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CHARACTERIZATION OF TRACES
OF THE WEIGHTED SOBOLEV SPACE $W^{1,p}(\Omega, d^*_{M})$ ON $M$

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1. INTRODUCTION

In this paper we shall use the following notation. Let $N > 0$, $k \geq 0$ be integers. Let \( \varepsilon, p \) be real numbers, $1 < p < \infty$. Denote by $p'$ the conjugate exponent, i.e. $p' = \frac{p}{p-1}$. Let \( \Omega \) be a non-empty, open, bounded subset of $\mathbb{R}^N$. Let $M$ be a closed subset of $\partial \Omega$ and let \( d_M(x) \) be the distance function, \( d_M(x) = \text{dist}(x, M) \). For simplicity we shall write \( d(x) \) instead of \( d_M(x) \). For an integer \( m, 1 \leq m \leq N \), we set \( Q_m = (0,1)^m \).

**Definition 1.1.** We shall write $(\Omega, M) \in B(k, N)$ for $1 \leq k \leq N - 1$, $N \geq 2$ if and only if there exists a bilipschitz mapping

$$B: Q_N \to \Omega$$

such that $B(\bar{Q}_k) = M$.

By $C^\infty(\bar{\Omega})$ we denote the set of real functions $u$ defined on $\bar{\Omega}$ such that the derivatives $D^\alpha u$ can be continuously extended to $\bar{\Omega}$ for all multiindices $\alpha$.

Define the weighted Sobolev space $W^{1,p}(\Omega, d^*_{M})$ as the closure of $C^\infty(\bar{\Omega})$ with respect to the norm

$$||u||_{W^{1,p}(\Omega, d^*_{M})} = \left( \int_\Omega |u(x)|^p d(x) dx + \int_\Omega \sum_{i=1}^N |D_i u(x)|^p d(x) dx \right)^{1/p}$$

where $D_i u = \frac{\partial u}{\partial x_i}$ stands for the generalized derivative of the function $u$. The space $H^{1,p}(\Omega, d^*_{M})$ is the class of all functions locally integrable on $\Omega$, with a finite norm

$$||u||_{H^{1,p}(\Omega, d^*_{M})} = \left( \int_\Omega |u(x)|^p d(x) dx + \int_\Omega \sum_{i=1}^N |D_i u(x)|^p d(x) dx \right)^{1/p}.$$
Now, let $(\Omega, M) \in B(k, N)$. Let $0 < s < 1$. Let us recall the definition of the Slobodeckij space $W^{s,p}(M)$ as the set of all functions $u$ defined on $M$ with a finite norm

$$
\|u|W^{s,p}(M)\| = \left( \int_M |u(x)|^p \, dx + \int_M \int_M \frac{|u(x) - u(y)|^p}{|x-y|^{k+s}} \, dx \, dy \right)^{1/p}.
$$

It is well known (see [1] and [4]) that for $\varepsilon \leq k - N$ or $\varepsilon > p + k - N$ the space $W^{1,p}(\Omega, d^\varepsilon)$ is isomorphically and topologically equivalent to the space $H^{1,p}(\Omega, d^\varepsilon)$. The space $H^{1,p}(\Omega, d^\varepsilon)$ has zero traces on $M$ in the case $\varepsilon \leq k - N$ while for $\varepsilon > p + k - N$ the traces on $M$ have no sense in general.

In this paper we show that in the case $k - N < \varepsilon < p + k - N$ the class of traces on $M$ of the space $W^{1,p}(\Omega, d^\varepsilon)$ for $(\Omega, M) \in B(k, N)$ is equal to the space $W^{1-k+p+s/p}_p(M)$. Section 2 contains the direct trace theorems, in Section 3 we find a corresponding extension operator.

2. Direct theorems

**Lemma 2.1.** Let $M = \{0\}$, $-N < \varepsilon < p - N$. Then there exists a constant $c > 0$ dependent only on $\varepsilon, p, N$ such that for all functions $u \in C^\infty(\overline{Q}_N)$ the inequality

$$
|u(0)| \leq c\|u|W^{1,p}(Q_N, d^\varepsilon)\|
$$

holds.

**Proof.** Let $u \in C^\infty(\overline{Q}_N)$. For $x \in Q_N$ we have

$$
u(x) = u(0) + \int_0^1 \sum_{i=1}^N x_i D_i u(tx) \, dt,$$

and so

$$
|u(0)| \leq |u(x)| + \int_0^1 \sum_{i=1}^N x_i |D_i u(tx)| \, dt.
$$

Multiplying the last inequality by $d^{\varepsilon/p}(x)$ and integrating over $Q_N$ we get

$$
|u(0)| \int_{Q_N} d^{\varepsilon/p}(x) \, dx \leq \int_{Q_N} |u(x)| d^{\varepsilon/p}(x) \, dx
$$

$$
+ \int_0^1 \int_{Q_N} \sum_{i=1}^N x_i |D_i u(tx)| d^{\varepsilon/p}(x) \, dx \, dt = I_1 + I_2.
$$

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Since $\epsilon/p > -N$, we have $c_1 = \int_{Q_N} d^{\epsilon/p}(x) \, dx < \infty$.

