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ON THE CLASSIFICATION OF ORIENTED VECTOR BUNDLES OVER 5-COMPLEXES

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1. INTRODUCTION

The effort to classify vector bundles over a fixed CW-complex has a long history. The first result in this direction is the assertion that every two-dimensional oriented vector bundle is uniquely determined by its Euler class. Complete characterization of oriented vector bundles over a 4-dimensional CW-complex was given in [2] using the difference cocycles. In [8] E. Thomas found conditions for a mapping \( f \in [X, Y] \) to be uniquely determined by its cohomology homomorphism \( f^* \in \text{Hom}(H^*(Y), H^*(X)) \) under the assumptions that \( X \) is a suspension or \( Y \) is an H-space. He also applied the result to \( Y = BO \), the classifying space for the group \( O \), and so he obtained conditions on \( H^*(X) \) under which stable vector bundles over \( X \) are determined by their Stiefel-Whitney and Pontrjagin classes. A further progress was made in [3] where the question how many \( n \)-dimensional vector bundles over a CW-complex of the same dimension are determined by a stable vector bundle \( \xi \). The results are given in terms of \( \xi \) and they allow successful application for \( n = 3 \) and \( 7 \). Earlier results concerning characterization of oriented vector bundles over low dimensional complexes were summarized and completed in [13]. Using elementary homotopy theoretic methods and relations among characteristic classes L. M. Woodward has given the classification of stable oriented vector bundles over CW-complexes of dimension \( \leq 8 \) and the classification of \( n \)-dimensional oriented vector bundles over CW-complexes of dimension \( n \) for \( n = 3, 4, 6, 7, 8 \), both in terms of characteristic classes. A typical condition on a CW-complex \( X \) to admit such a classification is: \( H^4(X, \mathbb{Z}) \) has no element of order 4.

In dimension 5 the situation is much more complicated as can be seen on the example of the sphere \( S^5 \). Both the trivial and the tangent bundle over \( S^5 \) have all characteristic classes equal to zero. Moreover, all conditions of Woodward's type are satisfied. The aim of our paper is to derive necessary and sufficient conditions
on a 5-dimensional CW-complex $X$ which make the classification of 5-dimensional oriented vector bundles over $X$ in terms of characteristic classes possible. This is carried out in Section 3 using a combination of the method of Postnikov tower and the Woodward method (see [9] and [13]).

The maximal number of linearly independent sections in a vector bundle $\xi$ is defined to be the span of $\xi$. As a consequence of the classification described above we compute the span of 5-dimensional oriented vector bundles over CW-complexes of the same dimension. These results complete computations of Thomas for tangent bundles over 5-dimensional manifolds given in [12] and also our results for the dimensions 6 and 7 obtained in [1]. Together with results on the existence of a 2-distribution and a 4-distribution with a complex structure they form the contents of Section 4.

2. Preliminaries

All vector bundles will be considered over a connected CW-complex $X$ and will be oriented. The letter $\varepsilon$ will stand for the trivial one-dimensional vector bundle. The mapping $\beta_k: H^*(X, \mathbb{Z}_k) \to H^*(X, \mathbb{Z})$ is the Bockstein homomorphism associated with the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_k \to 0$. The mappings $i_*: H^*(X, \mathbb{Z}_2) \to H^*(X, \mathbb{Z}_4)$ and $q_k: H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z}_k)$ are induced from the inclusion $\mathbb{Z}_2 \to \mathbb{Z}_4$ and reduction mod $k$, respectively.

An important role in our considerations is played by the Pontrjagin square $\Psi$, a cohomology operation from $H^{2k}(X, \mathbb{Z}_2)$ into $H^{4k}(X, \mathbb{Z}_4)$ satisfying the relations

(1) \[ \Psi x^2 = \varphi_4 x^2, \]
(2) \[ \Psi(u + v) = \Psi u + \Psi v + i_*(u \cdot v), \]

for $x \in H^{2k}(X, \mathbb{Z})$ and $u, v \in H^{2k}(X, \mathbb{Z}_2)$. See [5], chapter 2.

We will use $w_j(\xi)$ for the $j$-th Stiefel-Whitney class of the vector bundle $\xi$, $p_1(\xi)$ for the first Pontrjagin class, and $e(\xi)$ for the Euler class. For a complex vector bundle $\zeta$ the symbol $c_j(\zeta)$ denotes the $j$-th Chern class. The letters $w_j$, $p_1$, $e$ stand for characteristic classes of the universal oriented $n$-dimensional vector bundle over the classifying space $BSO(n)$. Our results given below are based on the following relations among the characteristic classes:

(3) \[ q_4 p_1(\xi) = \Psi w_2(\xi) + i_4 w_4(\xi), \]
(4) \[ w_6(\xi) = Sq^2 w_4(\xi) + w_2(\xi) w_4(\xi), \]

the former being proved in [4] and [7] and the latter being a special case of the Wu formula.

