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AN ITERATIVE CONSTRUCTION OF BASES FOR FINITELY GENERATED MODULES OVER PRINCIPAL IDEAL DOMAINS

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The existence of a set of linearly independent generators (i.e., a basis) for a finitely generated Module $V$ over a Principal Ideal Ring (i.e., a generalization of the Fundamental theorem of Abelian groups) is proved here in a well motivated way which starts by choosing from all possible sets of generators of $V$ a set $G$ of generators of $V$ such that $G$ has a smallest number of generators and such that $G$ also contains an element, say, $b$ of the minimal (as defined below) order. Then the process is repeated for the submodule of $V$ generated by $G - \{b\}$, etc. The completion of the process yields a basis of $V$. The proofs are considerably simpler and more lucid than those known in the existing literature and remain the same whether $V$ does or does not have elements of infinite order.

In what follows we shall use well known items and facts of any principal ideal domain $R$ such as the existence of a greatest common divisor of finitely many elements of $R$ (and its representation as a linear combination of these elements) the units and associates of $R$ and the fact that $R$ is a unique factorization domain, etc. [2, 3].

**Lemma 1.** Let $R$ be a principal ideal domain and let $a_n, \ldots, a_1$ be elements of $R$ with a greatest common divisor $g_n$, i.e.,

$$ (a_n, \ldots, a_1) = g_n. $$

Then there exists an $n$ by $n$ matrix $M_n$ with entries over $R$, whose first row is $a_n, \ldots, a_1$ and whose determinant is equal to $g_n$, i.e.,

$$ \det M_n = g_n. $$

**Proof.** We use induction to prove the Lemma. The statement of the Lemma is trivially true for $n = 1$. Let us assume that the Lemma is true for the $n - 1$ elements
\( a_{n-1}, \ldots, a_1 \) of \( R \), i.e.,

\[
(a_{n-1}, \ldots, a_1) = g_{n-1}
\]

and that there exists an \( n - 1 \) by \( n - 1 \) matrix \( M_{n-1} \) such that

\[
M_{n-1} = \begin{bmatrix}
a_{n-1} & \cdots & a_1 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}
\]

and \( \det M_{n-1} = g_{n-1} \).

Since \( R \) is a principal ideal domain from (1) and (3) it follows that

\[
g_n = pa_n + qg_{n-1}
\]

for some elements \( p \) and \( q \) of \( R \).

From (3) and (5) it follows that

\[
p(a_{n-1}/g_{n-1}), \ldots, p(a_1/g_{n-1}) \text{ are } n - 1\text{-elements of } R.
\]

Let \( M^*_{n-1} \) be an \( n - 1 \) by \( n - 1 \) matrix which is obtained by replacing the first row of the matrix \( M_{n-1} \) by the \( n - 1 \) elements of \( R \) given in (6). But then, clearly, from (4) and (6) it follows that

\[
\det M^*_{n-1} = p.
\]

Now, let us consider the \( n \) by \( n \) matrix \( M_n \) which extends the \( n - 1 \) by \( n - 1 \) matrix \( M^*_{n-1} \) on top by one row \( a_n, a_{n-1}, \ldots, a_1 \) (i.e., precisely \( a_n \) followed by the elements of the first row of matrix \( M_{n-1} \)) and on the left by one column as shown below:

\[
M_n = \begin{bmatrix}
a_n & a_{n-1} & \cdots & a_1 \\
-q & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & M^*_{n-1}
\end{bmatrix}.
\]

But then expanding the determinant of \( M_n \) along its first column, from (4), (5) and (7) it follows \( \det M_n = g_n \). Thus, \( M_n \) is an \( n \) by \( n \) matrix with entries over \( R \), whose first row is \( a_n, \ldots, a_1 \) and \( M_n \) satisfies (2). Hence, the proof of the Lemma is complete. \( \Box \)
Corollary 1. Let $a_1, \ldots, a_n$ be $n$ relatively prime elements of a principal ideal domain $R$. Then there exists an $n$ by $n$ matrix $M_n$ with entries over $R$ whose first row is $a_1, \ldots, a_n$ such that $\det M = 1$. Moreover, $M_n$ is invertible and $M_n^{-1}$ is an $n$ by $n$ matrix with entries over $R$.

Proof. By the assumption, $(a_1, \ldots, a_n) = 1$. Thus, from (1) and (2) it follows that $\det M_n = 1$. But then clearly, $M_n^{-1}$ exists and its entries are over $R$. □

Lemma 2. Let $R$ be a principal ideal domain and $V$ be an $R$-module generated by $n$ generators $g_1, \ldots, g_n$. Let $a_1, \ldots, a_n$ be $n$ relatively prime elements of $R$. Then $V$ can be also generated by a set of $n$ generators includes $a_1 g_1 + \ldots + a_n g_n$ as one of the generators.

