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ON THE CLASSIFICATION OF ORIENTED VECTOR BUNDLES
OVER 5-COMPLEXES

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1. INTRODUCTION

The effort to classify vector bundles over a fixed CW-complex has a long history. The first result in this direction is the assertion that every two-dimensional oriented vector bundle is uniquely determined by its Euler class. Complete characterization of oriented vector bundles over a 4-dimensional CW-complex was given in [2] using the difference cocycles. In [8] E. Thomas found conditions for a mapping \( f \in [X, Y] \) to be uniquely determined by its cohomology homomorphism \( f^* \in \text{Hom}(H^*(Y), H^*(X)) \) under the assumptions that \( X \) is a suspension or \( Y \) is an H-space. He also applied the result to \( Y = BO \), the classifying space for the group \( O \), and so he obtained conditions on \( H^*(X) \) under which stable vector bundles over \( X \) are determined by their Stiefel-Whitney and Pontrjagin classes. A further progress was made in [3] where the question how many \( n \)-dimensional vector bundles over a CW-complex of the same dimension are determined by a stable vector bundle \( \xi \). The results are given in terms of \( \xi \) and they allow successful application for \( n = 3 \) and 7. Earlier results concerning characterization of oriented vector bundles over low dimensional complexes were summarized and completed in [13]. Using elementary homotopy theoretic methods and relations among characteristic classes L. M. Woodward has given the classification of stable oriented vector bundles over CW-complexes of dimension \( \leq 8 \) and the classification of \( n \)-dimensional oriented vector bundles over CW-complexes of dimension \( n \) for \( n = 3, 4, 6, 7, 8 \), both in terms of characteristic classes. A typical condition on a CW-complex \( X \) to admit such a classification is: \( H^4(X, \mathbb{Z}) \) has no element of order 4.

In dimension 5 the situation is much more complicated as can be seen on the example of the sphere \( S^5 \). Both the trivial and the tangent bundle over \( S^5 \) have all characteristic classes equal to zero. Moreover, all conditions of Woodward’s type are satisfied. The aim of our paper is to derive necessary and sufficient conditions...
on a 5-dimensional CW-complex $X$ which make the classification of 5-dimensional oriented vector bundles over $X$ in terms of characteristic classes possible. This is carried out in Section 3 using a combination of the method of Postnikov tower and the Woodward method (see [9] and [13]).

The maximal number of linearly independent sections in a vector bundle $\xi$ is defined to be the span of $\xi$. As a consequence of the classification described above we compute the span of 5-dimensional oriented vector bundles over CW-complexes of the same dimension. These results complete computations of Thomas for tangent bundles over 5-dimensional manifolds given in [12] and also our results for the dimensions 6 and 7 obtained in [1]. Together with results on the existence of a 2-distribution and a 4-distribution with a complex structure they form the contents of Section 4.

2. PRELIMINARIES

All vector bundles will be considered over a connected CW-complex $X$ and will be oriented. The letter $e$ will stand for the trivial one-dimensional vector bundle. The mapping $\beta_k : H^*(X, \mathbb{Z}_k) \to H^*(X, \mathbb{Z})$ is the Bockstein homomorphism associated with the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_k \to 0$. The mappings $i_* : H^*(X, \mathbb{Z}_2) \to H^*(X, \mathbb{Z}_4)$ and $\varphi_k : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z}_k)$ are induced from the inclusion $\mathbb{Z}_2 \to \mathbb{Z}_4$ and reduction mod $k$, respectively.

An important role in our considerations is played by the Pontrjagin square $\mathfrak{P}$, a cohomology operation from $H^{2k}(X, \mathbb{Z}_2)$ into $H^{4k}(X, \mathbb{Z}_4)$ satisfying the relations

\begin{align*}
(1) \quad & \mathfrak{P}_{\mathbb{Z}_2} x = \varepsilon_{4x^2}, \\
(2) \quad & \mathfrak{P}(u + v) = \mathfrak{P}u + \mathfrak{P}v + i_*(u \cdot v),
\end{align*}

for $x \in H^{2k}(X, \mathbb{Z})$ and $u, v \in H^{2k}(X, \mathbb{Z}_2)$. See [5], chapter 2.