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The Eilenberg-MacLane space with the n-th homotopy group $G$ will be denoted by $K(G,n)$ and $i_n$ will stand for the fundamental class in $H^n(K(G,n),G)$. Writing the fundamental class it will be always clear which group $G$ we have in mind.

In the proof of Theorem 1 we will need suspension. Being defined for every fibration $F \xrightarrow{j} E \xrightarrow{p} B$, it is a natural mapping from a subgroup of $H^{k+1}(B)$ into $H^k(F)/\text{im } j^*$ which commutes with the Steenrod squares and $i_*$ (see [5]).

We say that $x \in H^*(X,\mathbb{Z})$ is an element of order $k$ ($k = 2, 3, 4, \ldots$) if and only if $x \neq 0$ and $k$ is the least positive integer such that $kx = 0$ (if it exists). Some results will involve the following hypotheses:

**Condition (A).** $H^4(X,\mathbb{Z})$ has no element of order 4.

**Condition (B).** $Sq^2 H^3(X,\mathbb{Z}_2) = H^5(X,\mathbb{Z}_2)$.

**Remark.** An important example of a CW-complex which satisfies Condition (B) is a 5-dimensional oriented smooth manifold $M$ with $w_2(M) \neq 0$. The Poincaré duality and the fact that the second Wu class is equal to $w_2(M)$ yields

$$Sq^2 H^3(M,\mathbb{Z}_2) = w_2(M)H^3(M,\mathbb{Z}_2) = H^5(M,\mathbb{Z}_2).$$

### 3. Classification theorem

Let $X$ be a connected CW-complex of dimension $\leq 5$. Our problem consists in finding conditions on $X$ such that for every $a \in H^2(X,\mathbb{Z}_2)$, $b \in H^4(X,\mathbb{Z}_2)$, $c \in H^4(X,\mathbb{Z})$ there is at most one oriented 5-dimensional vector bundle $\xi$ with $w_2(\xi) = a$, $w_4(\xi) = b$, $p_1(\xi) = c$. A necessary and sufficient condition on $a$, $b$, $c$ for the existence of such a vector bundle derived in [W] is given by the relation $e_4 c = \mathfrak{Pa} \circ i,*b$ (see (3)). Up to homotopy there is just one mapping $f: X \to K(\mathbb{Z}_2,2) \times K(\mathbb{Z}_2,4) \times K(\mathbb{Z},4)$ such that $f^*(\iota_2 \otimes 1 \otimes 1) = a$, $f^*(1 \otimes \iota_4 \otimes 1) = b$, $f^*(1 \otimes 1 \otimes \iota_4) = c$. Similarly, $w_2$, $w_4$, $p_1$, the cohomology classes of $BSO(5)$, determine a mapping $\alpha: BSO(5) \to K(\mathbb{Z}_2,2) \times K(\mathbb{Z}_2,4) \times K(\mathbb{Z},4)$ which can be considered to be a fibration. Now the problem described above can be formulated as a problem of lifting: when every mapping $f: X \to K(\mathbb{Z}_2,2) \times K(\mathbb{Z}_2,4) \times K(\mathbb{Z},4)$ has at most one lifting $\xi: X \to BSO(5)$ in the fibration $\alpha$. 

\[
\begin{array}{ccc}
X & \xrightarrow{f} & K(\mathbb{Z}_2,2) \times K(\mathbb{Z}_2,4) \times K(\mathbb{Z},4) \\
\xi & \mapsto & BSO(5) \\
\alpha & \downarrow & \\
& & \end{array}
\]
To solve this problem we will construct a Postnikov tower for the fibration \( \alpha: SSO(5) \to K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 4) \times K(\mathbb{Z}, 4) \). Put \( K = K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 4) \times K(\mathbb{Z}, 4) \) and denote the fibre of \( \alpha \) by \( V \). Let us recall that \( \pi_k(SSO(5)) \cong 0 \) for \( k = 1, 3 \), \( \pi_k(SSO(5)) \cong \mathbb{Z}_2 \) for \( k = 2, 5 \) and \( \pi_4(SSO(5)) \cong \mathbb{Z} \). Considering the characteristic classes as mappings from \( BSO(b) \) into appropriate Eilenberg-Mac Lane spaces, we get \( \iota_1: \pi_2(BSO(5)) \to \mathbb{Z}_2 \), \( \iota_4: \pi_4(BSO(5)) \to \mathbb{Z}_2 \), and \( p_1: \pi_4(BSO(5)) \to \mathbb{Z} \) is a multiplication by 2. See [13]. From the long exact homotopy sequence we compute:

\[
\mathbb{Z}_2 \xrightarrow{\iota_1} \pi_2(BSO(5)) = 0, \quad \mathbb{Z}_2 \xrightarrow{\iota_4} \pi_4(BSO(5)) = 0, \quad \pi_3(BSO(5)) = \mathbb{Z}_2, \quad \pi_4(BSO(5)) = \mathbb{Z}_2,
\]

and denote the fibre of \( \alpha \) by \( V \).

