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A CHARACTERIZATION OF THE INTERVAL FUNCTION OF A CONNECTED GRAPH

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0. By a graph we mean a finite undirected graph with no loop or multiple edge (i.e. a graph in the sense of [1] or [2], for example). Throughout the paper we assume, that a connected graph \( G \) is given. Let \( V \) and \( E \) denote its vertex set and its edge set, respectively. Moreover, we denote by \( d(u, v) \) the distance between \( u \) and \( v \) in \( G \), for any \( u, v \in V \). Note that \( d(u, v) \) is equal to the length of an arbitrary shortest \( u - v \) path in \( G \), for any \( u, v \in V \). Clearly, the vertex set \( V \) and the distance function \( d \) form a finite metric space. (Kay and Chartrand [2] found a necessary and sufficient condition for a finite metric space to be generated by the vertex set and the distance function of a connected graph).

Similarly as in [3], by the \( \text{interval function} \) \( I \) of \( G \) we mean the mapping of \( V \times V \) into the set of all subsets of \( V \) defined as follows (for every \( (u, v) \in V \times V \)):

\[
I(u, v) = \{x \in V; x \text{ belongs to a } u - v \text{ path of length } d(u, v) \text{ in } G\}.
\]

The interval function of a connected graph, which was defined and intensively studied in Mulder [3], is an important tool for the study of metric properties of graphs.

The definition of the interval function \( I \) of \( G \) depends on the notion of the distance in \( G \) (or on the notion of shortest paths in \( G \)). We are going to derive an essentially different characterization of the interval function.

1. Denote by \( J \) the set of all mappings \( J \) of \( V \times V \) into the set of all subsets of \( V \) such that \( J \) fulfills the following Axioms I–VI (for arbitrary \( u, v, w, x \in V \)):

\[
\begin{align*}
I & |J(u, v)| = 2 \text{ if and only if } \{u, v\} \in E; \\
II & u \in J(u, v); \\
III & \text{if } w \in J(u, v), \text{ then } |J(u, w) \cap J(w, v)| = 1;
\end{align*}
\]
IV if \( w \in J(u,v) \), then \( J(w,v) \subseteq J(u,v) \);

V if \( w \in J(u,v) \) and \( x \in J(w,v) \), then \( w \in J(u,x) \);

VI \( J(v,u) = J(u,v) \).

Put \( J = I \); it is clear that \( J \) fulfils Axiom I; using 1.1.2 from [3] we easily get

\[ I \in J. \]

We now make several observations concerning \( J \).

Using Axioms II and III we obtain \( J(u,u) = \{u\} \) for \( J \in J \) and \( u \in V \).

Let \( J \in J \). For \( u, v \in V \) we define the set \( \Sigma_J(u,v) \) as follows:

\[
\Sigma_J(u,v) = \begin{cases} 
\{(u)\} & \text{if } u = v; \\
\{ (x_1, \ldots, x_k, v); k \geq 1, x_k \in J(u,v), \\
\{ x_k, v \} \in E \text{ and } (x_1, \ldots, x_k) \in \Sigma_J(u,x_k) \} & \text{if } u \neq v.
\end{cases}
\]

**Lemma.** Let \( J \in J \) and \( u, v \in V \). Assume that \( u \neq v \). Then

1. \( \{u, v\} \subseteq J(u,v); \)
2. if \( w \in J(u,v) - \{u\} \), then \( J(w,v) \subseteq J(u,v) - \{u\} \);
3. there exists \( x \in J(u,v) \) such that \( \{x, v\} \in E \);
4. \( J(u,v) - \{v\} = \bigcup_{x \in J(u,v) - \{v\}} J(u,x); \)
5. if \( (w_1, \ldots, w_m) \in \Sigma_J(u,v) \), then \( w_1, \ldots, w_m \in J(u,v) \) and \( (w_1, \ldots, w_m) \) is a \( u - v \) path in \( G \) (i.e. a \( u - v \) path considered as a sequence of vertices);
6. \( \Sigma_J(u,v) \neq \emptyset. \)

**Proof.** (1) follows from Axioms II and VI.

Let \( w \in J(u,v) - \{u\} \). According to Axiom IV, \( J(w,v) \subseteq J(u,v) \). Suppose \( u \in J(w,v) \). Obviously, \( u \neq w \). As follows from Axioms IV and VI, \( J(w,u) = J(u,u) \subseteq J(v,v) = J(w,v) \subseteq J(u,v) \). Axiom III implies that \( |J(w,u)| = 1 \), which contradicts (1). Thus \( u \notin J(w,v) \) and we get (2).

(3) follows from (1), (2), and Axiom I.

First, let \( w \in J(u,v) - \{v\} \). Since \( w \neq v \), (3) implies that there exists \( x \in J(w,v) \) such that \( \{x, v\} \in E \). According to Axiom V, \( w \in J(u,x) \). Using (2) and Axiom VI, we get (4).