We will use $w_j(\xi)$ for the $j$-th Stiefel-Whitney class of the vector bundle $\xi$, $p_1(\xi)$ for the first Pontrjagin class, and $e(\xi)$ for the Euler class. For a complex vector bundle $\zeta$ the symbol $c_j(\zeta)$ denotes the $j$-th Chern class. The letters $w_j$, $p_1$, $e$ stand for characteristic classes of the universal oriented $n$-dimensional vector bundle over the classifying space $BSO(n)$. Our results given below are based on the following relations among the characteristic classes:

\begin{align*}
(3) \quad & \varphi_4 p_1(\xi) = \mathfrak{P}w_2(\xi) + i_*.w_4(\xi), \\
(4) \quad & w_6(\xi) = Sq^2 w_4(\xi) + w_2(\xi)w_4(\xi),
\end{align*}

the former being proved in [4] and [7] and the latter being a special case of the Wu formula.
The Eilenberg-MacLane space with the $n$-th homotopy group $G$ will be denoted by $K(G,n)$ and $i_n$ will stand for the fundamental class in $H^n(K(G,n),G)$. Writing the fundamental class it will be always clear which group $G$ we have in mind.

In the proof of Theorem 1 we will need suspension. Being defined for every fibration $F \xrightarrow{j} E \xrightarrow{p} B$, it is a natural mapping from a subgroup of $H^{k+1}(B)$ into $H^k(F)/\text{im } j^*$ which commutes with the Steenrod squares and $i_*$ (see [5]).

We say that $x \in H^*(X,\mathbb{Z})$ is an element of order $k$ ($k = 2, 3, 4, \ldots$) if and only if $x \neq 0$ and $k$ is the least positive integer such that $kx = 0$ (if it exists). Some results will involve the following hypotheses:

**Condition (A).** $H^4(X,\mathbb{Z})$ has no element of order 4.

**Condition (B).** $Sq^2H^3(X,\mathbb{Z}_2) = H^5(X,\mathbb{Z}_2)$.

**Remark.** An important example of a CW-complex which satisfies Condition (B) is a 5-dimensional oriented smooth manifold $M$ with $w_2(M) \neq 0$. The Poincaré duality and the fact that the second Wu class is equal to $w_2(M)$ yields

$$Sq^2H^3(M,\mathbb{Z}_2) = w_2(M)H^3(M,\mathbb{Z}_2) = H^5(M,\mathbb{Z}_2).$$

### 3. Classification theorem

Let $X$ be a connected CW-complex of dimension $\leqslant 5$. Our problem consists in finding conditions on $X$ such that for every $a \in H^2(X,\mathbb{Z}_2)$, $b \in H^4(X,\mathbb{Z}_2)$, $c \in H^4(X,\mathbb{Z})$ there is at most one oriented 5-dimensional vector bundle $\xi$ with $w_2(\xi) = a$, $w_4(\xi) = b$, $p_1(\xi) = c$. A necessary and sufficient condition on $a$, $b$, $c$ for the existence of such a vector bundle derived in [W] is given by the relation $a_4c = \varphi a + i_* b$ (see (3)). Up to homotopy there is just one mapping $f: X \rightarrow K(\mathbb{Z}_2,2) \times K(\mathbb{Z}_2,4) \times K(\mathbb{Z},4)$ such that $f^*(t_2 \otimes 1 \otimes 1) = a$, $f^*(1 \otimes t_4 \otimes 1) = b$, $f^*(1 \otimes 1 \otimes t_4) = c$. Similarly, $w_2$, $w_4$, $p_1$, the cohomology classes of $BSO(5)$, determine a mapping $\alpha: BSO(5) \rightarrow K(\mathbb{Z}_2,2) \times K(\mathbb{Z}_2,4) \times K(\mathbb{Z},4)$ which can be considered to be a fibration. Now the problem described above can be formulated as a problem of lifting: when every mapping $f: X \rightarrow K(\mathbb{Z}_2,2) \times K(\mathbb{Z}_2,4) \times K(\mathbb{Z},4)$ has at most one lifting $\xi: X \rightarrow BSO(5)$ in the fibration $\alpha$. 