Let us recall that \( \pi_k(BO(5)) = 0 \) for \( k = 1, 3, \pi_k(BO(5)) = \mathbb{Z}_2 \) for \( k = 2, 5 \) and \( \pi_4(BO(5)) = \mathbb{Z} \). Considering the characteristic classes as mappings from \( BSO(b) \) into appropriate Eilenberg-Mac Lane spaces, we get \( \iota_1: \pi_2(BSO(5)) \to \mathbb{Z}_2 \), \( \iota_4: \pi_4(BSO(5)) \to \mathbb{Z}_2 \), and \( p_1: \pi_4(BSO(5)) \to \mathbb{Z} \) is a multiplication by 2. See [13]. From the long exact homotopy sequence we compute:

\[
\mathbb{Z}_2 \xrightarrow{\iota_1} \pi_2(BSO(5)) = 0, \quad \mathbb{Z}_2 \xrightarrow{\iota_4} \pi_4(BSO(5)) = 0, \quad \pi_3(BSO(5)) = \mathbb{Z}_2, \quad \pi_4(BSO(5)) = \mathbb{Z}_2,
\]

and denote the fibre of \( \alpha \) by \( V \).

The first invariant in the Postnikov tower is the transgression of a fundamental class in \( H^3(V, \mathbb{Z}_4) \). It is a generator of \( \ker \alpha^* \subseteq H^4(K, \mathbb{Z}_4) \). Hence it is equal to

\[
\rho_4(1 \otimes 1 \otimes \iota_4) - \pi \iota_2 \otimes 1 \otimes 1 - 1 \otimes \iota_4 \otimes 1.
\]

Let \( E_1 \) be the first stage of the Postnikov tower and let the new mappings be denoted according to the diagram.

Consider \( \beta_1: SSO(5) \to E_1 \) as a fibration with a fibre \( F_1 \). This fibre is homotopy equivalent to the homotopy fibre \( F_1 \) of the mapping \( \beta \) (see [9]). Hence computing the homotopy groups of \( F_1 \) we get that \( F_1 \) is 4-connected and \( \pi_5(F_1) \cong \mathbb{Z} \). Consequently, \( \beta_1 \) is a 5-equivalence.

The next invariant \( \varphi \subseteq H^6(E_1, \mathbb{Z}_2) \) is the transgression of the generator of \( H^5(F_1, \mathbb{Z}_2) \) in the Serre exact sequence for the fibration \( \beta_1 \). \( E_1 \) is also the first stage in the Postnikov tower for the fibration \( \hat{\alpha}: BSO(6) \to K \) determined by \( w_2, w_4 \) and \( p_1 \). The mapping \( \hat{\beta}_1: BSO(6) \to E_1 \) in this Postnikov tower is a 6-equivalence (since \( \pi_6(BSO(6)) \cong 0 \)). Using the Serre exact sequence for the fibration \( \hat{\beta}_1 \), we get that \( \hat{\beta}_1 \) is an isomorphism between \( H^6(E_1, \mathbb{Z}_2) \) and \( H^6(BSO(6), \mathbb{Z}_2) \). The latter group has generators \( w_3^2, w_3, w_2w_4 \) and \( Sq^2w_4 = w_6 + w_2w_4 \). Hence the generators of \( H^6(E_1, \mathbb{Z}_2) \) are \( \pi_4^*(\iota_2 \otimes 1 \otimes 1) \), \( \pi_4^*((Sq^1\iota_2)^2 \otimes 1 \otimes 1) \), \( \pi_4^*(\iota_2 \otimes \iota_4 \otimes 1) \), \( \pi_4^*(1 \otimes Sq^2\iota_4 \otimes 1) \). The mapping \( \beta_1^*: H^6(E_1, \mathbb{Z}_2) \to H^6(BSO(5), \mathbb{Z}_2) \) maps them into \( w_3^2, w_3^3, w_2w_4 \) and \( Sq^2w_4 = w_2w_4 \), respectively. Consequently, using the Serre exact sequence for the fibration \( \beta_1 \) we get \( \varphi = \pi_4^*(\iota_2 \otimes \iota_4 \otimes 1 + 1 \otimes Sq^2\iota_4 \otimes 1) \). So we can build the second
stage $E_2$ of our Postnikov tower.

$$
\begin{array}{ccc}
\tilde{F}_2 & \longrightarrow & F_1 \\
\downarrow & & \downarrow i_2 \\
F_2 & \longrightarrow & BSO(5) \\
\downarrow \beta_1 & & \downarrow \pi_2 \\
E_1 & = & E_1 \\
\end{array}
\quad
\begin{array}{ccc}
\tilde{\beta}_2 & \longrightarrow & K(\mathbb{Z}_2, 5) \\
\end{array}
$$

Let the notation of new mappings accord with the diagram. We can consider $\beta_2$ to be a fibration with a fibre $F_2$. Similarly as for the first stage, we can compute the homotopy groups of $F_2$. So we get that $F_2$ is 5-connected and $\beta_2$ is a 6-equivalence.