The Hölder inequality yields

\[(2.3) \quad I_1 \leq \left( \int_{Q_N} \frac{1}{dx} \left( \int_{Q_N} |u(x)|^p \, dx \right)^{1/p} \right)^{1/p} \leq \|u\|W^{1,p}(Q_N, d^\epsilon).\]

Let us estimate the integral $I_2$. By the assumptions there exists $\alpha$, such that $\epsilon + N - 1 < \alpha p < p - 1$. We use the substitution $tx = y$ and the Hölder inequality and obtain

\[
I_2 \leq \int_0^1 \int_0^{1} \sum_{i=1}^N |D_i u(y)|d^{\epsilon/p}(y)t^{-N-\epsilon/p} \, dy \, dt
\]

\[
\leq \left( \int_0^1 t^{-\alpha p'} \, dt \right)^{1/p'} \left( \int_0^1 \left[ \int_0^1 \sum_{i=1}^N |D_i u(y)|d^{\epsilon/p}(y)t^{\alpha-N-\epsilon/p} \, dy \right]^p \right)^{1/p}.
\]

Since $-\alpha p' > -1$, we have $\left( \int_0^1 t^{-\alpha p'} \, dt \right)^{1/p'} = c_2 < \infty$.

Using again the Hölder inequality we obtain

\[(2.4) \quad I_2 \leq \left( \int_0^1 t^{\alpha-N-\epsilon} \, dt \right)^{1/p} \left( \int_{(0,1)^N} \sum_{i=1}^N |D_i u(y)|^p \, dy \right)^{1/p}
\]

\[
\leq c_3\|u\|W^{1,p}(Q_N, d^\epsilon),
\]

where $c_3 = c_2\left( \int_0^1 t^{\alpha-N-\epsilon} \, dt \right)^{1/p} < \infty$, because $p\alpha - N - \epsilon > -1$. Now (2.1) follows from (2.2), (2.3) and (2.4). \hfill \square

**Lemma 2.2.** Let $1 \leq k \leq N - 1$, $k - N < \epsilon < p + k - N$ and let $M = [0, 1]^k$. Then there exists a unique bounded linear operator

\[T: W^{1,p}(Q_N, d^\epsilon) \rightarrow L^p(M)\]

satisfying

\[Tu(x) = u(x_1, x_2, \ldots, x_k, 0, \ldots, 0), \quad x \in Q_N,\]

for every function $u \in C^\infty(\overline{Q}_N)$.

**Proof.** Let $u \in C^\infty(\overline{Q}_N)$. Fix $x_1, \ldots, x_k \in (0, 1)$ and define a function $v$ by

\[v(x_{k+1}, \ldots, x_N) = u(x), \quad (x_{k+1}, \ldots, x_N) \in (0, 1)^{N-k}.\]
Obviously, $v(x_{k+1}, \ldots, x_N) \in W^{1,p}(Q_{N-k}, d^\varepsilon)$.

By Lemma 2.1 there exists a constant $c$ dependent only on $\varepsilon, p, N, k$ such that

$$|u(x_1, x_2, \ldots, x_k, 0, \ldots, 0)|^p \leq c \left( \int_{Q_{N-k}} |u(x)|^p d^\varepsilon(x) dx_{k+1} \ldots dx_N + \sum_{i=k+1}^N |D_i u(x)|^p d^\varepsilon(x) dx_{k+1} \ldots dx_N \right).$$

Integrating over $x_1, \ldots, x_k$ we obtain

$$\|u(x_1, \ldots, x_k, 0, \ldots, 0)\|_{L^p(M)}^p \leq c\|u|_{W^{1,p}(Q_{N}, d^\varepsilon)}^p.$$

The operator $T$ is now the unique bounded linear extension of the mapping $u \mapsto u(x_1, x_2, \ldots, x_k, 0, \ldots, 0)$.

Using a similar argument we can prove a more general assertion:

**Theorem 2.3.** Let $1 \leq k \leq N - 1$, $k - N < \varepsilon < p + k - N$ and $(\Omega, M) \in B(k, N)$. Then there exists a unique bounded linear operator

$$T: W^{1,p}(\Omega, d^\varepsilon) \to L^p(M)$$

such that

$$Tu = u|_M$$

for all $u \in C^\infty(\overline{\Omega})$.

Now we shall prove that the trace operator is a bounded mapping from $W^{1,p}(\Omega, d^\varepsilon)$ into $W^{1-N+\frac{k+\varepsilon}{p}, p}(M)$ if $(\Omega, M) \in B(k, N)$ and $k - N < \varepsilon < p + k - N$.

The case $k = N - 1$.

**Lemma 2.4** (see [2]). Let $-1 < \varepsilon < p - 1$ and let $a$ be a real number such that $0 < a < 1$. Then there exists a constant $c > 0$ independent of $a$, such that for all functions $u \in C^\infty([0, a])$ and $v \in C^\infty([a, 1])$ the inequalities

$$\int_0^a \frac{1}{a - x} \int_x^a u(t) dt \, d(t-x)^{\varepsilon} dx \leq c \int_0^a |u(x)|^p (a - x)^\varepsilon dx,$$

$$\int_a^1 \frac{1}{x - a} \int_a^x v(t) dt \, d(t-a)^{\varepsilon} dx \leq c \int_a^1 |v(x)|^p (x - a)^\varepsilon dx$$

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Lemma 2.5. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x \in (0, 1), y \in (0, x)\}$, $M = \{(x, x) : 0 \leq x \leq 1\}$ and let $-1 < \varepsilon < p - 1$. Then there exists a constant $c > 0$ such that

\begin{equation}
\int_0^1 \int_0^t \frac{|u(t, t) - u(\tau, \tau)|^p}{|t - \tau|^{p-\varepsilon}} \, d\tau \, dt
\leq c \int_0^1 \int_0^t (|D_1 u(t, \tau)|^p + |D_2 u(t, \tau)|^p)(t - \tau)\varepsilon \, d\tau \, dt
\end{equation}

for all $u \in C^\infty(\overline{\Omega})$.