\[
\begin{array}{ccc}
X & \xrightarrow{f} & K(\mathbb{Z}_2,2) \times K(\mathbb{Z}_2,4) \times K(\mathbb{Z},4) \\
\downarrow & & \downarrow \alpha \\
BSO(5) & \xrightarrow{\xi} & \\
\end{array}
\]
To solve this problem we will construct a Postnikov tower for the fibration $\alpha : BSO(5) \to K(Z_2, 2) \times K(Z_2, 4) \times K(Z, 4)$. Put $K = K(Z_2, 2) \times K(Z_2, 4) \times K(Z, 4)$ and denote the fibre of $\alpha$ by $V$. Let us recall that $\pi_k(BSO(5)) \cong 0$ for $k = 1, 3$, $\pi_k(BSO(5)) \cong \mathbb{Z}$ for $k = 2, 5$ and $\pi_4(BSO(5)) \cong \mathbb{Z}$. Considering the characteristic classes as mappings from $BSO(6)$ into appropriate Eilenberg-Mac Lane spaces, we get $i_{\nu_2} : \pi_2(BSO(5)) \to \mathbb{Z}_2$, $i_{\nu_4} : \pi_4(BSO(5)) \to \mathbb{Z}_2$, and $p_1 : \pi_4(BSO(5)) \to \mathbb{Z}$ is a multiplication by 2. See [13]. From the long exact homotopy sequence we compute:

$$V \mapsto \pi_2(\nu_2) = 0, \pi_3 = \mathbb{Z}_2, \pi_4 = 0, \text{and } \pi_5(V) = \mathbb{Z}_2.$$

The first invariant in the Postnikov tower is the transgression of a fundamental class in $H^3(V, \mathbb{Z}_4)$. It is a generator of $\ker \alpha^* \subset H^4(K, \mathbb{Z}_4)$. Hence it is equal to

$$g_4(1 \otimes 1 \otimes \nu_4) - \nu_2 \otimes 1 \otimes 1 - 1 \otimes \nu_4 \otimes 1.$$

Let $E_1$ be the first stage of the Postnikov tower and let the new mappings be denoted according to the diagram.

Consider $\beta_1 : BSO(5) \to E_1$ as a fibration with a fibre $F_1$. This fibre is homotopy equivalent to the homotopy fibre $\tilde{F}_1$ of the mapping $\tilde{\beta}_1$ (see [9]). Hence computing the homotopy groups of $\tilde{F}_1$ we get that $F_1$ is 4-connected and $\pi_5(F_1) \cong \mathbb{Z}_2$. Consequently, $\beta_1$ is a 5-equivalence.

The next invariant $\varphi \in H^6(E_1, \mathbb{Z}_2)$ is the transgression of the generator of $H^5(F_1, \mathbb{Z}_2)$ in the Serre exact sequence for the fibration $\beta_1$. $E_1$ is also the first stage in the Postnikov tower for the fibration $\hat{\alpha} : BSO(6) \to K$ determined by $w_2$, $w_4$ and $p_1$. The mapping $\hat{\beta}_1 : BSO(6) \to E_1$ in this Postnikov tower is a 6-equivalence (since $\pi_5(BSO(6)) \cong 0$). Using the Serre exact sequence for the fibration $\hat{\beta}_1$, we get that $\hat{\beta}_1^* \cong \alpha^*$. The latter group has generators $w_3, w_5, w_2 w_4$ and $S^2 w_4 = w_6 + w_2 w_4$. The generators of $H^6(E_1, \mathbb{Z}_2)$ are $\pi_1^* (\nu_2 \otimes 1 \otimes 1)$, $\pi_1^* (w_3 \otimes 1 \otimes 1)$, $\pi_1^* (w_2 \otimes 1 \otimes 1)$, $\pi_1^* (\nu_4 \otimes 1 \otimes 1)$, $\pi_1^* (\nu_5 \otimes 1 \otimes 1)$, $\pi_1^* (\nu_2 \otimes 1 \otimes 1)$, $\pi_1^* (\nu_4 \otimes 1 \otimes 1)$, and $\pi_1^* (\nu_5 \otimes 1 \otimes 1)$. The mapping $\beta_1^* : H^6(E_1, \mathbb{Z}_2) \to H^6(BSO(5), \mathbb{Z}_2)$ maps them into $w_3, w_5, w_2 w_4$ and $S^2 w_4 = w_2 w_4$, respectively. Consequently, using the Serre exact sequence for the fibration $\beta_1$ we get $\varphi = \pi_1^* (\nu_2 \otimes 1 \otimes 1 + 1 \otimes S^2 \nu_4 \otimes 1)$.
stage $E_2$ of our Postnikov tower.

\[
\begin{array}{ccc}
F_2 & \longrightarrow & F_1 \\
\downarrow & & \downarrow \beta_2 \\
F_2 & \longrightarrow & BSO(5) \\
\downarrow \beta_1 & & \downarrow \pi_2 \\
E_1 & \longrightarrow & E_1 \\
\end{array}
\]

Due to Lemma 8.1 in [10], there is a homeomorphism $g: E_2 \rightarrow E$ where $\pi: E \rightarrow K$ is a principal fibration with the classifying map $k: K \rightarrow C$. Moreover, $\pi_1 \circ \pi_2 = \pi \circ g$ and the fibration $\beta = g \circ \beta_2: BSO(5) \rightarrow E$ is a 6-equivalence. Hence, we can consider the situation

\[
BSO(5) \xrightarrow{\beta} 6\text{-equiv} \quad E \\
\downarrow \alpha \qquad \downarrow \pi \\
K \xrightarrow{k = (k_1, k_2)} C
\]

which allows us to prove our main result.

