Jerry E. Vaughan
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ON FROLÍK'S CHARACTERIZATION OF CLASS P

JERRY E. VAUGHAN,[1] Greensboro

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1. INTRODUCTION

In this paper, we assume that *space* means $T_{3.5}$-space (i.e., completely regular, Hausdorff space), except in Theorem 1.2 which requires no separation axioms at all. A space is called *pseudocompact* provided every continuous, real-valued function on X is bounded, or equivalently, if there does not exist an infinite, locally finite family of non-void open sets in X. In [1], Z. Frolík defined the class P to be the class of all spaces X such that $X \times Y$ is pseudocompact for every pseudocompact space Y, and he gave a characterization of that class [1, Theorem 3.6]. M. Atsuji noticed that there is a gap in Frolík's proof of the characterization theorem. It is the main purpose of this paper to show that a minor modification to Frolík's proof results in a correct proof (thus, the characterization is valid), and to give an example to show that Frolík's proof does not work in all cases (so some modification of his proof is necessary in order to give a correct proof). We begin with the statement of Frolík's characterization.

Theorem 1.1. [1, 3.6] *For a $T_{3.5}$-space X, the following are equivalent*

1. *X belongs to class P,*

2. *If $\mathcal{U}$ is an infinite disjoint family of non-void open subsets of X, then there exists a disjoint sequence $\{U_n : n \in \omega\}$ in $\mathcal{U}$ such that for every filter $\mathcal{N}$ of infinite subsets of ω we have*

$$\bigcap_{F\in\mathcal{N}} \overline{\bigcup_{n\in F} U_n}^X \neq \emptyset.$$

The gap in Frolík's proof of Theorem 1.1 occurs in the step "1 → 2," where given an arbitrary space X that does not satisfy the condition in 2, he claims to construct

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a space $Y \subset \beta X$ which is pseudocompact, but $X \times Y$ is not pseudocompact (thus 1 fails). Our modification of the proof consists in constructing a similar Y, but our Y is a subset of $\beta\omega$ instead of βX. In §2, we show that Frolík's construction for Y does not work for every X.

Proof of Theorem 1.1. "1 → 2." Suppose that X does not satisfy the condition in 2. Then there exists a countably infinite pairwise disjoint family $\{U_n : n \in \omega\}$ of non-empty open subsets of X such that for any infinite $N \subset \omega$, there exists a filter $\mathscr{N}_N$ on N such that

$$\bigcap_{F \in \mathscr{N}_N} \overline{\bigcup_{n \in F} U_n}^X = \emptyset. \tag{1}$$

Define

$$Y = \omega \cup \{u \in \omega^\star : \exists N \in [\omega]^\omega (u \supset \mathscr{N}_N)\}. \tag{2}$$

Clearly $Y \subset \beta\omega$ is pseudocompact since every infinite subset of the dense set $\omega \subset Y$ has a limit point in Y. We show that the space $X \times Y$ is not pseudocompact. Consider the family $\mathscr{U} = \{U_n \times \{n\} : n \in \omega\}$ of non-empty open subsets of $X \times Y$. We shall prove that $\mathscr{U}$ is locally finite. Let $u \in Y - \omega$. By definition of Y there exist $N \subset \omega$, and a filter $\mathscr{N}_N$ on N such that $\mathscr{N}_N \subset u$, and (1) holds. Thus for any $x \in X$ there exist $F \in \mathscr{N}_N$ and an open set V containing x such that $V \cap (\bigcup_{n \in F} U_n) = \emptyset$. Then $V \times \overline{F}^{\beta\omega}$ is an open neighborhood of (x, u) which misses every set in $\mathscr{U}$ because if $n \in F$ then $U_n \cap V = \emptyset$. This completes the proof. □

We note that neither the above proof of "1 → 2" nor Frolík's proof of "2 → 1" requires any separation axioms on the spaces X or Y. Thus, Theorem 1.1 can now be reformulated as follows. Recall that a space is called *feebly compact* if every locally finite family of nonempty open sets in the space is finite.

