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On local joint capacities of operators


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Let $T = (T_1, \ldots, T_n)$ be an $n$-tuple of commuting operators in a Banach space $X$. Then the set of all $x \in X$ for which the local (Halmos-Stirling) capacity $\text{cap} (T, x)$ is equal to the capacity $\text{cap} T$ is dense in $X$. This generalizes the corresponding result for one operator [5].

Denote by $B(X)$ the algebra of all bounded operators in a Banach space $X$. Let $S \in B(X)$ and $x \in X$. The problem of describing the behaviour of all powers $S^n x$ (or all polynomials $p(S)x$) appears naturally in many questions of operator theory (e.g. local spectral theory or invariant subspace problem, cf. [1]).

The present paper was originally inspired by the paper of Halmos [2] and his notions of capacity in Banach algebras and quasialgebraic operators. He asked also whether every locally quasialgebraic operator is (globally) quasialgebraic, i.e. if there is a version of Kaplansky's theorem for quasialgebraic operators. An affirmative answer to this question was given in [4] and the result was improved in [5]. The present paper continues this study and generalizes the results for $n$-tuples of commuting operators.

Let $T = (T_1, \ldots, T_n)$ be an $n$-tuple of mutually commuting operators in a Banach space $X$.

We denote by $\sigma(T) \subseteq \mathbb{C}^n$ the Harte spectrum [3] of $T$, i.e. $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ does not belong to $\sigma(T)$ if and only if there exist operators $L_1, \ldots, L_n, R_1, \ldots, R_n \in B(X)$ such that

$$\sum_{i=1}^{n} L_i (T_i - \lambda_i) = I = \sum_{i=1}^{n} (T_i - \lambda_i) R_i.$$ 

Denote by $\sigma_e(T)$ the essential spectrum of $T$, i.e. the Harte spectrum of the commuting $n$-tuple $\pi(T) = (\pi(T_1), \ldots, \pi(T_n))$ in the Calkin algebra $B(X) \big| K(X)$, where

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$K(X)$ is the ideal of compact operators and $\pi: B(X) \to B(X)|K(X)$ is the canonical projection. We define formally $\sigma_e(T) = \emptyset$ for a commuting $n$-tuple $T$ of operators in a finite-dimensional Banach space.

For an operator $S_1 \in B(X)$ denote by $r_e(S_1)$ the essential spectral radius of $S_1$, i.e. $r_e(S_1) = \max\{|\mu|: \mu \in \sigma_e(S_1)\}$.

Denote further by $\sigma_{\pi e}(T)$ the essential approximate point spectrum of $T$, i.e. $\lambda \in \sigma_{\pi e}(T)$ if and only if

$$\inf\left\{\sum_{i=1}^{n}||T_i - \lambda_i||: x \in M, ||x|| = 1\right\} = 0$$

for every subspace $M \subset X$ of finite codimension.

We denote by $\mathcal{P}_r(n)$ the set of all polynomials in $n$ variables with degree $\deg p \leq r$. Every $p \in \mathcal{P}_r(n)$ can be written in the form

$$p(z) = \sum_{|\alpha| \leq r} c_{\alpha}(p)z^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an $n$-tuple of non-negative integers, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, the coefficients $c_{\alpha}(p)$ are complex, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. If $p$ is a polynomial in $n$ variables and $K \subset \mathbb{C}^n$ a compact set then $\|p\|_K = \max\{|p(z)|: z \in K\}$. We say that a set $K \subset \mathbb{C}^n$ is algebraic if $p(K) \subset \{0\}$ for some non-zero polynomial $p$.

The first lemma uses the idea of extremal points of Fekete-Leja, see [8]. The authors are indebted to Professor J. Siciak for supplying the proof of it. Our proof is slightly modified.