(5) follows from the definition of \( \Sigma_J(u,v) \), (2), and Axiom VI.

Combining (2), (3) and Axiom VI with the definition of \( \Sigma_J(u,v) \), we get (6), which completes the proof.  \( \square \)
2. Let \( J, J' \in \mathbf{J} \), let \( n \geq 0 \) be an integer. We write \( P_n(J, J') \) to express the fact that

\[ J(u, v) \subseteq J'(u, v) \]

for each pair of \( u \) and \( v \) in \( V \) such that \( d(u, v) = n \).

We now give a characterization of the interval function of \( G \), which is the main result of present paper.

**Theorem.** Let \( J \in \mathbf{J} \). Then \( J = I \) if and only if \( J \) fulfills the following Axioms VII and VIII (for arbitrary \( u, v, x, y \in V \)):

- **VII** if \( \{u, x\}, \{v, y\} \in E \), \( x \in J(u, v) \) and \( u \in J(x, y) \), then \( v \in J(x, y) \);
- **VIII** if \( \{u, x\}, \{v, y\} \in E \), \( x \notin J(u, v) \) and \( x \notin J(u, y) \), then \( v \in J(x, y) \).

**Proof.** (A) Assume that \( J = I \). We shall prove that \( J \) fulfills Axioms VII and VIII. Consider arbitrary \( u, v, x, y \in V \) such that \( \{u, x\}, \{v, y\} \in E \) and \( x \in J(u, v) \). Put \( n = d(u, v) \). Then \( d(x, v) = n - 1 \).

(Axiom VII) Assume that \( y \notin J(u, v) \) and \( x \in J(u, v) \). We want to prove that \( v \notin J(x, y) \). Since \( \{v, y\} \in E \) and \( y \notin J(u, v) \), we have \( d(u, y) = n - 1 \). Certainly, \( d(x, y) \leq n \). Since \( u \in J(x, y) \), we get \( d(x, y) = n \). Thus \( v \in J(x, y) \).

(Axiom VIII) Assume that \( y \notin J(u, v) \) and \( x \notin J(u, y) \). We want to prove that \( v \notin J(x, y) \). Since \( y \notin J(u, v) \), we have \( d(u, y) \geq n \). Since \( x \notin J(u, y) \), we have \( d(x, y) \geq d(u, y) \geq n \). Since \( d(x, v) = n - 1 \) and \( d(v, y) = 1 \), we get \( v \in J(x, y) \).

(B) Conversely, let us now assume that \( J \) fulfills Axioms VII and VIII. We shall prove that \( P_n(I, J) \) and \( P_n(J, I) \) for each integer \( n \) such that \( 0 \leq n \leq D \), where \( D \) denotes the diameter of \( G \). We proceed by induction on \( n \). It is clear that \( P_n(I, J) \) and \( P_n(J, I) \) for \( n = 0 \) and \( 1 \). Therefore, let us assume that \( 2 \leq n \leq D \) and

\[
P_k(I, J) \text{ and } P_k(J, I)
\]

for each \( k \in \{0, \ldots, n - 1\} \).

The rest of the proof will be divided into two steps.

**Step 1.** We shall prove that \( P_n(I, J) \). Consider arbitrary \( u, v \in V \) such that \( d(u, v) = n \). We want to prove that \( I(u, v) \subseteq J(u, v) \). Suppose, to the contrary, \( I(u, v) - J(u, v) \neq \emptyset \). Consider \( w \in I(u, v) - J(u, v) \). Since \( w \in I(u, v) \), there exist a \( v - u \) path \( (y_0, \ldots, y_n) \) in \( G \) and an integer \( i \) such that \( 0 \leq i \leq n \) and \( w = y_i \). Clearly, \( y_0 = v \) and \( y_n = u \). Since \( w \notin J(u, v) \), we have \( 0 < i < n \). Consider an arbitrary \( j \in \{1, \ldots, n-1\} \). It follows from (7) that \( I(v, y_j) = J(v, y_j) \) and \( I(y_j, u) = J(y_j, u) \).