Theorem 1.2. *For any feebly compact topological space X, the following are equivalent*

1. *$X \times Y$ is feebly compact for every feebly compact space Y,*
2. *If $\mathscr{U}$ is an infinite disjoint family of non-void open subsets of X, then there exists a disjoint sequence $\{U_n : n \in \omega\}$ in $\mathscr{U}$ such that for every filter $\mathscr{N}$ of infinite subsets of ω we have*

$$\bigcap_{F \in \mathscr{N}} \overline{\bigcup_{n \in F} U_n}^X \neq \emptyset.$$

2. Example

Let $[0,1]$ denote the closed unit interval with its usual topology. For each $n \in \omega$ let $I_n = \{n\} \times [0,1]$, and let $\mathscr{S} = \omega \times [0,1]$.

Theorem 2.1. *There is a space X with $\mathscr{S} \subset X \subset \beta\mathscr{S}$, and a family $\{U_n : n \in \omega\}$ of disjoint open subsets of X such that*

(a) *for any infinite $N \subset \omega$ there exists a filter $\mathscr{N}$ on N such that*

$$\bigcap_{F \in \mathscr{N}} \overline{\bigcup_{n \in F} U_n}^{X} = \emptyset,$$

and

(b) *for any choice of $z_n \in U_n$ for $n \in \omega$, the space*

$$Y = \overline{Z}^{\beta X} - (X - Z), \tag{3}$$

where $Z = \{z_n : n \in \omega\}$, is pseudocompact, and

(c) *$X \times Y$ is pseudocompact.*

Since the definition of the space Y in (3) is the one used by Frolík (see [1, page 345, line -2]), Theorem 2.1 shows that Frolík's proof of Theorem 1.1 ($1 \to 2$) does not always work. We now list two lemmas that we need in the proof of 2.1.

Recall that for $f : \omega \to X$ and $r \in \omega^\star$ we define $x = r\text{-}\lim f$ provided for every neighborhood U of $x \in X$, the set $\{n \in \omega : f(n) \in U\}$ is in r. A discussion of "r-limits" is given in [3].

Lemma 2.2. *Let π_X and π_Y denote the natural projection maps on $X \times Y$. If $f : \omega \to X \times Y$, and $r \in \omega^\star$, then $(x, y) = r\text{-}\lim f$ if and only if $x = r\text{-}\lim \pi_X \circ f$, and $y = r\text{-}\lim \pi_Y \circ f$.*

Lemma 2.3. *If $f : \omega \to K$ is one-to-one into a compact space K, and $f(\omega)$ is C^*-embedded in K, then the Stone extension $\bar{f} : \beta\omega \to \overline{f(\omega)}^{K}$ is a homeomorphism.*

Proof of Theorem 2.1. Let $\mathscr{A}$ denote the set of all one-to-one sequences $f : \omega \to \mathscr{S}$ such that $|f(\omega) \cap I_n| \leqslant 1$ for all $n \in \omega$. Let π denote the Stone extension of the map from $\mathscr{S}$ onto ω that collapses I_n onto n. Since $\mathscr{S}$ has cardinality c, so does $\mathscr{A}$. Let $\{(f_\alpha, g_\alpha) : \alpha < c\}$ be a listing of all pairs $(f, g) \in \mathscr{A} \times \mathscr{A}$ such that $f(\omega) \cap g(\omega) = \emptyset$. We now select some points of $\beta\mathscr{S} - \mathscr{S}$ by transfinite induction on c. Assume we have constructed distinct $x_\beta, y_\beta \in \beta\mathscr{S} - \mathscr{S}$ and ultrafilters $r_\alpha \in \omega^\star$ such that the following hold for all $\beta < \alpha$ where $\alpha < c$:

$$x_\beta = r_\beta\text{-}\lim f_\beta, \text{ and } y_\beta = r_\beta\text{-}\lim g_\beta, \tag{4}$$

(5) $$x_\beta, y_\beta \notin \{x_\gamma, y_\gamma : \gamma < \beta\}.$$

To pick x_α, y_α, and r_α, let $P = \{x_\beta, y_\beta : \beta < \alpha\}$. Since f_α and g_α are in $\mathscr{A}$, their ranges are closed subsets of the metric space $\mathscr{S}$; so by the Tietze extension theorem both $f_\alpha(\omega)$ and $g_\alpha(\omega)$ are C^*-embedded in $\mathscr{S}$, hence in $\beta\mathscr{S}$. By 2.3, their Stone extensions $\bar{f}_\alpha$ and $\bar{g}_\alpha$ are homeomorphisms, hence one-to-one. Since $|\omega^\star| = 2^c$,

$$Q = \omega^\star - (\bar{f}_\alpha^{-1}(P) \cup \bar{g}_\alpha^{-1}(P)) \neq \emptyset.$$

