar{<}}, \mathrel{\bar{\bar{<}}})$, where L is a nonempty set and $\leqslant$, $\leqq$, $\mathrel{\bar{<}}$, $\mathrel{\bar{\bar{<}}}$ are binary relations on L satisfying the conditions (e)–(h).

Definition 3. Let $(L, \leqslant, \leqq, \mathrel{\bar{<}}, \mathrel{\bar{\bar{<}}})$ be a V-poset and let a, $b \in L$. An element $i \in L$ satisfying the conditions

(i) $i \leqslant a$ and $i \leqq b$

and

(ii) if $v \leqslant a$ and $v \leqq b$, then $v \leqslant i$ for every $v \in L$, will be called the V-infimum of the ordered pair $[a, b] \in L^2$. If an element $s \in L$ satisfies the conditions

(j) $a \stackrel{=}{<} s$ and $b \stackrel{-}{<} s$

and

(jj) if $a \stackrel{=}{<} w$ and $b \stackrel{-}{<} w$, then $s \stackrel{-}{<} w$ for every $w \in L$,

then it is said to be the V-supremum of the ordered pair $[a, b] \in L^2$.

If both i_1 and i_2 are the V-infima of an ordered pair $[a, b] \in L^2$, then both the inequalities $i_1 \leqslant i_2$ and $i_2 \leqslant i_1$ hold. Hence $i_1 = i_2$.

By $\inf(a, b)$ we will denote the V-infimum of an ordered pair $[a, b]$, if it exists. Instead of the V-infimum we will briefly say the infimum. Analogously, we will write $\sup(a, b)$ for the V-supremum (briefly the supremum) of an ordered pair $[a, b]$.

From the definitions we immediately get

Lemma 3. *Let $(L, \leqslant, \leqq, \stackrel{-}{<}, \stackrel{=}{<})$ be a V-poset and let $a, b \in L$. If $a \leqq b$ then $\inf(a, b) = a$.*

Lemma 4. *Let $(L, \wedge, \vee)$ be a V-lattice and let $\leqslant$, $\leqq$, $\stackrel{-}{<}$, $\stackrel{=}{<}$ be binary relations on L defined by the conditions* (a)–(d), *respectively. Then for every ordered pair $[a, b] \in L^2$ both $\inf(a, b)$ and $\sup(a, b)$ exist and*

$$\inf(a, b) = a \wedge b, \quad \sup(a, b) = a \vee b.$$

P r o o f. From the definitions we have

$$a \wedge (a \wedge b) = a \wedge b \quad \text{yields} \quad a \wedge b \leqslant a,$$
$$(a \wedge b) \wedge b = a \wedge b \quad \text{yields} \quad a \wedge b \leqq b,$$

and from $v \leqslant a$ and $v \leqq b$ it follows that $(a \wedge b) \wedge v = (a \wedge b) \wedge (a \wedge v) = (a \wedge v) \wedge (a \wedge b) = (a \wedge v) \wedge b = v \wedge b = v$, so $v \leqslant a \wedge b$ and hence $a \wedge b = \inf(a, b)$. Analogously we can prove $a \vee b = \sup(a, b)$. □

Lemma 5. *Let $(L, \leqslant, \leqq, \stackrel{-}{<}, \stackrel{=}{<})$ be a V-poset in which for every ordered pair $[a, b] \in L^2$ there exist both $\inf(a, b)$ and $\sup(a, b)$. Define the operations $\wedge$ and $\vee$ on L by*

(k) $$a \wedge b = \inf(a, b) \quad \text{and} \quad a \vee b = \sup(a, b).$$

Then $(L, \wedge, \vee)$ is a V-lattice.