**Lemma 1.** Let $n, r$ be positive integers and $K \subset \mathbb{C}^n$ a compact set. Then there exists a finite subset $K' \subset K$ with $\text{card } K' = m \leq \binom{n+r}{n}$ such that

$$\|p\|_K \leq m \cdot \|p\|_{K'} \quad (p \in \mathcal{P}_r(n)).$$

**Proof.** Denote by $L = \{p \in \mathcal{P}_r(n): \|p\|_K = 0\}$ and let $M$ be a complementary space of $L$ in $\mathcal{P}_r(n)$, i.e. $M \cap L = \{0\}$ and $M + L = \mathcal{P}_r(n)$. Let $m = \dim M \leq \dim \mathcal{P}_r(n) = \binom{n+r}{r}$ and let $q_1, \ldots, q_m \in M$ be a basis of $M$. For $x_1, \ldots, x_m \in K$ denote by $V(x_1, \ldots, x_m) = \det(q_i(x_j))_{i,j=1}^{m}$. The polynomials $q_1, \ldots, q_m$ are linearly independent on $K$, so that there exist points $x_1, \ldots, x_m \in K$ such that the matrix $(q_i(x_j))_{i,j=1}^{m}$ is regular, i.e. $V(x_1, \ldots, x_m) \neq 0$. Let $k_1, \ldots, k_m \in K$ satisfy

$$|V(k_1, \ldots, k_m)| = \max\{|V(y_1, \ldots, y_m)|: y_1, \ldots, y_m \in K\}.$$
Then $V(k_1, \ldots, k_m) \neq 0$. For $j = 1, \ldots, m$ define polynomials $L^{(j)} \in \mathcal{P}_r(n)$ by

$$L^{(j)}(z) = V(k_1, \ldots, k_{j-1}, z, k_{j+1}, \ldots, k_m)/V(k_1, \ldots, k_m).$$

Clearly $|L^{(j)}(z)| \leq 1$ for every $z \in K$. The polynomials $L^{(j)}$ are linear combinations of polynomials $q_1, \ldots, q_m$, so that $L^{(j)} \in M$ ($j = 1, \ldots, m$). Further $L^{(j)}(k_i) = \delta_{ij}$ (the Kronecker symbol), so that the polynomials $L^{(1)}, \ldots, L^{(m)}$ are linearly independent and every polynomial $p \in M$ is a linear combination of them. Obviously

$$p(z) = \sum_{j=1}^m p(k_j)L^{(j)}(z) \quad (p \in M, z \in K).$$

Set $K' = \{k_1, \ldots, k_m\}$. Every polynomial $p \in \mathcal{P}_r(n)$ can be written in the form $p = p_1 + p_2$ for some $p_1 \in L$ and $p_2 \in M$, and $p_2 = \sum_{j=1}^m p_2(k_j)L^{(j)}$. Hence

$$\|p\|_K = \|p_2\|_K = \max\left\{\left|\sum_{j=1}^m p_2(k_j)L^{(j)}(z)\right| : z \in K\right\} \leq \sum_{j=1}^m |p_2(k_j)| \leq m \cdot \|p\|_{K'}.$$

Lemma 2. Let $E$ be a finite-dimensional subspace of an infinite dimensional Banach space $X$, let $\mathcal{M}$ be a finite-dimensional subspace of $B(X)$ and let $\varepsilon > 0$. Then there exists a subspace $Z \subset X$ with $\text{codim } Z < \infty$ such that

$$\|T(e + z)\| \geq (1 - \varepsilon) \max\{\|Te\|, \frac{1}{2}\|Tz\|\}$$

for every $e \in E$, $z \in Z$ and $T \in \mathcal{M}$.

Proof. Let $T_1, \ldots, T_r$ be a basis in $\mathcal{M}$. Set $F = \bigvee_{i=1}^r T_i E = \{Te : T \in \mathcal{M}, e \in E\}$. Clearly $F$ is a finite-dimensional subspace of $X$. By [5], Lemma 1 there exists a subspace $Y \subset X$ with $\text{codim } Y < \infty$ such that

$$\|f + y\| \geq (1 - \varepsilon) \max\{\|f\|, \frac{1}{2}\|y\|\} \quad (f \in F, y \in Y).$$

Set $Z = \bigcap_{i=1}^r T_i^{-1}Y$. As $\text{codim } S^{-1}Y < \infty$ for every $S \in B(X)$, we have $\text{codim } Z < \infty$.