Proof. Using the Fubini theorem we obtain

\begin{align*}
\int_0^1 \int_0^t \frac{|u(t, t) - u(\tau, \tau)|^p}{|t - \tau|^{p-\varepsilon}} \, d\tau \, dt \\
&\leq 2^{p-1} \left[ \int_0^1 \left( \int_0^t \frac{1}{|t - \tau|^{p-\varepsilon}} \int_0^t |D_2 u(t, \xi)\, d\xi \right)^p (t - \tau)^\varepsilon \, d\tau \right] \, dt \\
&\quad + \left[ \int_0^1 \left( \int_0^\tau \frac{1}{|t - \tau|^{p-\varepsilon}} \int_0^\tau |D_1 u(\xi, \tau)\, d\xi \right)^p (t - \tau)^\varepsilon \, d\tau \right] \\
&= 2^{p-1} [I_1 + I_2].
\end{align*}

According to Lemma 2.4 we have

\begin{align*}
I_1 &\leq c \int_0^1 \int_0^t |D_2 u(t, \tau)|^p (t - \tau)^\varepsilon \, d\tau \, dt, \\
I_2 &\leq c \int_0^1 \int_0^\tau |D_1 u(\xi, \tau)|^p (t - \tau)^\varepsilon \, dt \, d\tau.
\end{align*}

The inequality (2.5) follows. \qed
Lemma 2.6. Let $N \geq 2$, $-1 < \varepsilon < p - 1$. Define $A_i(u)$ by

$$A_i(u) = \int \int \ldots \int \left( \int \int_{\Omega} \frac{d\tau dt}{|t - \tau|^{p - \varepsilon}} \right)$$

for $(N - 2)$-fold
differentiation. Then there exists a positive constant $c$ such that

$$||u||_{W^{1, \frac{1 + \varepsilon}{p}}(Q_{N - 1})} \leq c \left(||u||_{L^p(Q_{N - 1})}^p + \sum_{i=1}^{N-1} A_i(u)\right)$$

for all functions $u \in L^p(Q_{N - 1})$ such that $A_i(u) < \infty$ for $i = 1, 2, \ldots, N - 1$.

Proof. The proof can proceed in a way similar to that of the proof of Lemma 6.8.10 in [3].

Lemma 2.7. Let $N \geq 2$, $-1 < \varepsilon < p - 1$ and $M = [0, 1]^{N - 1}$. Then there exists a constant $c > 0$ such that

$$||u(x_1, \ldots, x_{N - 1}, 0)||_{W^{1, \frac{1 + \varepsilon}{p}}(Q_{N - 1})} \leq c \left(||u||_{L^p(Q_{N - 1})} + \sum_{i=1}^{N-1} A_i(u)\right)$$

for all $u \in C^\infty(\overline{Q}_N)$.

Proof. Let $u \in C^\infty(\overline{Q}_N)$. Set $v(x_1, x_2, \ldots, x_{N - 1}) = u(x_1, x_2, \ldots, x_{N - 1}, 0)$. According to Lemma 2.6 and Lemma 2.2 we have

$$||u(x_1, \ldots, x_{N - 1}, 0)||_{W^{1, \frac{1 + \varepsilon}{p}}(Q_{N - 1})}^p = ||v||_{W^{1, \frac{1 + \varepsilon}{p}}(Q_{N - 1})}^p \leq c_1 \left(||v||_{L^p(Q_{N - 1})} + \sum_{i=1}^{N-1} A_i(v)\right) \leq c_2 \left(||u||_{W^{1, p}(Q_N, d^\varepsilon)}^p + \sum_{i=1}^{N-1} A_i(v)\right).$$

Obviously, it remains to estimate $A_i(v)$, $i = 1, 2, \ldots, N - 1$. Fix $i = 1, \ldots, N - 1$. Set $v_i(x_1, \ldots, x_N) = u(x_1, \ldots, x_{N - 1}, x_i - x_N)$ in the domain $0 \leq x_j \leq 1$ for
Lemma 2.5 and the direct calculation yield

\[ A_i(v) = 2 \int_0^1 \ldots \int_0^1 \int_0^t |u(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{N-1}, 0)|^p \frac{d\tau dt}{|t - \tau|^{p-\epsilon}} \]

\[ dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_{N-1} \]

\[ = 2 \int_0^1 \ldots \int_0^1 \int_0^t |v_i(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{N-1}, t)|^p \frac{d\tau dt}{|t - \tau|^{p-\epsilon}} \]

\[ dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_{N-1} \]

\[ \leq 2c \int_0^1 \ldots \int_0^1 \int_0^t (|D_i v_i(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{N-1}, t)|^p + |D_N v_i(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{N-1}, \tau)|^p) (t - \tau)^{\epsilon} d\tau dt \]

\[ dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_{N-1} \]

\[ \leq 2^{p+1}c \int_0^1 \ldots \int_0^1 \int_0^t (|D_i u(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{N-1}, t - \tau)|^p + |D_N u(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{N-1}, t - \tau)|^p) (t - \tau)^{\epsilon} d\tau dt \]

\[ dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_{N-1} \]

The substitution \( s = t - \tau \) gives

\[ A_i(v) \leq 2^{p+1}c \| u \|_{W^{1,p}(Q_N, d^\epsilon)} \|^p, \quad i = 1, 2, \ldots, N - 1, \]

which completes the proof. \( \square \)

As an immediate consequence we have

**Theorem 2.8.** Let \( N \geq 2, \ -1 < \epsilon < p - 1 \) and \( (\Omega, M) \in B(N - 1, N) \). Then there exists a unique bounded linear operator

\[ T : W^{1,p}(\Omega, d^\epsilon) \to W^{1-\frac{1+\epsilon}{p},p}(M) \]

such that

\[ Tu = u|_M \]
for all \( u \in C^\infty(\overline{Q}_N) \).