P r o o f. a) Clearly, $a \leqq a$ yields $\inf(a, a) = a$, so $a \wedge a = a$.

b) Let $i_1 = \inf(a,b)$, $i_2 = \inf(a,c)$, $i_3 = \inf(i_1,c)$, $i_4 = \inf(i_2,b)$. Then $i_3 \leqslant i_1$ and $i_1 \leqslant a$ yield $i_3 \leqslant a$. Further, $i_3 \leqslant a$ and $i_3 \leqq c$ yield $i_3 \leqslant i_2$. From $i_3 \leqslant i_1$ and $i_1 \leqq b$ it follows that $i_3 \leqq b$. Combining this with $i_3 \leqslant i_2$ we get $i_3 \leqslant i_4$. Analogously we can prove $i_4 \leqslant i_3$. Hence $i_3 = i_4$.

c) Set $i_1 = \inf(a,b)$, $i_2 = \inf(a,d)$, $i_3 = \inf(i_1,c)$, $i_4 = \inf(i_2,e)$, $i_5 = \inf(i_3,i_4)$, $i_6 = \inf(i_4,i_3)$. Clearly, $i_k \leqslant a$ for $k = 1, 2, \ldots, 6$. Therefore, if $i_k \leqq i_j$ for $k,j \in \{1,2,\ldots,6\}$, then also $i_k \leqslant i_j$. Thus from $i_5 \leqq i_4$ and $i_4 \leqq e$ we obtain.

(A) $i_5 \leqq e$.

Similarly, $i_5 \leqq i_4$ (which implies $i_5 \leqslant i_4$) and $i_4 \leqslant i_2$ yield

(B) $i_5 \leqslant i_2$.

From (A) and (B) we have $i_5 \leqslant i_4$. From this fact and from $i_5 \leqq i_3$ we get $i_5 \leqslant i_6$. Analogously it can be verified that $i_6 \leqslant i_5$, hence $i_5 = i_6$.

d) From $a \lessdot b \vee a$ we have $a \leqq b \vee a$ and then by Lemma 3, $a \wedge (b \vee a) = a$. The other identities can be proved dually. □

For a V-lattice $\mathbf{L} = (L, \wedge, \vee)$ let $\mathbf{L}^*$ denote the corresponding V-poset which is determined by the conditions (a)–(d). If $\mathbf{L} = (L, \leqslant, \leqq, \lessdot, \overline{\lessdot})$ is a V-poset such that for every ordered pair $[a,b] \in L^2$ both $\inf(a,b)$ and $\sup(a,b)$ exist (in $\mathbf{L}$), let $\mathbf{L}^+$ denote the V-lattice with operations given by (k). From Lemmas 4, 5 and their proofs we immediately get.

Theorem 1. *Let $\mathbf{L}_1$ be a V-lattice and $\mathbf{L}_2$ a V-poset such that for every ordered pair $[a,b] \in L^2$ both $\inf(a,b)$ and $\sup(a,b)$ exist. Then*

$$(\mathrm{m}) \qquad (\mathbf{L}_1^*)^+ = \mathbf{L}_1 \quad \textit{and} \quad (\mathbf{L}_2^+)^* = \mathbf{L}_2.$$

Thus, we are justified to speak of a V-lattice without specifying whether it is defined by relations or by operations.

Remark. Applying induction to (2) we can verify that the identity

$$(9) \qquad \Big(\ldots((x \wedge x_1) \wedge x_2) \wedge \ldots\Big) \wedge x_k = \Big(\ldots((x \wedge x_{i_1}) \wedge x_{i_2}) \wedge \ldots\Big) \wedge x_{i_k}$$

and the dual one hold, for any permutation $(i_1, \ldots, i_k)$ of the set $\{1, 2, \ldots, k\}$.

Example 1. We define binary relations $\leqslant, \leqq, \lessdot, \overline{\lessdot}$ on $L = \{0, 1, 2\}$ as follows:

$$a \leqslant b \quad \text{iff} \quad a = b \quad \text{or} \quad [a,b] \in \{[0,1],\ [0,2]\},$$
$$a \leqq b \quad \text{iff} \quad a = b \quad \text{or} \quad [a,b] \in \{[0,1],\ [0,2],\ [2,1],\ [1,2]\},$$
$$a \lessdot b \quad \text{iff} \quad a = b \quad \text{or} \quad [a,b] \in \{[0,1],\ [2,1]\},$$
$$a \overline{\lessdot} b \quad \text{iff} \quad a = b \quad \text{or} \quad [a,b] \in \{[0,1],\ [2,1],\ [0,2],\ [2,0]\}$$

(see Fig. 1 and Fig. 2).