Let $e \in E$, $z \in Z$ and $T \in \mathcal{M}$. Then $Te \in F$ and $T_iz \in Y$ ($i = 1, \ldots, r$) so that $Tz \in Y$. Hence

$$\|T(e + z)\| \geq (1 - \varepsilon) \max\{\|Te\|, \frac{1}{2}\|Tz\|\}.$$

$\square$
Lemma 3. Let $n, r$ be positive integers, let $T = (T_1, \ldots, T_n)$ be an $n$-tuple of mutually commuting operators on a Banach space $X$ such that $\sigma_e(T)$ is not algebraic. Let $Y$ be a subspace of $X$ with $\text{codim} Y < \infty$ and let $\varepsilon > 0$. Then there exists $x \in Y$ such that $\|x\| = 1$ and

\[ \|p(T)x\| \geq \frac{1 - \varepsilon}{2} \left( \frac{n + r}{n} \right)^{-2} r_e(p(T)) \quad (p \in \mathcal{P}_r(n)). \]

Proof. Clearly $X$ is infinite dimensional since $\sigma_e(T)$ is not algebraic.

Denote by $K = \sigma_{r_e}(T)$. As the polynomially convex hulls of $\sigma_{r_e}(T)$ and of $\sigma_e(T)$ coincide [7] and by the spectral mapping property for $\sigma_e$, we have, for every $p \in \mathcal{P}_r(n),

\|p\|_K = \max\{|p(z)|: z \in \sigma_{r_e}(T)\} = \max\{|p(z)|: z \in \sigma_e(T)\} = r_e(p(T)).

Further $\|p\|_K \neq 0$ for $p \neq 0$ as the set $\sigma_e(T)$ is not algebraic. For a polynomial $p \in \mathcal{P}_r(n)$, $p = \sum_{|\alpha| \leq r} c_\alpha(p)z^\alpha$ define a new norm by $|p| = \sum_{|\alpha| \leq r} |c_\alpha(p)|$. The norms $|\cdot|$ and $\|\cdot\|_K$ are equivalent on $\mathcal{P}_r(n)$ so that there exists a positive constant $c$ such that

(1) $|p| \leq c\|p\|_K$ \quad $(p \in \mathcal{P}_r(n)).$

By Lemma 1 there exist elements $\lambda_1, \ldots, \lambda_m \in K$, $m \leq (n+r)$ such that

(2) $\|p\|_K \leq m \cdot \max\{|p(\lambda_i)|: i = 1, \ldots, m\}$ \quad $(p \in \mathcal{P}_r(n)).$

We construct inductively points $x_1, \ldots, x_m \in Y$. Suppose $x_1, \ldots, x_k$ are already found. Let $E_k$ be the subspace generated by the vectors $x_1, \ldots, x_k$ and let $\mathcal{M} = \{p(T): p \in \mathcal{P}_r(n)\}$. By Lemma 2 there exists a subspace $Z_k \subset X$, $\text{codim} Z_k < \infty$ such that

(3) $\|p(T)(\varepsilon + z)\| \geq (1 - \varepsilon') \max\{|p(T)\varepsilon|, \frac{1}{2}\|p(T)z\|\}$ \quad $(e \in E_k, z \in Z_k, p \in \mathcal{P}_r(n))$

where $\varepsilon'$ is a positive number satisfying $\varepsilon' < 1$ and $(1 - \varepsilon')^2(1 - m\varepsilon') \geq 1 - \varepsilon$.