Let $C = K(\mathbb{Z}_4, 4) \times K(\mathbb{Z}_2, 6)$. Up to homotopy there is just one mapping $k = (k_1, k_2): K \to C$ given by

$$
\begin{align*}
    k_1^*(\iota_4) &= 1 \otimes 1 \otimes \varrho_4 \otimes 1 \otimes 1 + 1 \otimes \iota_4 \otimes 1 \\
    k_2^*(\iota_6) &= \iota_2 \otimes \iota_4 \otimes 1 + 1 \otimes \text{Sq}^2 \iota_4 \otimes 1.
\end{align*}
$$

Due to Lemma 8.1 in [10], there is a homeomorphism $g: E_2 \to E$ where $\pi: E \to K$ is a principal fibration with the classifying map $k: K \to C$. Moreover, $\pi_1 \circ \pi_2 = \pi \circ g$ and the fibration $\beta = g \circ \beta_2: BSO(5) \to E$ is a 6-equivalence. Hence, we can consider the situation

$$
\begin{array}{ccc}
BSO(5) & \xrightarrow{\beta = \text{6-equiv}} & E \\
\downarrow \alpha & & \downarrow \pi \\
K & \xrightarrow{k = (k_1, k_2)} & C
\end{array}
$$

which allows us to prove our main result.

**Theorem 1.** Let $X$ be a connected CW-complex of dimension $\leq 5$ and suppose

$$
\gamma: [X, BSO(5)] \to H^2(X, \mathbb{Z}_2) \oplus H^4(X, \mathbb{Z}_2) \oplus H^4(X, \mathbb{Z})
$$

is defined by $\gamma(\xi) = (w_2(\xi), w_4(\xi), p_1(\xi))$. Then

(i) $\text{im } \gamma = \{(a, b, c) | \varrho_4 c = \varrho a + \iota_4 b\}$,

(ii) $\gamma$ is injective if and only if Conditions (A) and (B) are satisfied.
Proof. (i) follows immediately from the fact that a mapping \( f: X \rightarrow K \) can be lifted in the fibration \( \alpha \) into \( BSO(5) \) if and only if \( f^*(1 \otimes 1 \otimes \varphi_4 \otimes 1 - 1 \otimes i_* \otimes 1) = 0 \). (See similar proofs in [1].)

(ii) Since the space \( E \) is a homotopy fibre of the mapping \( k: K \rightarrow C \), the Puppe sequence

\[
\begin{align*}
\Omega K &\xrightarrow{\Omega k} \Omega C \xrightarrow{q} E \xrightarrow{\pi} K \xrightarrow{k} C
\end{align*}
\]

yields the exact sequence

\[
\rightarrow [X, \Omega K] \xrightarrow{(\Omega k)_*} [X, \Omega C] \xrightarrow{q_*} [X, E] \xrightarrow{\pi_*} [X, K] \xrightarrow{k_*} [X, C].
\]

Moreover, \( \beta \) being a 6-equivalence, \( \beta_*: [X, BSO(5)] \rightarrow [X, E] \) is a bijection for every CW-complex of dimension \( \leq 5 \). The following statements are equivalent:

1. \( \gamma = \alpha_* = \pi_* \circ \beta_*: [X, BSO(5)] \rightarrow [X, K] \) is injective.
2. \( \pi_*: [X, E] \rightarrow [X, K] \) is injective.
3. \( q_* = 0 \)
4. \( (\Omega k)_*: [X, \Omega K] \rightarrow [X, \Omega C] \) is surjective.

Hence we need to compute \((\Omega k_1)^*: H^3(K(Z, 3), Z_4) \rightarrow H^3(\Omega K, Z_4)\) and \((\Omega k_2)^*: H^5(K(Z_2, 5), Z_2) \rightarrow H^5(\Omega K, Z_2)\).