**Theorem 1.** Let $X$ be a connected CW-complex of dimension $\leq 5$ and suppose

\[
\gamma: [X, BSO(5)] \rightarrow H^2(X, \mathbb{Z}_2) \oplus H^4(X, \mathbb{Z}_2) \oplus H^4(X, \mathbb{Z})
\]

is defined by $\gamma(\xi) = (w_2(\xi), w_4(\xi), p_1(\xi))$. Then

(i) $\text{im} \gamma = \{(a, b, c) \mid \varphi_4 c = \varphi a + i_* b\}$,

(ii) $\gamma$ is injective if and only if Conditions (A) and (B) are satisfied.
Proof. (i) follows immediately from the fact that a mapping \( f: X \to K \) can be lifted in the fibration \( \alpha \) into \( BSO(5) \) if and only if \( f^* \left( 1 \otimes 1 \otimes \varphi_4 t_4 - \varphi_{t_2} \otimes 1 - 1 \otimes i_* t_4 \otimes 1 \right) = 0 \). (See similar proofs in [1].)

(ii) Since the space \( E \) is a homotopy fibre of the mapping \( k: K \to C \), the Puppe sequence

\[
\Omega K \xrightarrow{\Omega k} \Omega C \xrightarrow{q} E \xrightarrow{\pi} K \xrightarrow{k} C
\]

yields the exact sequence

\[
\rightarrow [X, \Omega K] \xrightarrow{(\Omega k)_*} [X, \Omega C] \xrightarrow{q_*} [X, E] \xrightarrow{\pi_*} [X, K] \xrightarrow{k_*} [X, C].
\]

Moreover, \( \beta \) being a 6-equivalence, \( \beta_*: [X, BSO(5)] \to [X, E] \) is a bijection for every CW-complex of dimension \( \leq 5 \). The following statements are equivalent:

1. \( \gamma = \alpha_* = \pi_* \circ \beta_*: [X, BSO(5)] \to [X, K] \) is injective.
2. \( \pi_*: [X, E] \to [X, K] \) is injective.
3. \( q_* = 0 \)
4. \( (\Omega k)_*: [X, \Omega K] \to [X, \Omega C] \) is surjective.

Hence we need to compute \( (\Omega k_1)^*: H^3(K(\mathbb{Z}, 3), \mathbb{Z}_4) \to H^3(\Omega K, \mathbb{Z}_4) \) and \( (\Omega k_2)^*: H^5(K(\mathbb{Z}_2, 5), \mathbb{Z}_2) \to H^5(\Omega K, \mathbb{Z}_2) \).

First, let us consider \( k_1 \).

\[
\begin{array}{ccc}
\Omega K & \xrightarrow{\Omega k_1} & K(\mathbb{Z}_4, 3) \\
PK & \xrightarrow{P k_1} & PK(\mathbb{Z}_4, 4) \\
K & \xrightarrow{k_1} & K(\mathbb{Z}_4, 4)
\end{array}
\]

Every element in \( H^*(K, \mathbb{Z}_4) \) is suspensive. If we denote all suspensions by \( \sigma \), we get

\[
(\Omega k_1)^*(\iota_3) = (\Omega k_1)^*(\sigma \iota_4) = \sigma(k_1^* \iota_4) = \sigma(1 \otimes 1 \otimes \varphi_4 t_4 - \varphi_{t_2} \otimes 1 \otimes 1 - 1 \otimes i_* t_4 \otimes 1) - \sigma(1 \otimes i_* t_4 \otimes 1) - \sigma(\varphi_{t_2} \otimes 1 \otimes 1 - 1 \otimes \sigma(i_* t_4) \otimes 1)
\]

the last equality being a consequence of the definition of suspension and coboundary operator. In the fibration \( K(\mathbb{Z}, 3) \to PK(\mathbb{Z}, 4) \to K(\mathbb{Z}, 4) \) we get

\[
\sigma(\varphi_4 t_4) = \varphi_4(\sigma t_4) = \varphi_4 t_3.
\]
In the fibration $K(\mathbb{Z}_2, 3) \to PK(\mathbb{Z}_2, 4) \to K(\mathbb{Z}_2, 4)$ we have

$$\sigma(i_3) = i_*(\sigma_4) = i_3$$

and finally, in the fibration $K(\mathbb{Z}_2, 1) \to PK(\mathbb{Z}_2, 2) \to K(\mathbb{Z}_2, 2)$ we obtain

(5) $\sigma(P_4) = i_4^3$.