Pick any $r \in Q$, and define $r_\alpha = r$, $x_\alpha = r\text{-}\lim f_\alpha$ and $y_\alpha = r\text{-}\lim g_\alpha$. In $\beta\omega$ it is obvious that $r = r\text{-}\lim i_\omega$, where i_ω denotes the identity map on ω. By continuity,

$$\bar{f}_\alpha(r) = r\text{-}\lim \bar{f}_\alpha \circ i_\omega = r\text{-}\lim f_\alpha,$$

and similarly for $\bar{g}_\alpha$. Thus (5) holds because $x_\alpha = \bar{f}_\alpha(r)$, and $y_\alpha = \bar{g}_\alpha(r)$. This completes the induction. Now we define

$$X = \mathscr{S} \cup \{x_\alpha : \alpha < c\}.$$

Note that X is pseudocompact because every sequence in the dense set $\mathscr{S} \subset X$ has a limit point in X. Define

$$U_n = \pi^{-1}(\{n\}) \cap X = I_n$$

for all $n \in \omega$. We show that 2.1(a) holds. Since $\pi(X)$ has cardinality c, the set $D = \omega^\star - \pi(X)$ is dense in $\omega^\star$, and therefore for any infinite $N \subset \omega$ there exists $u \in D \cap N^\star$. The ultrafilter u is the filter on N required in 2.1(a): Since $\pi^{-1}(F) = \bigcup\{U_i : i \in F\}$,

(6) $$\bigcap_{F \in u} \overline{\bigcup_{n \in F} U_n}^{X} = \bigcap_{F \in u} \overline{\pi^{-1}(F)}^{X}.$$

To show that the set in (6) is empty, it suffices to show

(7) $$\bigcap_{F \in u} \overline{\pi^{-1}(F)}^{\beta\mathscr{S}} \subset \pi^{-1}(u).$$

If (7) did not hold, then there would exist a point $x \in \overline{\pi^{-1}(F)}^{\beta\mathscr{S}}$ for all $F \in u$ and $\pi(x) \neq u$. Thus, there would be an $F \in u$ such that $\pi(x) \notin \overline{F}^{\beta\omega}$; so $\pi^{-1}(\overline{\omega - F}^{\beta\omega})$ would be a clopen neighborhood of x missing $\pi^{-1}(F)$ (i.e., $x \notin \overline{\pi^{-1}(F)}^{\beta\mathscr{S}}$). Thus 2.1(a) holds.

Now we turn to 2.1(b), and in fact take *any* choice of $z_n \in U_n$ for $n \in \omega$, put $Z = \{z_n : n \in \omega\}$, and define, as in Frolík's proof [1, page 345, line −2],

$$Y = \overline{Z}^{\beta X} - (X - Z).$$

The subspace Y is pseudocompact because every sequence in the dense subset Z has a limit point in Y. Thus, 2.1(b) holds.

To complete the proof we have to show that 2.1(c) holds, i.e., that $X \times Y$ *is* pseudocompact. To do this, we take an arbitrary infinite family $\mathscr{W}$ of disjoint open subsets of $X \times Y$, and show it is not locally finite in $X \times Y$. We project this family into X and Y, and consider three cases. Since $\bigcup\{I_n : n \in \omega\} = \mathscr{S}$ is dense in X, and projection maps are open, we know that for every $W \in \mathscr{W}$ there exists $m \in \omega$ such that $\pi_X(W) \cap I_m \neq \emptyset$.

Case 1. There exists $m \in \omega$ such that for infinitely many $W \in \mathscr{W}$, $\pi_X(W) \cap I_m \neq \emptyset$. In this case, infinitely many sets in $\mathscr{W}$ intersects $I_m \times Y$ which is the product of a compact and a pseudocompact space; hence is pseudocompact [2, 9.14]. Thus, $\mathscr{W}$ is not locally finite.