The case \( 1 \leq k \leq N - 2 \).

**Lemma 2.9.** Let \( K, L, s > 0 \). Then there exists \( c > 0 \) such that

\[
\frac{c}{r^{1+s}} \leq \int_0^L \int_0^L \frac{1}{(r + |\ell_1 - \ell_2|)^{2+s}} \, d\ell_1 \, d\ell_2
\]

for every \( r \in (0, K] \).

**Proof.** Obviously,

\[
\int_0^L \int_0^L \frac{d\ell_1 \, d\ell_2}{(r + |\ell_1 - \ell_2|)^{2+s}} = 2 \int_0^L \int_0^L \frac{d\ell_1 \, d\ell_2}{(r + \ell_1 - \ell_2)^{2+s}} = 2I.
\]

Direct calculation yields

\[
I = \frac{1}{1 + s} \frac{L}{r^{1+s}} + \frac{1}{s(s + 1)} \left( \frac{1}{(r + L)^s} - \frac{1}{r^s} \right).
\]

By the Lagrange Mean Value Theorem there exists \( \xi \in (r, r + L) \) such that

\[
I = \frac{L}{1 + s} \left( \frac{1}{r^{1+s}} - \frac{\xi^{s-1}}{r^s(r + L)^s} \right).
\]

If \( s \leq 1 \) we have

\[
I \geq \frac{L}{1 + s} \left( \frac{1}{r^{1+s}} - \frac{r^{s-1}}{r^s(r + L)^s} \right) = \frac{L}{1 + s} \frac{1}{r^{1+s}} \left( 1 - \left( \frac{r}{r + L} \right)^s \right)
\]

\[
\geq \frac{L}{1 + s} \left( 1 - \left( \frac{K}{K + L} \right)^s \right) \frac{1}{r^{1+s}}.
\]

For \( s \geq 1 \) we get

\[
I \geq \frac{L}{1 + s} \left( \frac{1}{r^{1+s}} - \frac{(r + L)^{s-1}}{r^s(r + L)^s} \right) = \frac{L}{1 + s} \frac{1}{r^{1+s}} \left( 1 - \frac{r}{r + L} \right)
\]

\[
\geq \frac{L}{1 + s} \left( 1 - \frac{K}{K + L} \right) \frac{1}{r^{1+s}}.
\]

Thus, (2.6) holds with \( c = 2 \min \left( \frac{L}{1 + s} \left( 1 - \left( \frac{K}{K + L} \right)^s \right), \frac{L}{1 + s} \left( 1 - \frac{K}{K + L} \right) \right) \).
Lemma 2.10. Let \( N \geq 2, k - N < \varepsilon < p + k - N \) and \( M = [0, 1]^k \). Then there exists a unique bounded linear operator

\[ T : W^{1,p}(Q_N, d^\varepsilon) \rightarrow W^{1, \frac{N-k-\varepsilon}{p}}(M) \]

such that

\[ Tu = u \big|_M \]

for all functions \( u \in C^\infty(\overline{Q}_N) \).

Proof. Let \( u \in C^\infty(\overline{Q}_N) \). We introduce general cylindrical coordinates \((x', r, \varrho), x' = (x_1, \ldots, x_k), \varrho = (\varrho_1, \ldots, \varrho_{N-k-1})\), by

\[
\begin{align*}
x_1 &= x_1, \\
\vdots \, n_k &= x_k, \\
x_{k+1} &= r \cos \varrho_1, \\
x_{k+2} &= r \sin \varrho_1 \cos \varrho_2, \\
\cdots \\
x_N &= r \sin \varrho_1 \cdots \sin \varrho_{N-k-2} \cos \varrho_{N-k-1}.
\end{align*}
\]

Let \( \Omega' = \{(x', r, \varrho) : x' \in (0,1)^k, r \in (0,1), \varrho \in (\frac{\pi}{6}, \frac{\pi}{3})^{N-k-1}\} \). Obviously \( M = \{(x', r, \varrho) \in \overline{\Omega'} : r = 0\} \). Let \( \Omega \) be the set of points on \( \mathbb{R}^N \) whose cylindrical coordinates \((x', r, \varrho)\) belong to \( \Omega' \). Then \( \Omega \subset Q_N, M \subset \overline{\Omega}, d(x) = r \) and there exist two constants \( c_1, c_2 > 0 \) such that

\[ (2.7) \quad c_1 r^{N-k-1} \leq J(x', r, \varrho) \leq c_2 r^{N-k-1}, (x', r, \varrho) \in \Omega', \]

where \( J(x', r, \varrho) \) is the Jacobian of the mapping \((x', r, \varrho) \mapsto x\). Define a function \( v \) by

\[ v(x', r, \varrho) = u(x). \]