Since this fact is not generally known, we will prove it at the end of this section. As a result of these computations we get

$$(\Omega k_1)_* : [X, \Omega K] \to [X, K(\mathbb{Z}_4, 3)] : (a, b, c) \mapsto \varrho_4 c - i_4 a^3 - i_4 b.$$ 

Hence $(\Omega k_1)_*$ is surjective if and only if

(6) $H^3(X, \mathbb{Z}_4) = \varrho_4 H^3(X, \mathbb{Z}) + i_4 H^3(X, \mathbb{Z}_2)$.

We show that (6) is equivalent to the condition (A).

(A) $\Rightarrow$ (6). Let $x \in H^3(X, \mathbb{Z}_2)$, then $4\beta_4 x = 0$. (A) implies that $2\beta_4 x = 0$. Consequently, there is a $y \in H^3(X, \mathbb{Z}_2)$ such that $\beta_4 x = \beta_2 y = \beta_4 i_4 y$. That is why $\beta_4 (x - i_4 y) = 0$, which implies $x = i_4 y + \varrho_4 z$ for some $z \in H^3(X, \mathbb{Z})$.

(6) $\Rightarrow$ (A). Let $v \in H^4(X, \mathbb{Z})$ satisfy $4v = 0$. Then $v = \beta_4 x$ where $x = \varrho_4 z + i_4 y \in H^3(X, \mathbb{Z}_4)$ so that $v = \beta_4 \varrho_4 z + \beta_4 i_4 y = \beta_4 i_4 y = \beta_2 y$. Hence $2v = 0$ and $v$ is not an element of order 4.

Now consider the mapping $k_2$. The computation of $(\Omega k_2)^* : H^5(K(\mathbb{Z}_2, 5), \mathbb{Z}_2) \to H^5(\Omega K, \mathbb{Z}_2)$ gives

$$(\Omega k_2)^*(\iota_5) = (\Omega k_2)^*(\sigma_6) = \sigma k_2^*(\iota_6) = 1 \otimes \sigma(Sq^2 \iota_4) \otimes 1 + \sigma(\iota_2 \otimes \iota_4) \otimes 1$$

$$= 1 \otimes Sq^2 \iota_3 \otimes 1 + \sigma(\iota_2 \otimes \iota_4) \otimes 1.$$ 

We are going to prove that $\sigma(\iota_2 \otimes \iota_4) = 0$. Consider the fibration

$$\Omega B \to PB \xrightarrow{p} B$$

where $B = K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 4)$. Let $\tilde{p}^* : H^6(B, \mathbb{Z}_2) \to H^6(PB, \Omega B; \mathbb{Z}_2)$ be determined by the mapping $p$. It is sufficient to show $\tilde{p}^*(\iota_2 \otimes \iota_4) = 0$. Using the Serre spectral sequence with coefficients $\mathbb{Z}_2$ for the above fibration, we have

$$\tilde{p}^* : H^6(B, \mathbb{Z}_2) \cong E_2^{6,0} \to E_8^{6,0} \to H^6(PB, \Omega B; \mathbb{Z}_2).$$
We compute $d_2: E_2^{4,1} \to E_2^{6,0}$. Since $E_2^{4,1} \cong E_2^{4,0} \otimes E_2^{0,1}$, for the generators of $E_2^{4,1}$ we obtain

\[ d_2(\iota_2 \otimes \iota_1) = d_2(\iota_2^2) \cdot \iota_1 + \iota_2^2 \cdot d_2(\iota_1) = \iota_2^2 \cdot \iota_2 = \iota_2^3, \]

\[ d_2(\iota_4 \otimes \iota_1) = d_2(\iota_4) \cdot \iota_1 + \iota_4 \cdot d_2(\iota_1) = \iota_4 \cdot \iota_2. \]

Hence $\iota_4 \cdot \iota_2$ vanishes in $E_3^{6,0}$ and $\tilde{p}^*(\iota_2 \otimes \iota_4) = 0$.