Write $\lambda_{k+1} = (\lambda_{k+1,1}, \ldots, \lambda_{k+1,n})$ and consider the subspace $W_k = Y \cap \bigcap_{i=0}^{k} Z_i$. Clearly $\text{codim} W_k < \infty$. By the definition of $\sigma_{r_e}(T)$ we have

$$\inf\left\{\sum_{i=1}^{n} \|(T_i - \lambda_{k+1,i})w\| : w \in W_k, \|w\| = 1\right\} = 0$$

746
so that there exists \( x_{k+1} \in W_k, \|x_{k+1}\| = 1 \) such that
\[
\| (T^\alpha - \lambda_{k+1}^\alpha) x_{k+1} \| \leq c^{-1} \varepsilon'
\]
for every multiindex \( \alpha, |\alpha| \leq r \). Let \( p = \sum_{|\alpha| \leq r} c_\alpha(p) z^\alpha \in \mathcal{P}_r(n) \). Then by (1),
\[
\| (p(T) - p(\lambda_{k+1})) x_{k+1} \| = \left\| \sum_{|\alpha| \leq r} c_\alpha(p)(T^\alpha - \lambda_{k+1}^\alpha) x_{k+1} \right\|
\]
\[
\leq \sum_{|\alpha| \leq r} |c_\alpha(p)| \max \{ \| (T^\alpha - \lambda_{k+1}^\alpha) x_{k+1} \| : |\alpha| \leq r \} \leq |p| \cdot c^{-1} \varepsilon' \leq \varepsilon' \| p \|_K.
\]

Suppose that we have found elements \( x_1, \ldots, x_m \) in this way. Set \( x = a^{-1} \sum_{i=1}^m x_i \), where \( a = \| \sum_{i=1}^m x_i \| \). Then
\[
a \leq \sum_{i=1}^m \| x_i \| = m \quad \text{and} \quad a \geq (1 - \varepsilon') \| x_1 \| = 1 - \varepsilon'
\]
as \( x_1 \in E_1 \) and \( x_2, \ldots, x_m \in Z_1 \). Clearly \( x \in Y \) and \( \| x \| = 1 \). Let \( p \in \mathcal{P}_r(n) \). Then, for \( k = 1, \ldots, m \), we have
\[
\| p(T) x \| = \left\| a^{-1} \sum_{i=1}^m p(T) x_i \right\| \geq (1 - \varepsilon') a^{-1} \left\| \sum_{i=1}^k p(T) x_i \right\| \geq \frac{1}{2} (1 - \varepsilon')^2 a^{-1} \| p(T) x_k \|
\]
\[
\geq \frac{(1 - \varepsilon')^2}{2m} \left( \| p(\lambda_k) x_k \| - \| (p(T) - p(\lambda_k)) x_k \| \right)
\]
\[
\geq \frac{(1 - \varepsilon')^2}{2m} \left( |p(\lambda_k)| - \varepsilon' \| p \|_K \right)
\]
so that
\[
\| p(T) x \| \geq \frac{(1 - \varepsilon')^2}{2m} \left( \max\{ |p(\lambda_k)| : k = 1, \ldots, m \} - \varepsilon' \| p \|_K \right)
\]
\[
\geq \frac{(1 - \varepsilon')^2}{2m} \| p \|_K (m^{-1} - \varepsilon')
\]
\[
\geq \frac{1 - \varepsilon}{2m^2} \| p \|_K = \frac{1 - \varepsilon}{2m^2} r_e(p(T)).
\]

\[\square\]

**Theorem 4.** Let \( T = (T_1, \ldots, T_n) \) be an n-tuple of mutually commuting operators in a Banach space \( X \) such that \( \sigma_e(T) \) is not algebraic, let \( x \in X \) and \( \varepsilon > 0 \). Then there exists \( y \in X \) and a constant \( C = C(\varepsilon) \) such that \( \| y - x \| < \varepsilon \) and
\[
\| p(T) y \| \geq C(1 + \deg p)^{-(2n+\varepsilon)} r_e(p(T))
\]

747
for every polynomial $p$.

**Proof.** Find $k_0 \geq 1$ such that $\sum_{i=k_0}^{\infty} \frac{1}{i^2} < \varepsilon$, $2^{k_0} \geq n$ and $k^2 \leq 2^{(k-1)} (k \geq k_0)$.

Denote by $C = \frac{1}{8k_0^2} (n + 2^{k_0})^{-2n}$. Choose positive numbers $\varepsilon_i (i \geq k_0)$ such that $\varepsilon_i < 1$ and $\prod_{i=k_0}^{\infty} (1 - \varepsilon_i) \geq \frac{1}{2}$. We construct inductively points $y_{k_0}, y_{k_0+1}, \ldots \in X$, $\|y_i\| = 1$. Suppose that $y_{k_0}, \ldots, y_{k-1}$ are already given. Set $E_k = \bigvee \{x, y_{k_0}, \ldots, y_{k-1}\}$.