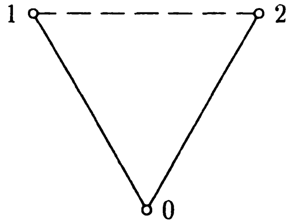

Fig. 1

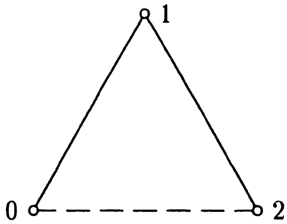

Fig. 2

Then $(L, \leqslant, \leqq, \bar{<}, \bar{\bar{<}})$ is a V-lattice in which $\wedge$ and $\vee$ are defined by Table 1 and Table 2, respectively.

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

Table 1

$\vee$	0	1	2
0	0	1	2
1	1	1	1
2	0	1	2

Table 2

Example 2. Let $\mathbf{L}_1 = (L_1, \wedge, \vee)$ and $\mathbf{L}_2 = (L_2, \wedge, \vee)$ be lattices for which $L_1 \cap L_2 = \emptyset$. Let $f\colon L_1 \to L_2$, $g\colon L_2 \to L_1$ be any mappings. Define relations $\leqslant$, $\leqq$, $\bar{<}$, $\bar{\bar{<}}$ on $L = L_1 \cup L_2$ in the following way:

$a \leqslant b$ iff $(a, b \in L_1$ and $a \wedge b = a$ in $\mathbf{L}_1)$ or
$(a, b \in L_2$ and $a \wedge b = a$ in $\mathbf{L}_2)$,

$a \bar{<} b$ iff $a \leqslant b$,

$a \leqq b$ iff at least one of the following conditions is fulfilled:

(i) $a \leqslant b$,

(ii) $a \wedge g(b) = a$ if $a \in L_1$, $b \in L_2$,

(iii) $a \wedge f(b) = a$ if $a \in L_2$, $b \in L_1$;

$a \bar{\bar{<}} b$ iff at least one of the following conditions is fulfilled:

(j) $a \bar{<} b$,

(jj) $f(a) \vee b = b$ if $a \in L_1$, $b \in L_2$,

(jjj) $g(a) \vee b = b$ if $a \in L_2$, $b \in L_1$.

Then $(L, \leqslant, \leqq, \lessdot, \bar{\lessdot})$ is a V-lattice and its operations $\frown$ and $\smile$ satisfy

$$\begin{array}{llll} a \frown b = a \wedge b, & a \smile b = a \vee b & \text{if } a, b \in L_1 & \text{or } a, b \in L_2, \\ a \frown b = a \wedge g(b), & a \smile b = f(a) \vee b & \text{if } a \in L_1, & b \in L_2, \\ a \frown b = a \wedge f(b), & a \smile b = g(a) \vee b & \text{if } a \in L_2, & b \in L_1. \end{array}$$

Lemma 6. *Let* $\mathbf{L}$ *be a* V*-lattice. Then*

(n) $$a \leqslant b \iff a \bar{\lessdot} b \quad \text{for every } a, b \in L$$

holds if and only if $\mathbf{L}$ *satisfies the identities*

(10) $$(y \vee x) \wedge x = x, \quad x \vee (x \wedge y) = x.$$

Proof. a) Let the condition (n) hold. For all a, $b \in L$ we have $a \leqslant b \vee a$, hence $(b \vee a) \wedge a = a$. The second identity in (10) follows from duality.

b) To prove the converse suppose that the identities (10) hold. If $a \leqslant b$, then $b \wedge a = a$, i.e. $b \vee a = b \vee (b \wedge a) = b$ and this implies $a \bar{\lessdot} b$. The converse implication can be obtained by a similar argument. □

3. V-LATTICES OF VARIETIES

Let $T_\alpha = (t_1, t_2, \ldots)$, $\alpha \in N \cup \{\infty\}$, be a sequence of natural numbers. Here $T_\alpha = (t_1, \ldots, t_n)$ if $\alpha = n$, $T_\alpha = (t_1, \ldots, t_n, \ldots)$ if $\alpha = \infty$.