Proof. Let $M_n$ be the matrix mentioned in Corollary 1. Clearly,

\[
\begin{pmatrix}
    g_1 \\
    \vdots \\
    g_n
\end{pmatrix} = M_n^{-1} M_n \begin{pmatrix}
    g_1 \\
    \vdots \\
    g_n
\end{pmatrix} = M_n^{-1} \begin{pmatrix}
    a_1 g_1 + \ldots + a_n g_n \\
    \vdots
\end{pmatrix}.
\]

Obviously, the elements of the rightmost column appearing in (9) form a set of generators of $V$. Indeed, as (9) shows everyone of the $n$ generators $g_1, \ldots, g_n$ of $V$ is a linear combination of the elements of the rightmost column appearing in (9). But then since $a_1 g_1 + \ldots + a_n g_n$ is one of the elements of the rightmost column appearing in (9), we see that there exists a set of $n$ generators of $V$ which includes $a_1 g_1 + \ldots + a_n g_n$ (which could be 0) as one of the generators. Thus, Lemma 2 is proved. □

Remark 1. We note that the proof of Lemma 1 gives us a constructive method of building of the matrix $M_n$ and that Lemma 2 gives us a constructive method of replacing a set of generators of $R$ with another set of generators of $R$ [cf. 1].

Let $R$ be a principal ideal domain, we recall that elements $x$ and $y$ of $R$ are called associates (denoted by $x \simeq y$) iff $x = uy$ for some unit $u$ of $R$. We define order $<$ (read: less than) in $R$ as follows:

\[
x < y \text{ if and only if } x \mid y \text{ and } x \not\simeq y,
\]

i.e., $x$ divides $y$ and $x$ and $y$ are not associates. This means that $x$ and $y$ are not associates and that $y$ is an elements of the ideal generated by $x$. Since $R$ has no infinite properly ascending chain of ideals [3, p. 121], we have:

\[
\text{every nonempty subset of } R \text{ has a minimal (i.e., } < \text{-minimal) element.}
\]

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Let \( V \) be a module over a principal ideal domain \( R \). As expected, a minimal annihilator (if it exists) of an element \( v \) of \( R \) is called order of \( v \) (denoted by \( \text{ord} \ v \)); otherwise, \( v \) is said to be of infinite order. Clearly \( \text{ord} \ v \) is defined up to an associate. We observe that \( \text{ord} \ v \) coincides with its classical definition [3, p. 165]. Let \( a_1v_1 + \ldots + a_nv_n \) be a linear combination of the elements \( v_i \) of \( V \) with \( a_i \) elements of \( R \). We say that \( a_1v_1 + \ldots + a_nv_n \) is nontrivial in \( v_n \) if and only if

\[
a_1v_1 + \ldots + a_nv_n = 0 \quad \text{and} \quad a_nv_n \neq 0.
\]

**Lemma 3.** Let, as in (12), \( a_1v_1 + \ldots + a_nv_n \) be nontrivial in \( v_n \) and \( v_n \) be not of infinite order. Then there exists a linear combination \( b_1v_1 + \ldots + b_nv_n \) such that

\[
b_1v_1 + \ldots + b_nv_n \quad \text{is nontrivial in} \ v_n \quad \text{and} \quad b_n < \text{ord} \ v_n
\]

**Proof.** Indeed, let

\[
b_n = (a_n, \text{ord} \ v_n) = xa_n + y(\text{ord} \ v_n).
\]

Clearly, \( b_n \neq \text{ord} \ v_n \) since otherwise, in view of (14), \( \text{ord} \ v_n \) would divide \( b_n \) and also would divide \( a_n \) contradicting (12). On the other hand, since \( b_n \) divides \( \text{ord} \ v_n \) from (10) it follows that \( b_n < v_n \). But then, from (12) and (14) we obtain

\[
0 = x(a_1v_1 + \ldots + a_nv_n) + y(\text{ord} \ v_n)v_n
\]

\[
= x a_1v_1 + \ldots + (xa_n + \ldots + y(\text{ord} \ v_n))v_n = b_1v_1 + \ldots + b_nv_n
\]

where \( b_i = xa_i \) for \( i < n \). Clearly, in the above \( b_nv_n \neq 0 \) since \( b_n < \text{ord} \ v_n \). Thus, (13) is established, and the Lemma is proved. \( \Box \)

Let \( R \) be a principal ideal domain and \( V \) be an \( R \)-module generated by \( n \) pairwise distinct nonzero generators \( g_1, \ldots, g_n \). We recall that these \( n \) generators form a basis of \( V \) if and only if 0 (the zero of \( V \)) cannot be equal to a linear combination of \( g_1, \ldots, g_n \) over \( R \) with some nonzero summands.