By Lemma 2 for $\mathcal{H} = \{p(T) : p \in \mathcal{P}_{2^n}(n)\}$ there exists a subspace $Z \subset X$ with codim $Z < \infty$ such that

$$\|p(T)(e + z)\| \geq \left(1 - \frac{\varepsilon_k}{2}\right) \max\left\{\|p(T)e\|, \frac{1}{2}\|p(T)z\|\right\}$$

for every $e \in E_k$, $z \in Z$ and $p \in \mathcal{P}_{2^n}(n)$. By Lemma 3 there exists $y_k \in Z$ such that $\|y_k\| = 1$ and

$$\|p(T)y_k\| \geq \frac{1}{2} \left(1 - \frac{\varepsilon_k}{2}\right) \left(n + 2^k\right)^{-2} r_\varepsilon(p(T)) \quad (p \in \mathcal{P}_{2^n}(n)).$$

Thus

$$\|p(T)(e + y_k)\| \geq \left(1 - \frac{\varepsilon_k}{2}\right) \max\left\{\|p(T)e\|, \frac{1}{4} \left(1 - \frac{\varepsilon_k}{2}\right) \left(n + 2^k\right)^{-2} r_\varepsilon(p(T))\right\}$$

(5)

$$\geq (1 - \varepsilon_k) \max\left\{\|p(T)e\|, \frac{1}{4} \left(n + 2^k\right)^{-2} r_\varepsilon(p(T))\right\}$$

for every $e \in E_k$ and $p \in \mathcal{P}_{2^n}(n)$.

Set $y = x + \sum_{i=k_0}^{\infty} \frac{y_i}{i^2}$. Clearly $\|y - x\| \leq \sum_{i=k_0}^{\infty} \frac{1}{i^2} < \varepsilon$. Let $p$ be a polynomial of degree $r$. We distinguish two cases:

1) Let $r \leq 2^{k_0}$. Then, by (5), we have for $N \geq k_0$

$$\left\|p(T)x + \sum_{i=k_0}^{N} \frac{1}{i^2} p(T)y_i\right\| \geq (1 - \varepsilon_N) \left\|p(T)x + \sum_{i=k_0}^{N-1} \frac{1}{i^2} p(T)y_i\right\| \geq \ldots$$

$$\geq \prod_{i=k_0+1}^{N} (1 - \varepsilon_i) \left\|p(T)x + \frac{1}{k_0^2} p(T)y_{k_0}\right\| \geq \prod_{i=k_0}^{N} (1 - \varepsilon_i) \cdot \frac{1}{4k_0^2} \left(n + 2^{k_0}\right)^{-2} r_\varepsilon(p(T))$$

$$\geq \frac{1}{8k_0^2} (n + 2^{k_0})^{-2n} r_\varepsilon(p(T)) \geq C \cdot r_\varepsilon(p(T)).$$

748
2) Let $2^{k-1} < r \leq 2^k$ for some $k > k_0$. Then for $N \geq k$ we have

$$
\left\| p(T)x + \sum_{i=k_0}^{N} \frac{1}{i^2} p(T)y_i \right\| \geq \prod_{i=k+1}^{N} (1 - \epsilon_i) \cdot \left\| p(T)x + \sum_{i=k_0}^{k} \frac{1}{i^2} p(T)y_i \right\|
$$

$$
\geq \prod_{i=k}^{N} (1 - \epsilon_i) \cdot \frac{1}{4k^2} \left( n + 2^k \right)^{-2} r_e(p(T))
$$

$$
\geq \frac{1}{8} 2^{-\epsilon(k-1)} (n + 2^k)^{-2n} r_e(p(T))
$$

$$
\geq \frac{1}{8} r^{-\epsilon} (3r)^{-2n} r_e(p(T))
$$

$$
\geq C r^{-2n+\epsilon} r_e(p(T)).
$$

So for every polynomial $p$ we have

$$
\left\| p(T)y \right\| = \lim_{N \to \infty} \left\| p(T)x + \sum_{i=k_0}^{N} \frac{1}{i^2} p(T)y_i \right\| \geq C (1 + \deg p)^{-2n+\epsilon} r_e(p(T)).
$$

The notion of capacity for elements of a Banach algebra was introduced by Halmos [2] and extended to commuting $n$-tuples by Stirling [9].