So we conclude that

\[ (\Omega k_2)_* [X, \Omega K] \to [X, K(\mathbb{Z}_2, 5)] : (a, b, c) \mapsto S q^2 b, \]

and its surjectivity is given directly by Condition (B).

It remains to prove the relation (5). Consider the Serre spectral sequence for the fibration $K(\mathbb{Z}_2, 1) \to PK(\mathbb{Z}_2, 2) \to K(\mathbb{Z}_2, 2)$ with coefficients $\mathbb{Z}_4$. For brevity we will again denote this fibration by $\Omega B \to PB \to B$. It is not difficult to show that $H^4(B, \mathbb{Z}_4) \cong \mathbb{Z}_4$ with the generator $\mathcal{P}\iota_2$ and $H^3(\Omega B, \mathbb{Z}_4) \cong \mathbb{Z}_2$ with the generator $i_*\iota_1^3$. The coboundary operator in the long exact sequence for the couple $(PB, \Omega B)$ is an isomorphism, hence it is sufficient to prove that $\tilde{p}^*(\mathcal{P}\iota_2) \neq 0, \tilde{p}^*: H^4(B, \mathbb{Z}_4) \to H^4(PB, \Omega B; \mathbb{Z}_4)$ being induced by $p$. Since

\[ E_4^{4,0} \cong H^4(B, \mathbb{Z}_4)/\ker \tilde{p}^*, \]

it is sufficient to show that $E_4^{4,0} \neq 0$. We have

\[ E_2^{2,1} \cong H^2(B, H^1(\Omega B, \mathbb{Z}_4)) \cong \mathbb{Z}_2 \cong E_2^{2,0} \otimes E_2^{0,1}, \]

\[ E_4^{4,0} \cong H^4(B, H^0(\Omega B, \mathbb{Z}_4)) \cong \mathbb{Z}_4. \]

Moreover, $d_2: E_2^{2,1} \to E_2^{4,0}$ is injective because

\[ d_2(i_*\iota_2 \otimes i_*\iota_1) = d_2(i_*\iota_2) \cdot i_*\iota_1 + i_*\iota_2 \cdot d_2(i_*\iota_1) = i_*\iota_2 \cdot \tau(i_*\iota_1) = i_*\iota_2^2 \]

where $\tau$ is a transgression. Hence $E_3^{4,0} \cong \mathbb{Z}_2$. Further, $E_3^{1,2} \cong 0, E_3^{7,-2} \cong 0$ and consequently, $E_4^{4,0} \cong \mathbb{Z}_2$. \hfill \Box
In this section we compute the span of oriented 5-dimensional vector bundles over a 5-dimensional CW-complex satisfying Conditions (A) and (B) of Theorem 1. Under the same conditions we find all oriented 5-dimensional vector bundles which admit a 2-distribution, i.e. an oriented 2-dimensional subbundle, and all oriented 5-dimensional vector bundles which admit a 4-distribution endowed with a complex structure, i.e. a complex 2-dimensional subbundle. For these purposes we need

**Theorem 2.** Let $X$ be a connected CW-complex of dimension $\leq 5$ and let $W \in H^2(X, \mathbb{Z}_2)$, $P \in H^4(X, \mathbb{Z})$. Then there exists an oriented 3-dimensional vector bundle $\xi$ over $X$ with

$$w_2(\xi) = W, \quad p_1(\xi) = P$$

if and only if

$$s_4P = \mathcal{P}W.$$

**Proof.** is very similar to the proof of the first part of Theorem 1. See also [13].

**Corollary 1.** Let $X$ be a connected CW-complex of dimension $\leq 5$ satisfying Conditions (A) and (B). Then an oriented 5-dimensional vector bundle $\xi$ has a 2-distribution with Euler class $U$ if and only if

$$(7) \quad s_2 U^2 + w_2(\xi) s_2 U + w_4(\xi) = 0.$$

**Proof.** ($\Rightarrow$) Let $\xi = \zeta \oplus \tau$ where $\tau$ is an oriented 2-dimensional vector bundle over $X$ with the Euler class $U$ and $\zeta$ is an oriented 3-dimensional vector bundle over $X$. Then

$$w_2(\xi) = w_2(\zeta) + w_2(\tau) = w_2(\zeta) + s_2 U,$$

$$w_4(\xi) = w_2(\zeta) \cdot w_2(\tau) = w_2(\zeta) \cdot s_2 U.$$

Substituting from here into the expression $s_2 U^2 + w_2(\xi) \cdot s_2 U + w_4(\xi)$, we get (7).