It is easy to see that \( v \in C^\infty(\overline{\Omega'}) \) and

\[
\begin{align*}
\left| \frac{\partial v}{\partial x_i}(x', r, \varrho) \right| &= \left| \frac{\partial u}{\partial x_i}(x) \right| \quad \text{for} \quad i = 1, 2, \ldots, k, \\
\left| \frac{\partial v}{\partial r}(x', r, \varrho) \right| &\leq \sum_{j=k+1}^{N} \left| \frac{\partial u}{\partial x_j}(x) \right|, \\
\left| \frac{\partial v}{\partial \varrho_j}(x', r, \varrho) \right| &\leq r \sum_{i=k+1}^{N} \left| \frac{\partial u}{\partial x_i}(x) \right| \quad \text{for} \quad j = 1, \ldots, N-k-1.
\end{align*}
\]
These inequalities and (2.7) yield

\[
(2.8) \quad I(v) := \int_{\Omega'} \left| v(x', r, \varrho) \right|^p r^{N-k-1} r^e \, dx' \, dr \, d\varrho
\]

\[
+ \int_{\Omega'} \left( \sum_{i=1}^k \left| \frac{\partial v}{\partial x_i} (x', r, \varrho) \right|^p + \left| \frac{\partial v}{\partial r} (x', r, \varrho) \right|^p \right) r^{N-k-1} r^e \, dx' \, dr \, d\varrho
\]

\[
+ \int_{\Omega'} \sum_{i=1}^{N-k-1} \left| \frac{\partial v}{\partial \varrho_i} (x', r, \varrho) \right|^p r^{N-k-1} r^{e-p} \, dx' \, dr \, d\varrho
\]

\[
\leq c_3 ||u||_{W^{1,p}(Q_N, d\varrho)}^p.
\]

Set \( \varepsilon_1 = \varepsilon - N - k - 1 \) and define the anisotropic weighted space \( V^{1,p}(\Omega', r^{\varepsilon_1}) \) as the closure of \( C^\infty(\Omega') \) with respect to the norm \( ||.|| = (I(.))^{1/p} \). Obviously

\[
(2.9) \quad V^{1,p}(\Omega', r^{\varepsilon_1}) \hookrightarrow W^{1,p}(\Omega', r^{\varepsilon_1}).
\]

Putting \( \varepsilon = \varepsilon_1 \) in Theorem 2.8 we obtain the existence of a constant \( c_4 > 0 \) such that

\[
(2.10) \quad c_4 ||v||_{W^{1,p}(\Omega', r^{\varepsilon_1})}^p \geq \int_{(0,1)^k} \int_{(0,1)^k} \int_{\left(\frac{3}{5}, \frac{5}{5}\right)^N-k-1} \int_{\left(\frac{5}{5}, \frac{3}{5}\right)^N-k-1} \frac{|v(x',0,\varrho) - v(y',0,\psi)|^p}{(|x' - y'| + |\varrho - \psi|)^{N+p-2-\varepsilon_1}} \, d\varrho \, d\psi \, dx' \, dy'.
\]

Denote the right-hand side of (2.10) by \( I_1 \). Since \( v(x',0,\varrho) = u(x_1, \ldots, x_k, 0, \ldots, 0) = u(x',0) \), we can write

\[
(2.11) \quad I_1 = \int_{(0,1)^k} \int_{(0,1)^k} |u(x',0) - u(y',0)|^p J(|x' - y'|) \, dx' \, dy',
\]

where

\[
J(|x' - y'|) \geq c_5 \int_{\left(\frac{3}{5}, \frac{5}{5}\right)^{N-k-1}} \int_{\left(\frac{5}{5}, \frac{3}{5}\right)^{N-k-1}} \frac{d\varrho_1 \cdots d\varrho_{N-k-1} d\psi_1 \cdots d\psi_{N-k-1}}{(|x' - y'| + |\varrho_1 - \psi_1| + \cdots + |\varrho_{N-k-1} - \psi_{N-k-1}|)^{N+p-2-\varepsilon_1}}.
\]

Using \((N-k-1)\)-times Lemma 2.9 we arrive at the estimate

\[
(2.12) \quad J(|x' - y'|) \geq \frac{c_6}{|x' - y'|^{p+k-1-\varepsilon_1}} = \frac{c_6}{|x' - y'|^{p+2k-N-\varepsilon_1}}.
\]
From (2.11) and (2.12) we obtain
\[ I_1 \geq c_6 \int_{(0,1)^k} \int_{(0,1)^k} \frac{|u(x',0) - u(y',0)|^p}{|x' - y'|^{p + 2k - N - \varepsilon}} \, dx' \, dy'. \]

The estimates (2.8), (2.9), (2.10) and (2.12) imply the assertion of the lemma. \( \square \)

Again, it is not difficult to prove the following more general statement:

**Theorem 2.11.** Let \( N \geq 2, k - N < \varepsilon < p + k - N \) and let \((\Omega, M) \in B(k, N)\). Then there exists a unique bounded linear operator
\[ T : W^{1,p}(\Omega, d^\varepsilon) \to W^{1, \frac{N-k+\varepsilon}{p}}(M) \]
such that
\[ Tu = u \big|_M \]
for all \( u \in C^\infty(\Omega) \).