Denote by $\mathcal{P}^1_k(n)$ the set of all polynomials $p(z) = \sum_{|\mu| \leq k} a_\mu(p) z^\mu \in \mathcal{P}_k(n)$ with $\sum_{|\mu| = k} |a_\mu(p)| = 1$. These polynomials were called monic in [9].

Let $T = (T_1, \ldots, T_n)$ be an $n$-tuple of mutually commuting operators in a Banach space $X$. The joint capacity of $T$ was defined in [9] by

$$
cap (T) = \liminf_{k \to \infty} \frac{\cap_k(T)}{k}
$$

where

$$
\cap_k(T) = \inf \{ ||p(T)|| : p \in \mathcal{P}^1_k(n) \}
$$

(note that the liminf in the definition of $\cap T$ can be replaced by limit by [6]). For a compact subset $K \subset \mathbb{C}^n$ define the corresponding capacity by

$$
\cap K = \liminf_{k \to \infty} \frac{\cap_k(K)}{k}
$$

where

$$
\cap_k K = \inf \{ ||p||_K : p \in \mathcal{P}^1_k(n) \}.
$$

749
This capacity was studied in [10] and called the homogeneous Tshebyshev constant of a compact set $K$.

By [6] $\text{cap} \ T = \text{cap} \ \sigma(T) = \text{cap} \ \sigma_\epsilon(T)$.

Let $T = (T_1, \ldots, T_n) \in B(X)^n$ be a commuting $n$-tuple and let $x \in X$. We define the local capacity $\text{cap}(T, x)$ by

$$\text{cap}(T, x) = \liminf_{k \to \infty} \text{cap}_k(T, x)^{1/k}$$

where

$$\text{cap}_k(T, x) = \inf\{||p(T)x|| : p \in D_k(n)\}.$$ 

Clearly $\text{cap}(T, x) \leq \text{cap} \ T$ for every $x \in X$.

**Theorem 5.** Let $T = (T_1, \ldots, T_n)$ be an $n$-tuple of mutually commuting operators in a Banach space $X$. Then the set of all $y \in X$ with $\text{cap}(T, y) = \text{cap} \ T$ is dense in $X$.

**Proof.** If $\sigma_\epsilon(T)$ is an algebraic set then $\text{cap} \ \sigma_\epsilon(T) = 0$ so that $\text{cap} \ T = 0$ and the assertion of Theorem 5 is satisfied trivially for every $y \in X$.

Suppose $\sigma_\epsilon(T)$ is not algebraic. Let $x \in X$ and $\epsilon > 0$. Then there exists $y \in X$ with $||y - x|| < \epsilon$ and

$$||p(T)y|| \geq C(1 + \deg p)^{-(2n+\epsilon)}r_\epsilon(p(T))$$

for every polynomial $p$. Thus

$$\text{cap}_k(T, y) = \inf\{||p(T)y|| : p \in D_k(n)\} \geq C(1 + k)^{-(2n+\epsilon)}\inf\{r_\epsilon(p(T)) : p \in D_k(n)\}$$

where

$$r_\epsilon(p(T)) = \sup\{|p(z)| : z \in \sigma_\epsilon(T)\}$$

so that

$$\text{cap}_k(T, y) \geq C(1 + k)^{-(2n+\epsilon)}\text{cap}_k(\sigma_\epsilon(T)).$$

Hence

$$\text{cap}(T, y) = \liminf_{k \to \infty} \text{cap}_k(T, y)^{1/k} = \text{cap}(\sigma_\epsilon(T)) = \text{cap} \ T.$$ 

$\square$
References


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