**Theorem.** Let \( R \) be a principal ideal domain and \( V \) be a finitely generated \( R \)-module. Then \( V \) has a basis.

**Proof.** We prove the Theorem in its following version. Let \( V \) be such that it can be generated by \( n \) generators \( g_1, \ldots, g_n \) and not by less than \( n \) generators, where (to avoid the trivial case) we let \( n > 1 \). We use induction. Thus, we assume that any \( R \)-module which can be generated by \( n - 1 \) generators and not by less than
$n - 1$ generators has a basis. Now, by (11), among all possible sets of $n$ generators of $V$ we choose a set \{\(g_1, \ldots, g_{n-1}, b\)\} such that no set of $n$ generators of $V$ has an element of order (which could be infinite) less than the order of $b$. Clearly, the submodule $S$ of $V$ which is generated by the set \{\(g_1, \ldots, g_{n-1}\)\} of $n - 1$ generators cannot be generated by less than $n - 1$ generators since $V$ cannot be generated by less than $n$ generators. Hence, by our assumption, $S$ has a basis, say, \{\(b_1, \ldots, b_{n-1}\)\}. We prove the Theorem by showing that \{\(b_1, \ldots, b_{n-1}, b\)\} is a basis of $V$. Obviously, \{\(b_1, \ldots, b_{n-1}, b\)\} generates $V$. Let us assume to the contrary, and therefore

\[(15) \quad a_1 b_1 + \ldots + a_{n-1} b_{n-1} + a_n b = 0 \quad \text{and} \quad a_n b \neq 0.\]

But then $a_1, \ldots, a_{n-1}, a_n$ cannot be relatively prime since otherwise from Lemma 2 it would follow that $a_1 b_1 + \ldots + a_{n-1} b_{n-1} + a_n b$ could be a member of a set of $n$ generators of $V$ which by (15) would imply that 0 would be a member of a set of $n$ generators of $V$ and therefore $V$ could be generated by less than $n$ generators which is impossible. Hence, $a_1, \ldots, a_{n-1}, a_n$ are not relatively prime and \((a_1, \ldots, a_{n-1}, a_n) = d \neq 1\). But then from (15) we have

\[(16) \quad d((a_1/d) b_1 + \ldots + (a_{n-1}/d) b_{n-1} + (a_n/d) b) = 0\]

where \((a_1/d), \ldots, (a_{n-1}/d), (a_n/d)\) are now relatively prime. But then, again, from Lemma 2 it would follow that \(b^* = (a_1/d) b_1 + \ldots + (a_{n-1}/d) b_{n-1} + (a_n/d) b\) could be a member of a set of $n$ generators of $V$ which by (16) would lead to a contradiction if $b$ were of infinite order. Thus, in (15), we let $b$ be not of infinite order, and, in view of (13), without loss of generality we may assume that in (15) it is the case that $a_n < \text{ord} b$. But then, from (16) we see that $\text{ord} b^*$ divides $d$ which in turn divides $a_n$ and therefore by (10) we have $\text{ord} b^* < \text{ord} b$, contradicting the choice of $b$. Thus, the Theorem is proved. 

Remark 2. From the proof of the Theorem it follows that if $V$ is a finitely generated module over a principal ideal domain such that no set of generators of $V$ has an element of not of infinite order then any set with least number of generators of $V$ is a base of $V$. Also, since every finitely generated Abelian group is a finitely generated module over the integral domain of integers, the above Theorem and its proof implies the following Fundamental Theorem of Abelian Groups with a proof which does not consider two cases of Torsion and Torsion free subgroups of the group.

**Corollary 2.** Every Finitely Generated Abelian group has a basis and therefore is a direct sum of its cyclic subgroups.

Remark 3. The central lines of ideas and proofs given above are generalized version of the ideas in [4] to the case of Modules over principal ideal domains. The
generalization is nontrivial as witnessed by Lemma 3 and the succeeding proofs. Also, it can be shown that based on Lemmas 1, 2, 3 an iterative process can be devised which starting with a set of generators of $V$ will yield a basis of $V$ in finitely many steps.

References


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