First, let us consider \( k_1 \).

\[
\begin{align*}
\Omega K &\xrightarrow{\Omega k_1} K(Z_4, 3) \\
PK &\xrightarrow{P k_1} PK(Z_4, 4) \\
K &\xrightarrow{k_1} K(Z_4, 4)
\end{align*}
\]

Every element in \( H^*(K, Z_4) \) is suspensive. If we denote all suspensions by \( \sigma \), we get

\[
(\Omega k_1)^*(\iota_3) = (\Omega k_1)^*(\sigma \iota_4) = \sigma(k_1^* \iota_4) = \sigma(1 \otimes 1 \otimes \varphi_4 \otimes 1 - 1 \otimes 1 \otimes \varphi_4 \otimes 1 - 1 \otimes 1) - \sigma(1 \otimes i_* \otimes 1) = 1 \otimes 1 \otimes \sigma(\varphi_4 \otimes 1) - \sigma(\varphi_2 \otimes 1) \otimes 1 - 1 \otimes \sigma(i_* \otimes 1) \otimes 1
\]

the last equality being a consequence of the definition of suspension and coboundary operator. In the fibration \( K(Z, 3) \rightarrow PK(Z_4, 4) \rightarrow K(Z_4, 4) \) we get

\[
\sigma(\varphi_4) = \varphi_4(\sigma_4) = \varphi_4 \iota_3.
\]
In the fibration $K(\mathbb{Z}_2, 3) \rightarrow PK(\mathbb{Z}_2, 4) \rightarrow K(\mathbb{Z}_2, 4)$ we have

$$\sigma(i_*\iota_4) = i_*(\sigma\iota_4) = i_*\iota_3$$

and finally, in the fibration $K(\mathbb{Z}_2, 1) \rightarrow PK(\mathbb{Z}_2, 2) \rightarrow K(\mathbb{Z}_2, 2)$ we obtain

(5) $$\sigma(\mathcal{P}\iota_2) = i_*\iota_1^3.$$ 

Since this fact is not generally known, we will prove it at the end of this section. As a result of these computations we get

$$(\Omega k_1)_* : [X, \Omega K] \rightarrow [X, K(\mathbb{Z}_4, 3)] : (a, b, c) \mapsto g_4c - i_*a^3 - i_*b.$$ 

Hence $(\Omega k_1)_*$ is surjective if and only if

(6) $$H^3(X, \mathbb{Z}_4) = g_4H^3(X, \mathbb{Z}) + i_*H^3(X, \mathbb{Z}_2).$$

We show that (6) is equivalent to the condition (A).

(A) $\Rightarrow$ (6). Let $x \in H^3(X, \mathbb{Z}_4)$, then $4\beta_4x = 0$. (A) implies that $2\beta_4x = 0$. Consequently, there is a $y \in H^3(X, \mathbb{Z}_2)$ such that $\beta_4x = \beta_2y = \beta_4i_*y$. That is why

$$\beta_4(x - i_*y) = 0,$$

which implies $x = i_*y + g_4z$ for some $z \in H^3(X, \mathbb{Z})$.

(6) $\Rightarrow$ (A). Let $v \in H^4(X, \mathbb{Z})$ satisfy $4v = 0$. Then $v = \beta_4x$ where $x = g_4z + i_*y \in H^3(X, \mathbb{Z}_4)$ so that $v = \beta_4g_4z + \beta_4i_*y = \beta_4i_*y = \beta_2y$. Hence $2v = 0$ and $v$ is not an element of order 4.

Now consider the mapping $k_2$. The computation of $(\Omega k_2)^* : H^5(K(\mathbb{Z}_2, 5), \mathbb{Z}_2) \rightarrow H^5(\Omega K, \mathbb{Z}_2)$ gives

$$(\Omega k_2)^*(\iota_5) = (\Omega k_2)^*(\sigma\iota_6) = \sigma k_2^*(\iota_6) = 1 \otimes \sigma(Sq^2\iota_4) \otimes 1 + \sigma(\iota_2 \otimes \iota_4) \otimes 1$$

$$= 1 \otimes Sq^2\iota_3 \otimes 1 + \sigma(\iota_2 \otimes \iota_4) \otimes 1.$$ 

We are going to prove that $\sigma(\iota_2 \otimes \iota_4) = 0$. Consider the fibration

$$\Omega B \rightarrow PB \xrightarrow{p} B$$

where $B = K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 4)$. Let $\hat{p}^* : H^6(B, \mathbb{Z}_2) \rightarrow H^6(PB, \Omega B; \mathbb{Z}_2)$ be determined by the mapping $p$. It is sufficient to show $\hat{p}^*(\iota_2 \otimes \iota_4) = 0$. Using the Serre spectral sequence with coefficients $\mathbb{Z}_2$ for the above fibration, we have

$$\hat{p}^* : H^6(B, \mathbb{Z}_2) \cong E_2^{6,0} \Rightarrow E_8^{6,0} \Rightarrow H^6(PB, \Omega B; \mathbb{Z}_2).$$
We compute \( d_2: E_2^{4,1} \to E_2^{6,0} \). Since \( E_2^{4,1} \cong E_2^{4,0} \otimes E_2^{0,1} \), for the generators of \( E_2^{4,1} \) we obtain

\[
d_2(\iota_2^2 \otimes \iota_1) = d_2(\iota_2^2) \cdot \iota_1 + \iota_2^2 \cdot d_2(\iota_1) = \iota_2^2 \cdot \iota_2 = \iota_2^3,
\]

\[
d_2(\iota_4 \otimes \iota_1) = d_2(\iota_4) \cdot \iota_1 + \iota_4 \cdot d_2(\iota_1) = \iota_4 \cdot \iota_2.
\]