Case 2. There exists $z \in Z$ such that $\{W \in \mathscr{W} : z \in \pi_Y(W)\}$ is infinite. As in Case 1, we are done since $X \times \{z\}$ is pseudocompact.

Case 3. Not Case 1, and not Case 2. Assume we have constructed $W_i \in \mathscr{W}$, $x_i \in X$, and $n_i, m_i \in \omega$ for all $i < k$ such that

$$n_0 < n_1 < \cdots < n_{k-1} \text{ and } m_0 < m_1 < \cdots < m_{k-1}, \tag{8}$$

$$x_i \in \pi_X(W_i) \cap (I_{n_i} - Z), \tag{9}$$

$$(x_i, z_{m_i}) \in W_i. \tag{10}$$

To construct n_k, m_k, W_k, and x_k, we note that by "not Case 1" there are only finitely many $W \in \mathscr{W}$ such that $\pi_X(W)$ intersects $\bigcup\{I_i : i \leqslant n_{k-1}\}$, and by "not Case 2" there are only finitely many $W \in \mathscr{W}$ such that $\pi_X(W)$ intersect $\{z_i : i \leqslant m_{k-1}\}$. Pick $W_k \in \mathscr{W}$ which is not in either of the mentioned finite subsets of $\mathscr{W}$. There exists $n_k \in \omega$ such that $\pi_X(W_k) \cap I_{n_k} \neq \emptyset$. Clearly $n_k > n_{k-1}$. Since I_n has no isolated points and $|Z \cap I_n| = 1$ we may pick $x_k \in \pi_X(W_k) \cap (I_{n_k} - Z)$. Since $\{x_k\} \times Z$ is dense in $\{x_k\} \times Y$, there exists $z \in Z$ such that $(x_k, z) \in W_k$. Let $m_k \in \omega$ be such that $z = z_{m_k}$. Clearly $m_k > m_{k-1}$. This completes the induction.

Define $f(i) = x_i$ and $g(i) = z_{m_i}$ for all $i \in \omega$. Now (8) and (9) imply that $(f, g) \in \mathscr{A}$, say $(f, g) = (f_\alpha, g_\alpha)$ for some $\alpha < c$. We have, therefore,

$$x_\alpha = r_\alpha\text{-}\lim f_\alpha, \text{and } y_\alpha = r_\alpha\text{-}\lim g_\alpha.$$

Hence by 2.2, $(x_\alpha, y_\alpha) = r_\alpha\text{-}\lim(f(i), g(i))$. By definition of X, we have $x_\alpha \in X$, and since g is a mapping into Z, we have $y_\alpha \in \overline{Z}^{\beta X} - Z$. By the construction, $y_\alpha \notin X$; so we have $y_\alpha \in \overline{Z}^{\beta X} - (X - Z) = Y$. Thus $(x_\alpha, y_\alpha) \in X \times Y$. This shows by (10) that $\mathscr{W}$ is not locally finite at the point (x_α, y_α), and this completes the proof of Theorem 2.1.

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Added 12 May 1993: Today we learned that a proof of Theorem 1.1 (1 — 2), essentially the same as above, was given earlier by J. L. Blasco Olcina in "Pseudocompacidad y compacidad numerable del producto de dos espacios topológicos," Collectanea Mathematica 29 (1978) 89–96, and in "Compacidad numerable y pseudocompacidad del producto de dos espacios topologícos. Productos finitos de espacios con topologias proyectivas de funciones reales," Serie Universitaria 39, Fundacion Juan March, Madrid 1978.

References

[1] *Zdeněk Frolík*: The topological product of two pseudocompact spaces. Czech. Math. J. *10* (1960), 339–348.

[2] *L. Gillman and M. Jerison*: Rings of Continuous Functions. van Nostrand, New York, 1960.

[3] *J.E. Vaughan*: Countably compact and sequentially compact spaces. Handbook of Set-theoretic Topology (K. Kunen and J. Vaughan, ed.). North-Holland, Amsterdam, 1984.

Author's address: University of North Carolina at Greensboro, Greensboro, NC 27412, U.S.A.