Let $J_\alpha = (f_1, f_2, \ldots)$ be a language of type T_α and let V_α be a fixed variety of algebras with the language J_α. By $I(V_\alpha, J_\alpha)$ we denote the set of all identities (written in J_α) which are satisfied in the variety V_α. Let V_1, V_2, ... be varieties of algebras with languages $J_1 = (f_1)$, $J_2 = (f_1, f_2)$, ..., where the variety V_i, $i \in \{1, 2, \ldots, \alpha\}$, is given by the set of those identities from $I(V_\alpha, J_\alpha)$ which are written in the language J_i. Thus V_i is a variety of algebras with the language J_i given by the set of identities $I(V_\alpha, J_\alpha) \cap I_i$, where I_i is the set of all identities written in J_i. In the sequel we denote this set by $I(V_i, J_i)$.

For example, if V_5 is the variety of all ortholattices with the language $(\wedge, \vee, 0, 1, ')$ of type $(2, 2, 0, 0, 1)$ (for our purposes it is suitable to change the order of operation symbols in comparison with that in [3]), then V_1 is the variety of all semilattices with the language $(\wedge)$ of type (2), V_2 is the variety of all lattices with the language $(\wedge, \vee)$ of type $(2, 2)$, V_3 is the variety of all lattices with the least element, etc.

Further, we will suppose that $n < \infty$ for any natural number n.

Let W be a subvariety of the variety V_i, $i \in N$. By $W(j)$, $1 \leqq i < j$, we denote the class of all algebras $(A, f_1, f_2, \ldots) \in V_j$ such that $(A, f_1, \ldots, f_i) \in W$. (In practice we prefer to write f_i for an operation as well as for an operation symbol – this convention creates an ambiguity, but it seldom causes a problem.) We will call the class $W(j)$ an extension of the variety W in the language J_j.

Lemma 7. *The algebras of the class $W(j)$ form a variety.*

Proof. Evidently, $W(j)$ is the class of algebras with the language J_j that is given by the set of identities $I(V_j, J_j) \cup I(W, J_i)$, where $I(W, J_i)$ is the set of all identities (written in J_i), which determine the variety W. □

If W is a subvariety of the variety V_j, we denote by $W[i]$, $1 \leqq i < j$, the class of all algebras $(A, f_1, \ldots, f_i)$ such that $(A, f_1, f_2, \ldots) \in W$. The class $W[i]$ will be called a restriction of the variety W in the language J_i.

In general, a class $W[i]$ is not a variety. For example, if W is the variety of all Boolean algebras with the language $(\wedge, \vee, 0, 1, ')$, then $W[2]$ is the class of all Boolean lattices with the language $(\wedge, \vee)$.

Lemma 8. *If W_1 is a subvariety of the variety V_i and W_2 is a subvariety of the variety V_j, $1 \leqq i < j$, then*

$$(11) \qquad W_2[i] \subseteq W_1 \quad \textit{iff} \quad W_2 \subseteq W_1(j).$$

Proof. Let $W_2[i] \subseteq W_1$ and let $(A, f_1, f_2, \ldots)$ be an algebra in W_2. Then $(A, f_1, \ldots, f_i) \in W_2[i]$, so by the assumption we have $(A, f_1, \ldots, f_i) \in W_1$, and thus $(A, f_1, f_2, \ldots) \in W_1(j)$. The converse statement can be established in the same manner. □

In the sequel, let L_i be the set of all subvarieties of the variety V_i and let

$$L = \bigcup_{i=1}^{\alpha} L_i.$$

We are going to show that the set L can be equipped with relations $\leqslant$, $\leqq$, $\lessapprox$, $\lesseqqgtr$ such that $(L, \leqslant, \leqq, \lessapprox, \lesseqqgtr)$ is a V-poset. To this end introduce binary relation $\leqslant$, $\leqq$, $\lessapprox$, $\lesseqqgtr$ as follows.