Hence \( \iota_4 \cdot \iota_2 \) vanishes in \( E_3^{6,0} \) and \( \tilde{p}^*(\iota_2 \otimes \iota_4) = 0 \).

So we conclude that

\[
(\Omega k_2)_* [X, \Omega K] \to [X, K(\mathbb{Z}_2, 5)]: (a, b, c) \mapsto S q^2 b
\]

and its surjectivity is given directly by Condition (B).

It remains to prove the relation (5). Consider the Serre spectral sequence for the fibration \( K(\mathbb{Z}_2, 1) \to PK(\mathbb{Z}_2, 2) \to K(\mathbb{Z}_2, 2) \) with coefficients \( \mathbb{Z}_4 \). For brevity we will again denote this fibration by \( \Omega B \to PB \xrightarrow{p} B \). It is not difficult to show that \( H^4(B, \mathbb{Z}_4) \cong \mathbb{Z}_4 \) with the generator \( \Omega \iota_2 \) and \( H^3(\Omega B, \mathbb{Z}_4) \cong \mathbb{Z}_2 \) with the generator \( \iota_1^3 \). The coboundary operator in the long exact sequence for the couple \( (PB, \Omega B) \) is an isomorphism, hence it is sufficient to prove that \( \tilde{p}^*(\Omega \iota_2) \neq 0, \tilde{p}^*: H^4(B, \mathbb{Z}_4) \to H^4(PB, \Omega B; \mathbb{Z}_4) \) being induced by \( p \). Since

\[
E_4^{4,0} \cong H^4(B, \mathbb{Z}_4)/\ker \tilde{p}^*,
\]

it is sufficient to show that \( E_4^{4,0} \neq 0 \). We have

\[
E_2^{2,1} \cong H^2(B, H^1(\Omega B, \mathbb{Z}_4)) \cong \mathbb{Z}_2 \cong E_2^{2,0} \otimes E_2^{0,1},
\]

\[
E_2^{4,0} \cong H^4(B, H^0(\Omega B, \mathbb{Z}_4)) \cong \mathbb{Z}_4.
\]

Moreover, \( d_2: E_2^{2,1} \to E_2^{4,0} \) is injective because

\[
d_2(i_* \iota_2 \otimes i_* \iota_1) = d_2(i_* \iota_2) \cdot i_* \iota_1 + i_* \iota_2 \cdot d_2(i_* \iota_1) = i_* \iota_2 \cdot \tau(i_* \iota_1) = i_* \iota_2^2
\]

where \( \tau \) is a transgression. Hence \( E_3^{4,0} \cong \mathbb{Z}_2 \). Further, \( E_3^{1,2} \cong 0, E_3^{7,-2} \cong 0 \) and consequently, \( E_4^{4,0} \cong \mathbb{Z}_2 \). \( \square \)
4. SPAN AND THE EXISTENCE OF DISTRIBUTIONS

In this section we compute the span of oriented 5-dimensional vector bundles over a 5-dimensional CW-complex satisfying Conditions (A) and (B) of Theorem 1. Under the same conditions we find all oriented 5-dimensional vector bundles which admit a 2-distribution, i.e. an oriented 2-dimensional subbundle, and all oriented 5-dimensional vector bundles which admit a 4-distribution endowed with a complex structure, i.e. a complex 2-dimensional subbundle. For these purposes we need

**Theorem 2.** Let $X$ be a connected CW-complex of dimension $\leq 5$ and let $W \in H^2(X, \mathbb{Z}_2)$, $P \in H^4(X, \mathbb{Z})$. Then there exists an oriented 3-dimensional vector bundle $\xi$ over $X$ with

$$w_2(\xi) = W, \quad p_1(\xi) = P$$

if and only if

$$\varphi_4 P = \mathfrak{P} W.$$

**Proof.** is very similar to the proof of the first part of Theorem 1. See also [13].

**Corollary 1.** Let $X$ be a connected CW-complex of dimension $\leq 5$ satisfying Conditions (A) and (B). Then an oriented 5-dimensional vector bundle $\xi$ has a 2-distribution with Euler class $U$ if and only if