3. **Inverse Theorems**

Let \( N \geq 2 \) and let \( k \) be fixed, \( 1 \leq k \leq N - 1 \). For \( x \in (x_1, \ldots, x_N) \in \mathbb{R}^N \) we shall write \( x = (x', x'') \), \( x' = (x_1, \ldots, x_k) \), \( x'' = (x_{k+1}, \ldots, x_N) \). Let \( \Phi \) be a function of \( C^\infty(\mathbb{R}^k) \) such that \( \Phi(x') = 0 \) for \( |x'| \geq 1 \), \( \Phi(x') > 0 \) for \( |x'| < 1 \) and \( \int_{|x'| \leq 1} \Phi(x') \, dx' = 1 \). We define an operator \( R \) by
\[ (3.1) \quad Ru(x', x'') = |x''|^{-k} \int_{|x' - t'| \leq |x''|} \Phi\left(\frac{x' - t'}{|x''|}\right) u(t') \, dt', \quad u \in L^1_{\text{loc}}(\mathbb{R}^k). \]

**Lemma 3.1.** Let \( k - N < \varepsilon < p + k - N \) and \( K > 0 \). Let \( Q^k = (-K, K)^k \),
\[ D = \{ x \in \mathbb{R}^N : x = (x', x''), x' \in Q^k, |x''| \leq K - \max_{i=1,\ldots,k} |x_i|, x_j \geq 0 \text{ for } k+1, \ldots, N \} \]
and \( M = \overline{Q^k} \).

Then the operator \( R \) is a bounded linear mapping of \( W^{1, \frac{N-k+\varepsilon}{p}}(Q^k) \) in \( W^{1,p}(D, d^\varepsilon) \).

**Proof.** The linearity of the operator \( R \) is obvious. Now, we shall follow the idea of the proof of Lemma 6.9.1 in [3]. Without loss of generality we may assume \( K = 1 \). From (3.1) we get
\[ (3.2) \quad D_i Ru(x', x'') = \frac{1}{|x''|^{k+1}} \int_{|x' - t'| < |x''|} D_i \Phi\left(\frac{x' - t'}{|x''|}\right) u(t') \, dt', \quad i = 1, 2, \ldots, k, \]
(3.3) \( D_i R u(x',x'') = \frac{1}{|x''|^{k+1}} \int _{|x'-t'| \leq |x''|} \left( -\frac{k x_i}{|x''|} \Phi \left( \frac{x'-t'}{|x''|} \right) \right) - \sum _{j=1} ^k D_j \Phi \left( \frac{x'-t'}{|x''|} \right) \frac{(x_j - t_j) x_i}{|x''|^2} u(t') \, dt', \)
\[ i = k + 1, \ldots, N. \]

If we take \( u = 1 \) in (3.1) we get \( R u(x',x'') = 1 \) and from (3.2) and (3.3) we obtain

\[
\begin{align*}
0 &= \frac{1}{|x''|^{k+1}} \int _{|x'-t'| \leq |x''|} D_i \Phi \left( \frac{x'-t'}{|x''|} \right) \, dt', \quad i = 1, 2, \ldots, k, \\
0 &= \frac{1}{|x''|^{k+1}} \int _{|x'-t'| \leq |x''|} \left( -\frac{k x_i}{|x''|} \Phi \left( \frac{x'-t'}{|x''|} \right) \right) - \sum _{j=1} ^k D_j \Phi \left( \frac{x'-t'}{|x''|} \right) \frac{(x_j - t_j) x_i}{|x''|^2} u(t') \, dt', \\
&\quad i = k + 1, \ldots, N.
\end{align*}
\]

Thus, (3.2) and (3.3) can be rewritten in the following way:

\[
D_i R u(x',x'') = \frac{1}{|x''|^{k+1}} \int _{|x'-t'| \leq |x''|} D_i \Phi \left( \frac{x'-t'}{|x''|} \right) (u(t') - u(x')) \, dt', \]
\[ i = 1, \ldots, k, \]

\[
D_i R u(x',x'') = \frac{1}{|x''|^{k+1}} \int _{|x'-t'| \leq |x''|} \left( -\frac{k x_i}{|x''|} \Phi \left( \frac{x'-t'}{|x''|} \right) \right) - \sum _{j=1} ^k D_j \Phi \left( \frac{x'-t'}{|x''|} \right) \frac{(x_j - t_j) x_i}{|x''|^2} (u(t') - u(x')) \, dt', \\
&\quad i = k + 1, \ldots, N.
\]

We shall estimate \( R u \) and \( D_i R u \) in the norm of \( L^p(D,d^x) \). Note that \( d(x) = |x''| \) for \( x \in D \).

Let us estimate \( I_0 = \|R u\|_{L^p(D,d^x)}\|^p\):

We put \( b(x') = 1 - \max _{i=1,\ldots,k} |x_i| \). According to (3.1), we have

\[
I_0 = \int _{Q_h} \int _{|x''| \leq b(x')} \left( \int _{|x'-t'| \leq |x''|} \Phi \left( \frac{x'-t'}{|x''|} \right) u(t') \, dt' \right) |x''|^p \, dx'' \, dx'.
\]

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Using the substitution \( s' = \frac{x'' - t}{|x''|} \), the Hölder inequality, the spherical coordinates for \( x'' \) and the Fubini theorem we obtain

\[
I_0 = \int \left( \int_{Q^k} \int_{|s'| < b(x')} \int_{|s'| \leq 1} \Phi(s') u(x' - s'|x''|) ds' |x''|^p dx'' \right) dx'
\]

\[
\leq c_1 \int_{Q^k} \int_{0 < r < b(x')} \int_{|s'| \leq 1} |u(x' - s'|r)|^p ds' r^{N-k-1+\varepsilon} dr dx'
\]

\[
= c_1 \int_{|s'| \leq 1} \int_0^1 \int_{(r-1, 1-r)^k} |u(x' - s'|r)|^p dx' r^{N-k-1+\varepsilon} dr ds'.
\]

Now we use the substitution \( y' = x' - s'r \). Obviously, \(|y'| - |s'|r \leq |y'| + |s'|r = |x'| \leq 1 - r \), and so \(|y'| \leq 1 - r(1 - |s'|) \leq 1\).