For every $W_1 \in L_i$, $W_2 \in L_j$, $1 \leqq i \leqq j$ we define

1. $W_1 \leqslant W_2 \iff W_1 \lessapprox W_2 \iff i = j$ and $W_1 \subseteq W_2$,

2. if $i = j$ then

$$W_1 \leqq W_2 \iff W_1 \overset{=}{<} W_2 \iff W_1 \subseteq W_2,$$

3. if $i \neq j$ then

$$W_1 \leqq W_2 \iff W_1 \subseteq \langle W_2[i] \rangle$$

($\langle W_2[i] \rangle$ denotes the variety generated by the class of algebras of $W_2[i]$)

$$\begin{aligned} W_2 \leqq W_1 &\iff W_2 \subseteq W_1(j), \\ W_1 \overset{=}{<} W_2 &\iff W_1(j) \subseteq W_2, \\ W_2 \overset{=}{<} W_1 &\iff W_2[i] \subseteq W_1 \quad (\text{i.e. iff } W_2 \leqq W_1). \end{aligned}$$

Theorem 2. *Let L be the set and $\leqslant$, $\leqq$, $\bar{<}$, $\overset{=}{<}$ the binary relations defined above. Then $(L, \leqslant, \leqq, \bar{<}, \overset{=}{<})$ is a V-poset and in the corresponding V-lattice $(L, \frown, \smile)$ we have for $W_1 \in L_i$, $W_2 \in L_j$, $i \leqq j$:*

1. *if $i < j$ then*

$$\begin{aligned} W_1 \frown W_2 &= W_1 \wedge \langle W_2[i] \rangle \quad (\text{in } \mathbf{L}_i), \\ W_2 \frown W_1 &= W_2 \wedge W_1(j) \quad (\text{in } \mathbf{L}_j), \\ W_1 \smile W_2 &= W_1(j) \vee W_2 \quad (\text{in } \mathbf{L}_j), \\ W_2 \smile W_1 &= \langle W_2[i] \rangle \vee W_1 \quad (\text{in } \mathbf{L}_i); \end{aligned}$$

2. *if $i = j$ then $\frown$ and $\smile$ are the same as in $\mathbf{L}_i$ (i.e. $W_1 \frown W_2 = W_1 \wedge W_2$ etc.).*

P r o o f. It can be easily shown that for $\leqslant$, $\leqq$, $\bar{<}$, $\overset{=}{<}$ the conditions (e) and (h) are fulfilled and $\leqslant$, $\bar{<}$ are obviously partial orders. We will prove (f) of Lemma 2. If $W_1 \leqslant W_2$ and $W_2 \leqq W_3$, $W_2 \in L_i$, $W_3 \in L_j$ then $W_1 \subseteq W_2$ and

$$\begin{aligned} W_2 &\subseteq \langle W_3[i] \rangle \quad \text{if } i < j, \\ W_2 &\subseteq W_3(i) \quad \text{if } j < i. \end{aligned}$$

This implies $W_1 \subseteq \langle W_3[i] \rangle$, $W_1 \subseteq W_3(i)$, respectively, and so $W_1 \leqq W_3$. Similarly, $W_1 \overset{=}{<} W_2$, $W_2 \bar{<} W_3$ implies $W_1 \overset{=}{<} W_3$.

It remains to prove that all pairs of elements of L have both an infimum and a supremum. We are going to show that

$$\inf(W_1, W_2) = W_1 \cap \langle W_2[i] \rangle \quad \text{if } W_1 \in L_i,\ W_2 \in L_j,\ i < j.$$

It follows from $W_1 \cap \langle W_2[i] \rangle \subseteq W_1$ that $W_1 \cap \langle W_2[i] \rangle \leqslant W_1$, and from $W_1 \cap \langle W_2[i] \rangle \subseteq \langle W_2[i] \rangle$ we have $W_1 \cap \langle W_2[i] \rangle \leqq W_2$. If $W \leqslant W_1$, $W \leqq W_2$ then W, W_1 are of the same type (i.e. $W \in L_i$) and $W \subseteq W_1$, $W \subseteq \langle W_2[i] \rangle$. Hence $W \subseteq W_1 \cap \langle W_2[i] \rangle$, i.e. $W \leqslant W_1 \cap \langle W_2[i] \rangle$. A similar argument shows that $\inf(W_1, W_2) = W_1 \cap W_2(i)$ if $W_1 \in L_i$, $W_2 \in L_j$, $i > j$. The existence of a supremum can be proved dually. □

R e m a r k . We could suppose that $J_\alpha = \{f_\beta\,;\ \beta < \alpha\}$ for a fixed ordinal α. Then the proof of Theorem 2 also works.

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