$$\varphi_2 U^2 + w_2(\xi) \varphi_2 U + w_4(\xi) = 0. \tag{7}$$

**Proof.** ($\Rightarrow$) Let $\xi = \zeta \oplus \tau$ where $\tau$ is an oriented 2-dimensional vector bundle over $X$ with the Euler class $U$ and $\zeta$ is an oriented 3-dimensional vector bundle over $X$. Then

$$w_2(\xi) = w_2(\zeta) + w_2(\tau) = w_2(\zeta) + \varphi_2 U,$$

$$w_4(\xi) = w_2(\zeta) \cdot w_2(\tau) = w_2(\zeta) \cdot \varphi_2 U.$$

Substituting from here into the expression $\varphi_2 U^2 + w_2(\xi) \cdot \varphi_2 U + w_4(\xi)$, we get (7).

($\Leftarrow$) Let $U \in H^2(X, \mathbb{Z})$ satisfy (7). There is an oriented 2-dimensional vector bundle $\tau$ over $X$ with the Euler class $U$. Put

$$W = w_2(\xi) + \varphi_2 U, \quad P = p_1(\xi) - U^2.$$
Then

$$e_4P - \mathfrak{P}W = e_4p_1(\xi) - e_4U^2 - \mathfrak{P}(w_2(\xi) + e_2U) =$$

$$= e_4p_1(\xi) - e_4U^2 - \mathfrak{P}w_2(\xi) - \mathfrak{P}e_2U - i_*(w_2(\xi)e_2U) =$$

$$= i_*(e_2U^2 + w_2(\xi)e_2U + w_4(\xi)) = 0.$$  

According to Theorem 2, there is an oriented 3-dimensional vector bundle $\zeta$ over $X$ with $w_2(\zeta) = W$ and $p_1(\zeta) = P$. We compute the characteristic classes of the vector bundle $\zeta \oplus \tau$.

$$w_2(\zeta \oplus \tau) = w_2(\zeta) + w_2(\tau) = W + e_2U = w_2(\xi),$$

$$w_4(\zeta \oplus \tau) = w_2(\zeta) \cdot w_2(\tau) = W \cdot e_2U = w_2(\xi)e_2U + e_2U^2 =$$

$$= w_4(\xi),$$

$$p_1(\zeta \oplus \tau) = p_1(\zeta) + p_1(\tau) = P + U^2 = p_1(\xi).$$

(See [13] for the additivity of $p_1$ in this case.) Theorem 1 now implies that $\xi = \zeta \oplus \tau$, which completes the proof.

Remark. As far as it is known to the authors there are only two general results concerning 2-distributions in 5 or $4k + 1$-dimensional vector bundles. See [11], Theorems 1.3 and 4.1. The former deals with spin manifolds (i.e. $w_1(X) = w_2(X) = 0$) and tangent bundles while the latter requires $\text{span} \geq 2$. Both examine the existence of 2-distributions with the Euler class $2U \in H^2(X, \mathbb{Z})$.

Corollary 2. Let $X$ be a connected CW-complex of dimension $\leq 5$ and let $\xi$ be an oriented 5-dimensional vector bundle over $X$. Then

1. span$\xi \geq 1$ if and only if $e(\xi) = 0$.
2. span$\xi \geq 2$ if and only if $w_4(\xi) = 0$.
3. span$\xi \geq 3$ if and only if $w_4(\xi) = 0$ and there is a $U \in H^2(X, \mathbb{Z})$ such that $w_2(\xi) = e_2U$, $p_1(\xi) = U^2$.
4. span$\xi = 5$ if and only if $w_2(\xi) = 0$, $w_4(\xi) = 0$, $p_1(\xi) = 0$.

Proof. (1) is well known and is included only for completeness.

(2) is the immediate consequence of Corollary 1 for $U = 0$.

(3) $\Rightarrow$ Let $\xi = \zeta \oplus 3\varepsilon$ where $\zeta$ is an oriented 2-dimensional vector bundle over $X$. Then $w_4(\xi) = w_4(\zeta) = 0$ and for $U = e(\zeta)$ we get $w_2(\xi) = w_2(\zeta) = e_2U$, $p_1(\xi) = p_1(\zeta) = U^2$.

$\Leftarrow$ For $U \in H^2(X, \mathbb{Z})$ there is an oriented 2-dimensional vector bundle $\zeta$ over $X$ such that $e(\zeta) = U$. Then $w_2(\zeta \oplus 3\varepsilon) = w_2(\zeta) = e_2U = w_2(\xi)$, $w_4(\zeta \oplus 3\varepsilon) = w_4(\zeta) = 0$.  