If \( y_j \in J(u, v) \), then Axioms IV and VI imply that \( I(v, y_j) \subseteq J(u, v) \) and \( I(y_j, u) \subseteq J(u, v) \), and thus \( w \in J(u, v) \), which is a contradiction. We conclude that \( y_1, \ldots, y_{n-1} \notin J(u, v) \).
As follows from (6), there exist \( x_0, \ldots, x_m \in V \) \((m \geq 1)\) such that \((x_0, \ldots, x_m) \in \Sigma_J(u, v)\). According to (5), \((x_0, \ldots, x_m)\) is a \(u-v\) path in \(G\). Thus \(x_0 = u\) and \(x_m = v\). Since \(n = d(u, v)\), \(m \geq n\). Since \((x_0, \ldots, x_{i+1}) \in \Sigma_J(x_0, x_{i+1})\), it follows from (5) and Axioms V and VI that

\[(8) \quad x_{i+1} \in J(x_i, v)\]

for each \(i \in \{0, \ldots, m - 1\}\). Since \((y_0, \ldots, y_n)\) is a \(v-u\) path in \(G\) and \(y_1, \ldots, y_{n-1} \notin J(u, v)\), we see that

\[(9) \quad (y_i, \ldots, y_n = x_0, \ldots, x_i) \text{ is a path in } G\]

for each \(i \in \{0, \ldots, n\}\).

Put \(x_{n-1} = y_{n-1}\). Certainly, the following statements \((10_i), (11_i)\) and \((12_i)\) hold for \(i = 0:\)

\[(10_i) \quad d(x_i, y_i) = n;\]
\[(11_i) \quad v \in J(x_i, y_i);\]
\[(12_i) \quad x_{i-1} \notin J(x_i, y_i).\]

Clearly, \(x_{n-1} \in J(x_0, x_n)\). Since \(y_n = x_0, x_{n-1} \in J(x_n, y_n)\). Thus \((12_n)\) does not hold. This means that there exists \(h \in \{0, \ldots, n-1\}\) such that each of the statements \((10_h), (11_h)\) and \((12_h)\) holds but at least one of the statements \((10_{h+1}), (11_{h+1})\) and \((12_{h+1})\) does not.

Combining \((8_h)\) and \((11_h)\) with Axioms IV–VI, we get

\[(13) \quad x_{h+1} \in J(x_h, y_h);\]
\[(14) \quad v \in J(x_{h+1}, y_h).\]

It follows from \((9_h)\) and \((10_h)\) that \(d(x_h, y_{h+1}) = n - 1\). According to (7), \(J(x_h, y_{h+1}) = I(x_h, y_{h+1})\). Obviously, \(x_{h-1} \in I(x_h, y_{h+1})\). Thus \(x_{h-1} \in J(x_h, y_{h+1})\).

If \(y_{h+1} \in J(x_h, y_h)\), then it follows from Axioms IV and VI that \(x_{h-1} \in J(x_h, y_h)\), which contradicts \((12_h)\). Therefore,

\[(15) \quad y_{h+1} \notin J(x_h, y_h).\]

We now want to show that \(x_{h+1} \notin J(x_h, y_{h+1})\). Suppose, to the contrary, \(x_{h+1} \in J(x_h, y_{h+1})\). Since \(d(x_h, y_{h+1}) = n - 1\), it follows from (7) that \(x_{h+1} \in I(x_h, y_{h+1})\). Thus \(d(x_{h+1}, y_{h+1}) = n - 2\). It follows from \((10_h)\) that \(d(x_{h+1}, y_h) = n - 1\) and
According to (7), \( y_{h+1} \in J(x_{h+1}, y_h) \). Combining this fact with (13) and Axiom IV, we get \( y_{h+1} \in J(x_h, y_h) \), which contradicts (15). Therefore, \( x_{h+1} \notin J(x_h, y_{h+1}) \).

Since \( x_{h+1} \in J(x_h, y_h) \) and \( y_{h+1} \notin J(x_h, y_h) \), Axioms VIII implies that

\[
y_h \in J(x_{h+1}, y_{h+1}).
\]

Combining (14) and (16) with Axioms IV and VI, we get (11_{h+1}).

As follows from (9_{h+1}), \( d(x_{h+1}, y_{h+1}) \leq n \). Suppose \( d(x_{h+1}, y_{h+1}) \leq n - 1 \). According to (7), \( J(x_{h+1}, y_{h+1}) = I(x_{h+1}, y_{h+1}) \). It follows from (16) that \( y_h \in I(x_{h+1}, y_{h+1}) \). This implies that \( d(x_{h+1}, y_h) \leq n - 2 \). Hence, \( d(x_h, y_h) \leq n - 1 \), which is a contradiction. Thus we have (10_{h+1}).

Since (10_{h+1}) and (11_{h+1}) hold, it follows from the definition of \( h \) that (12_{h+1}) does not hold. Thus we have \( x_h \notin J(x_{h+1}, y_{h+1}) \). Combining this fact with (13), (16) and Axiom VII, we get \( y_{h+1} \notin J(x_h, y_h) \), which contradicts (15).