($\Leftarrow$) Let $U \in H^2(X, \mathbb{Z})$ satisfy (7). There is an oriented 2-dimensional vector bundle $\tau$ over $X$ with the Euler class $U$. Put

$$W = w_2(\xi) + s_2 U, \quad P = p_1(\xi) - U^2.$$
Then
\[

g_4 P - \mathcal{P} W = g_4 p_1(\xi) - g_4 U^2 - \mathcal{P}(w_2(\xi) + g_2 U) = \\
= g_4 p_1(\xi) - g_4 U^2 - \mathcal{P} w_2(\xi) - \mathcal{P} g_2 U - i_*(w_2(\xi) g_2 U) = \\
= i_*(g_2 U^2 + w_2(\xi) g_2 U + w_4(\xi)) = 0.
\]

According to Theorem 2, there is an oriented 3-dimensional vector bundle \( \zeta \) over \( X \) with \( w_2(\zeta) = W \) and \( p_1(\zeta) = P \). We compute the characteristic classes of the vector bundle \( \zeta \oplus \tau \).

\[

w_2(\zeta \oplus \tau) = w_2(\zeta) + w_2(\tau) = W + g_2 U = w_2(\xi), \\
w_4(\zeta \oplus \tau) = w_2(\zeta) \cdot w_2(\tau) = W \cdot g_2 U = w_2(\xi) g_2 U + g_2 U^2 = \\
= w_4(\xi), \\
p_1(\zeta \oplus \tau) = p_1(\zeta) + p_1(\tau) = P + U^2 = p_1(\xi).
\]

(See [13] for the additivity of \( p_1 \) in this case.) Theorem 1 now implies that \( \xi = \zeta \oplus \tau \), which completes the proof. \( \square \)

**Remark.** As far as it is known to the authors there are only two general results concerning 2-distributions in 5 or 4\( k + 1 \)-dimensional vector bundles. See [11], Theorems 1.3 and 4.1. The former deals with spin manifolds (i.e. \( w_1(X) = w_2(X) = 0 \)) and tangent bundles while the latter requires \( \text{span} \geq 2 \). Both examine the existence of 2-distributions with the Euler class \( 2U \in H^2(X, \mathbb{Z}) \).

**Corollary 2.** Let \( X \) be a connected CW-complex of dimension \( \leq 5 \) and let \( \xi \) be an oriented 5-dimensional vector bundle over \( X \). Then

1. \( \text{span} \xi \geq 1 \) if and only if \( e(\xi) = 0 \).
2. \( \text{span} \xi \geq 2 \) if and only if \( w_4(\xi) = 0 \).
3. \( \text{span} \xi \geq 3 \) if and only if \( w_4(\xi) = 0 \) and there is a \( U \in H^2(X, \mathbb{Z}) \) such that \( w_2(\xi) = g_2 U, \ p_1(\xi) = U^2 \).
4. \( \text{span} \xi = 5 \) if and only if \( w_2(\xi) = 0, \ w_4(\xi) = 0, \ p_1(\xi) = 0 \).

**Proof.** (1) is well known and is included only for completeness.
(2) is the immediate consequence of Corollary 1 for \( U = 0 \).
(3) (\( \Rightarrow \)) Let \( \xi = \zeta \oplus 3\varepsilon \) where \( \zeta \) is an oriented 2-dimensional vector bundle over \( X \). Then \( w_4(\xi) = w_4(\zeta) = 0 \) and for \( U = e(\zeta) \) we get \( w_2(\xi) = w_2(\zeta) = g_2 U, \ p_1(\xi) = p_1(\zeta) = U^2 \).
(\( \Leftarrow \)) For \( U \in H^2(X, \mathbb{Z}) \) there is an oriented 2-dimensional vector bundle \( \zeta \) over \( X \) such that \( e(\zeta) = U \). Then \( w_2(\zeta \oplus 3\varepsilon) = w_2(\zeta) = g_2 U = w_2(\xi), \ w_4(\zeta \oplus 3\varepsilon) = w_4(\zeta) = \)
0 = w_4(\xi) and p_1(\zeta + 3\epsilon) = p_1(\zeta) = U^2 = p_1(\xi). Theorem 1 implies that \zeta + 3\epsilon = \xi since the characteristic classes of both vector bundles are the same.

(4) follows immediately from Theorem 1.