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0 = \omega_4(\xi) and \rho_1(\zeta \oplus 3\varepsilon) = \rho_1(\zeta) = U^2 = \rho_1(\xi). Theorem 1 implies that \zeta \oplus 3\varepsilon = \xi since the characteristic classes of both vector bundles are the same.

(4) follows immediately from Theorem 1.

Remark. Statements (3) and (4) of Corollary 2 under a little bit different conditions were already known to E. Thomas [12]. Statement (2) under Conditions (A) and (B) is new. It deals with the cases which are not covered in [12]. The condition \omega_4(\xi) = 0 coincides with the condition for the stable span of 4k + 1-dimensional vector bundles over a CW-complex of the same dimension to be \geq 2. See [6], Theorem 2.1.1.

Now we will investigate the existence of distributions with complex structure. The case of 2-distributions is treated in Corollary 1. Here we will deal with 4-distributions. For this purpose we need the following

**Theorem 3.** Let X be a connected CW-complex of dimension \leq 5 and let \zeta_1 \in H^2(X, \mathbb{Z}), \zeta_2 \in H^4(X, \mathbb{Z}). Then there exists a 2-dimensional complex vector bundle \zeta over X with the Chern classes

\[ c_1(\zeta) = \zeta_1, \quad c_2(\zeta) = \zeta_2. \]

**Proof** of this theorem follows the same lines as in [13].

**Corollary 3.** Let X be a connected CW-complex of dimension \leq 5 satisfying the conditions (A) and (B). Then an oriented 5-dimensional vector bundle \xi over X has a 4-distribution with a complex structure if and only if

(i) \epsilon(\xi) = 0,

(ii) \beta_2\omega_2(\xi) = 0.

**Proof.** (\Rightarrow) Let \eta be a 4-distribution in \xi with complex structure. Then obviously \epsilon(\xi) = 0 and \beta_2\omega_2(\xi) = \beta_2\omega_2(\eta \oplus \varepsilon) = \beta_2\omega_2(\eta) = \beta_2\varepsilon_2 c_1(\eta) = 0.

(\Leftarrow) We have \beta_2\omega_2(\xi) = 0 and \beta_2\omega_4(\xi) = \epsilon(\xi) = 0. Consequently, we can find \alpha_1 \in H^2(X, \mathbb{Z}) and \alpha_2 \in H^4(X, \mathbb{Z}) such that \varepsilon_2\alpha_1 = \omega_2(\xi) and \varepsilon_2\alpha_2 = \omega_4(\xi). Then

\[ \varepsilon_4(\alpha_1^2 - 2\alpha_2) = \mathbb{P}\varepsilon_2\alpha_1 + i_*\varepsilon_2\alpha_2 = \mathbb{P}\omega_2(\xi) + i_*\omega_4(\xi) = \varepsilon_4\rho_1(\xi). \]

Hence there is a b \in H^4(X, \mathbb{Z}) such that \alpha_1^2 - 2\alpha_2 - 4b = \rho_1(\xi). Put \zeta_1 = \alpha_1 and \zeta_2 = \alpha_2 + 2b. According to Theorem 3 there exists a complex vector bundle \eta over X of complex dimension 2 with

\[ c_1(\eta) = \zeta_1 \quad \text{and} \quad c_2(\eta) = \zeta_2. \]
Let us now consider the 5-dimensional real vector bundle $\eta \oplus \varepsilon$. We get

$$w_2(\eta \oplus \varepsilon) = w_2(\eta) = \varphi_2 c_1(\eta) = \varphi_2 C_1 = w_2(\xi),$$
$$w_4(\eta \oplus \varepsilon) = w_4(\eta) = \varphi_2 c_2(\eta) = \varphi_2 C_2 = w_4(\xi),$$
$$p_1(\eta \oplus \varepsilon) = p_1(\eta) = c_1(\eta)^2 - 2c_2(\eta) = C_1^2 - 2C_2 = p_1(\xi).$$

Theorem 1 implies that $\xi = \eta \oplus \varepsilon$. This completes the proof. 

**Remark.** Let us recall that an $f$-structure on a vector bundle $\xi$ is an endomorphism $f : \xi \to \xi$ satisfying the polynomial equation $f^3 + f = 0$ with $\dim \ker f$ constant. It can be easily seen that if $f$ is an $f$-structure then $\xi = \zeta \oplus \eta$ where $\zeta = \ker f$ and $\eta = \ker(f^2 + \text{id})$. This means that on a vector bundle $\xi$ there exists an $f$-structure if and only if there exists a distribution $\eta \subset \xi$ endowed with a complex structure. If $\xi$ is an oriented 5-dimensional vector bundle over a connected CW-complex $X$ of dimension 5, we can distinguish two cases. In the first case of $\dim \eta = 2$ the existence problem for an $f$-structure is covered by Corollary 1. The second case of $\dim \eta = 4$ is treated in Corollary 3.

**References**


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