Increasing the integration domain we obtain

\[
I_0 \leq c_1 \int_{|s'| \leq 1} \int_0^1 \int_{|y'| \leq 1} |u(y')|^p dy' r^{N-k-1+\varepsilon} dr ds' \leq c_2 \|u\|_{L^p(Q^k)}^p,
\]

where \( c_2 = c_1 \int_{|s'| \leq 1} ds' \int_0^1 r^{N-k-1+\varepsilon} dr \).

To estimate \( I_i = \|D_i Ru|L^p(D, d^s)|^p \) for \( i = 1, 2, \ldots, N \) we proceed in the same way as in the previous case. We obtain

\[
I_i \leq c_3 \int_{Q^k} \int_{0 < r < b(x')} \int_{|s'| \leq 1} \frac{|u(x' - s'|r) - u(x')|^p}{r^p} r^{N-k-1+\varepsilon} ds' dr dx'
\]

\[
\leq c_3 \int_{Q^k} \int_{|t' - x'| \leq b(x')} \frac{|u(t') - u(x')|^p}{|t' - x'|^{p+2k-N-\varepsilon}} \int_{|t' - x'|} \frac{|t' - x'|^{p+2k-N-\varepsilon}}{|t' - x'|^{p+2k-N-\varepsilon-1}} dr dt' dx'.
\]

Direct calculation yields

\[
\int_{|t' - x'|} \frac{|t' - x'|^{p+2k-N-\varepsilon}}{|t' - x'|^{p+2k-N-\varepsilon-1}} dr \leq \frac{1}{p + 2k - N - \varepsilon}.
\]

Since \( \{t': |x' - t'| \leq 1 - \max_{i=1,\ldots,k} |x_i| \} \subset Q^k \), we have

\[
I_i \leq c_4 \int_{Q^k} \int_{Q^k} \frac{|u(t') - u(x')|^p}{|t' - x'|^{p+2k-N-\varepsilon}} dt' dx' \leq c_4 \|u\|_{W^{1,\frac{N-k-\varepsilon}{r}}(Q^k)}^p,
\]

which completes the proof. \( \square \)
Lemma 3.2. Let $G \subset \mathbb{R}^k$ have a Lipschitz boundary, i.e. $G \in C^{0,1}$ in the sense of Definition 5.5.6 in [3], $s \in \mathbb{R}$, $0 < s < 1$. Then there exists a bounded linear operator $S: W^{s,p}(G) \to W^{s,p}(\mathbb{R}^k)$ such that $Su = u$ on $\Omega$.

Proof. We shall write $x = (x', x_k)$ where $x' = (x_1, \ldots, x_{k-1})$. Let $Q_k^- = \{x = (x', x_k), x' \in Q_{k-1}, -1 < x_k < 0\}$. Define an extension operator $S_0$ in the following way:

$$(S_0 u)(x', x_k) = \begin{cases} u(x', x_k) & \text{for } x \in Q_k^+ \\ u(x', -x_k) & \text{for } x \in Q_k^- \end{cases}$$

It is not difficult to prove that $S_0$ is a bounded linear operator from $W^{s,p}(Q_k)$ into $W^{s,p}(Q_k \cup Q_k^-)$.

Using the local description of $\partial G$, the corresponding partition of unity and the standard technique we obtain the assertion of the lemma. 

Lemma 3.3. Let $N \geq 2$, $k - N < \varepsilon < p + k - N$ and let $M = [0, 1]^k$. Then there exists a bounded linear operator

$$R_0: W^{1-\frac{N-k+\varepsilon}{p}, p}(M) \to W^{1,p}(Q_N, d^e)$$

such that $T(R_0 u) = u$.

Proof. Take $K$ sufficiently large so that $D \supset Q_N$ where $D$ is the domain defined in Lemma 3.1. Set $Q^k = [-K, K]^k$. Let $S$ be the operator from Lemma 3.2 and $R$ the operator from Lemma 3.1 (with $G = Q^k$ and $s = 1 - \frac{N-k+\varepsilon}{p}$). Let $u \in W^{1-\frac{N-k+\varepsilon}{p}, p}(M)$. Obviously,

$$||RSu||_{W^{1,p}(Q_N, d^e)} \leq c_1 ||u||_{W^{1-\frac{N-k+\varepsilon}{p}, p}(M)}.$$ 

It remains to prove that $T(RSu) = u$. Note that $Su \in L^p(Q^k)$ and $Su = u$ on $M$. Put

$$I(x'') = \int_M |(RSu)(x', x'') - u(x')|^p \, dx'.$$
Then
\[
I(x''') = \int_M \left[ \int_{|s'| < 1} \Phi(s')((Su)(x' - s' |x''') - u(x')) ds' \right]^p dx'
\]
\[
\leq c_2 \int_M \left[ \int_{|s'| < 1} |(Su)(x' - s' |x''') - (Su)(x')|^p dx' \right].
\]

The p-mean continuity of Su yields
\[
\lim_{|x'''| \to 0} I(x''') = 0.
\]