**Remark.** Statements (3) and (4) of Corollary 2 under a little bit different conditions were already known to E. Thomas [12]. Statement (2) under Conditions (A) and (B) is new. It deals with the cases which are not covered in [12]. The condition \( w_4(\xi) = 0 \) coincides with the condition for the stable span of \( 4k+1 \)-dimensional vector bundles over a CW-complex of the same dimension to be \( \geq 2 \). See [6], Theorem 2.1.1.

Now we will investigate the existence of distributions with complex structure. The case of 2-distributions is treated in Corollary 1. Here we will deal with 4-distributions. For this purpose we need the following

**Theorem 3.** Let \( X \) be a connected CW-complex of dimension \( \leq 5 \) and let \( C_1 \in H^2(X, \mathbb{Z}), C_2 \in H^4(X, \mathbb{Z}) \). Then there exists a 2-dimensional complex vector bundle \( \zeta \) over \( X \) with the Chern classes

\[
c_1(\zeta) = C_1, \quad c_2(\zeta) = C_2.
\]

**Proof.** of this theorem follows the same lines as in [13]. \( \square \)

**Corollary 3.** Let \( X \) be a connected CW-complex of dimension \( \leq 5 \) satisfying the conditions (A) and (B). Then an oriented 5-dimensional vector bundle \( \xi \) over \( X \) has a 4-distribution with a complex structure if and only if

(i) \( e(\xi) = 0 \),
(ii) \( \beta_2 w_2(\xi) = 0 \).

**Proof.** \( (\Rightarrow) \) Let \( \eta \) be a 4-distribution in \( \xi \) with complex structure. Then obviously \( e(\xi) = 0 \) and \( \beta_2 w_2(\xi) = \beta_2 w_2(\eta + \epsilon) = \beta_2 w_2(\eta) = \beta_2 \varphi_2 c_1(\eta) = 0 \).

\( (\Leftarrow) \) We have \( \beta_2 w_2(\xi) = 0 \) and \( \beta_2 w_4(\xi) = e(\xi) = 0 \). Consequently, we can find \( a_1 \in H^2(X, \mathbb{Z}) \) and \( a_2 \in H^4(X, \mathbb{Z}) \) such that \( \varphi_2 a_1 = w_2(\xi) \) and \( \varphi_2 a_2 = w_4(\xi) \). Then

\[
\varphi_4(a_1^2 - 2a_2) = \varphi \varphi_2 a_1 + i_{\ast} \varphi_2 a_2 = \varphi w_2(\xi) + i_{\ast} w_4(\xi) = \varphi_4 p_1(\xi).
\]

Hence there is a \( b \in H^4(X, \mathbb{Z}) \) such that \( a_1^2 - 2a_2 - 4b = p_1(\xi) \). Put \( C_1 = a_1 \) and \( C_2 = a_2 + 2b \). According to Theorem 3 there exists a complex vector bundle \( \eta \) over \( X \) of complex dimension 2 with

\[
c_1(\eta) = C_1 \quad \text{and} \quad c_2(\eta) = C_2.
\]
Let us now consider the 5-dimensional real vector bundle $\eta \oplus \varepsilon$. We get

$$w_2(\eta \oplus \varepsilon) = w_2(\eta) = \varepsilon_2c_1(\eta) = \varepsilon_2C_1 = w_2(\xi),$$

$$w_4(\eta \oplus \varepsilon) = w_4(\eta) = \varepsilon_2c_2(\eta) = \varepsilon_2C_2 = w_4(\xi),$$

$$p_1(\eta \oplus \varepsilon) = p_1(\eta) = c_1(\eta)^2 - 2c_2(\eta) = C_1^2 - 2C_2 = p_1(\xi).$$

Theorem 1 implies that $\xi = \eta \oplus \varepsilon$. This completes the proof. □

**Remark.** Let us recall that an $f$-structure on a vector bundle $\xi$ is an endomorphism $f: \xi \to \xi$ satisfying the polynomial equation $f^3 + f = 0$ with $\text{dim} \ker f$ constant. It can be easily seen that if $f$ is an $f$-structure then $\xi = \zeta \oplus \eta$ where $\zeta = \ker f$ and $\eta = \ker(f^2 + \text{id})$. This means that on a vector bundle $\xi$ there exists an $f$-structure if and only if there exists a distribution $\eta \subset \xi$ endowed with a complex structure. If $\xi$ is an oriented 5-dimensional vector bundle over a connected CW-complex $X$ of dimension 5, we can distinguish two cases. In the first case of $\text{dim} \eta = 2$ the existence problem for an $f$-structure is covered by Corollary 1. The second case of $\text{dim} \eta = 4$ is treated in Corollary 3.

**References**


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