By virtue of the trivial imbedding
\[
L^p(M) \hookrightarrow L^1(M)
\]
we have
\[
(3.4) \quad \lim_{|x'''| \to 0} \|(RSu)(x', x''') - u(x')|L^1(M)\| = 0.
\]

Since RSu \in W^{1,p}(D, d^s), by Theorem 2.11 the trace of RSu on M exists. Denote this trace by v. We shall prove that
\[
\lim_{|x'''| \to 0} \|(RSu)(x', x''') - v(x'')|L^1(M)\| = 0.
\]

Let \( u_n \in C^\infty(\overline{Q_N}), u_n \to RSu \) in \( W^{1,p}(Q_N, d^s) \). We can write
\[
(3.5) \quad \int_M |(RSu)(x', x''') - v(x')| dx' \leq \int_M |(RSu)(x', x''') - (RSu_n)(x', x''')| dx'
\]
\[
+ \int_M |(RSu_n)(x', x''') - u_n(x', 0)| dx' + \int_M |u_n(x', 0) - v(x')| dx'
\]
\[
= J_1(n, x''') + J_2(n, x''') + J_3(n).
\]

Set \( \sigma_{N-k} = \frac{1}{N-k}s_{N-k} \) where \( s_{N-k} \) is the \((N-k-1)\)-dimensional Hausdorff measure of the \((N-k)\)-dimensional unit sphere. For simplicity we write \( f_n \) instead of \( RSu - RSu_n \). From the definition of the operator \( RS \) we have \( f_n(x', x_1''') = f_n(x', x_2''') \) for \( |x_1'''| = |x_2'''| \). Thus, we can rewrite the integral \( J_1(u, x''') \) in the following way:
\[
(3.6) \quad J_1(u, x''') = \int_M |f_n(x', x''')| dx'
\]
\[
= \frac{1}{\sigma_{N-k}|x'''|^{N-k-1}} \int_M \int_{|y'''|=|x'''|} |f_n(x', y''')| dx' dy'''.
\]
Put
\[ M(|x''|) = \{ y'' : |y''| < |x''|, y_i > 0 \text{ for } i = k + 1, \ldots, N \}. \]

Since the traces of $W^{1,1}(\Omega)$ are in $L^1(\partial \Omega)$, there exists a constant $c_3 > 0$ independent of $|x''|$ and such that
\[
J_1(u, x'') \leq \frac{c_3}{\sigma_{N-k}|x''|^{N-k-1}} \int_{M M(|x''|)} \left( |f_n(x', y'')| \right. \\
\left. + \sum_{i=1}^N |D_i f_n(x', y'')| \right) dx' dy''.
\]

We use (3.5), (3.6), the Hölder inequality and the general cylindrical coordinates to obtain
\[
J_1(n, x'') \leq c_4 \int_{M M(|x''|)} \left( \left( |f_n(x', y'')|^p \right. \\
\left. + \sum_{i=1}^N |D_i f_n(x', y'')|^p \right)^{1/p} \right)^{1/p'} dx' dy'' \\
\leq c_5 |x''|^{1 - \frac{N-k+k}{p}} \| f_n \|_{W^{1,p}(Q_N, d^\epsilon)}^p.
\]

Consequently,
\[
(3.7) \quad J_1(n, x'') \leq c_5 |x''|^{1 - \frac{N-k+k}{p}} \| RSu - RSu_n \|_{W^{1,p}(Q_N, d^\epsilon)}.
\]

Note that evidently $1 - \frac{N-k+k}{p} > 0$.

Let $\delta > 0$ be a fixed real number. Then for every $n$ there exists $r(n) > 0$ such that
\[
J_2(n, x'') = \int_M \|(RSu_n)(x', x'') - u_n(x', 0)\| dx' < \delta / 3 \quad \text{for } |x''| < r(n).
\]

The proof is analogous to the estimate of $I(x'')$.

Finally, from the trivial imbedding
\[ W^{1,p}(Q_N, d^\epsilon) \hookrightarrow W^{1,1}(Q_N) \]
and from Lemma 2.2 we have
\[ \lim_{n \to \infty} J_3(n) = 0. \]
Now, fix \( n_0 \) such that \( J_3(n_0) < \delta/3 \). From (3.8) we can find a corresponding number \( r(n_0) \) such that \( J_2(n_0, x'') < \delta/3 \) for \( |x''| < r(n_0) \). According to (3.7) there exists a number \( r_1 > 0 \) such that
\[
J_1(n_0, x'') < \delta/3 \quad \text{for} \quad |x''| < r_1.
\]

Let \( r = \min(r_1, r(n_0)) \).

The estimate (3.5) implies
\[
J(x'') < \delta \quad \text{for} \quad |x''| < r.
\]

Therefore,
\[
(3.9) \quad \lim_{|x''| \to 0} J(x'') = 0.
\]

By (3.4) and (3.9) we have \( u = v \) in \( L^1(M) \). Setting \( R_0 = RS \) we complete the proof. \( \square \)

**Theorem 3.4.** Let \( N \geq 2, 1 \leq k \leq N - 1, \varepsilon \in \mathbb{R}, k - N < \varepsilon < p + k - N \) and let \((\Omega, M) \in B(k, N)\). Then there exists a bounded linear operator
\[
R: W^{1-\frac{k+k+\varepsilon}{p}, p}(M) \to W^{1,p}(\Omega, d^\varepsilon)
\]
such that
\[
T(Ru) = u.
\]

**Proof.** The theorem is an easy consequence of Lemma 3.3. \( \square \)

**References**


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