Thus \( I(u, v) \subseteq J(u, v) \) and we have

\[
P_n(I, J).
\]

**Step 2.** We shall prove that \( P_n(J, J) \). Consider arbitrary \( u, v \in V \) such that \( d(u, v) = n \). We want to prove that \( J(u, v) \subseteq I(u, v) \). Suppose, to the contrary, \( J(u, v) \neq I(u, v) \). It follows from (4) that \( d(w, v) < n \) and \( J(u, v) = I(u, v) \). Assume that there exists \( w' \in J(u, w) \neq \emptyset \) such that \( d(w', v) \leq n \). According to Axioms V and VI, \( w \in J(w', v) \). Thus \( w \in I(w', v) \). Since \( w' \in I(u, v) \), \( w \in I(u, v) \). This means that \( d(u, w) = n - 1 \). As follows from (7), \( J(u, w) = I(u, w) \).

We get \( J(u, w) \subseteq I(u, v) \), which is a contradiction. Thus we have obtained that \( (J(u, w) \neq \emptyset) \cap I(u, v) = \emptyset \). According to (6), \( \Sigma_J(u, w) \neq \emptyset \). There exist \( x_0, \ldots, x_{m-1} \in V \) (\( m \geq 2 \)) such that \( (x_0, \ldots, x_{m-1}) \in \Sigma_J(u, w) \). Clearly, \( x_0 = u, x_{m-1} = w, \) and \( x_1, \ldots, x_{m-1} \notin I(u, v) \). Put \( x_m = v \). Certainly, \( (x_0, \ldots, x_m) \) is a \( u - v \) path in \( G \). Since \( x_{m-1} \notin I(u, v) \), we see that \( m > n \). Moreover, we have (8i) for each \( i \in \{0, \ldots, m - 1\} \).

Since \( d(u, v) = n \), there exist \( y_0, \ldots, y_n \in V \) such that \( y_0 = v, y_n = u, \) and \( (y_0, \ldots, y_n) \) is a \( u - v \) path of length \( n \) in \( G \). Clearly, \( y_0, \ldots, y_n \in I(u, v) \). We get (9i) for each \( i \in \{0, \ldots, n\} \).

 Obviously, both (10_0) and (11_0) hold. Since \( m > n, x_n \neq v \). Since \( y_n = u, (2) \) implies that \( v \notin J(x_n, y_n) \). Thus (11_n) does not hold. This means there exists \( h \in \{0, \ldots, n - 1\} \) such that both (10_h) and (11_h) hold but at least one of the statements (10_{h+1}) and (11_{h+1}) does not.

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Similarly as in Step 1, we have (13) and (14).

We want to show that \( d(x_{h+1}, y_h) \geq n \). Suppose to the contrary \( d(x_{h+1}, y_h) \leq n - 1 \). Since \( d(x_h, y_h) = n \), \( d(x_{h+1}, y_h) = n - 1 \). According to (7), \( J(x_{h+1}, y_h) = I(x_{h+1}, y_h) \). Since \( v \in J(x_{h+1}, y_h) \), we have \( v \in I(x_{h+1}, y_h) \). Obviously, \( d(v, y_h) = h \). Thus \( d(x_{h+1}, v) = n - h - 1 \). According to (7), \( J(x_{h+1}, v) = l(x_{h+1}, v) \). Combining (8) and (7), we see that \( d(x_{k}, v) = n - k \) and \( J(x_{k}, v) = l(x_{k}, v) \) for each integer \( k \) such that \( h + 1 < k \leq n \). This means that \( d(x_{h}, v) = 0 \) and therefore \( m = n \), which is a contradiction. Thus we have \( d(x_{h+1}, y_h) \geq n \).

As follows from (9), \( d(x_{h+1}, y_{h+1}) < n \). We want to show that \( 10 \). To the contrary, let \( d(x_{h+1}, y_{h+1}) = n \). Since \( d(x_{h+1}, y_h) = n \), we have \( d(x_{h+1}, y_{h+1}) = n - 1 \). Then \( y_{h+1} \in l(x_{h+1}, y_h) \). It follows from (17) that \( y_{h+1} \in J(x_{h+1}, y_h) \). Combining this fact and (13) with Axioms V and VI, we get \( x_{h+1} \in J(x_h, y_{h+1}) \). Since \( d(x_h, y_h) = n \), we see that \( d(x_h, y_{h+1}) = n - 1 \). It follows from (7) that \( x_{h+1} \in I(x_h, y_{h+1}) \). Hence \( d(x_{h+1}, y_{h+1}) = n - 2 \), which is a contradiction. Thus we have \( 10 \).

Combining (9) and (10), we see that \( y_{h+1} \in I(x_h, y_h) \). As follows from (17), \( x_{h+1} \in J(x_h, y_{h+1}) \). According to (10), \( d(x_{h+1}, y_{h+1}) = n \). Therefore, \( x_{h} \in I(x_{h+1}, y_{h+1}) \). As follows from (17), \( x_{h} \in J(x_{h+1}, y_{h+1}) \). According to (13), \( x_{h+1} \in J(x_{h}, y_{h}) \). Since \( x_{h} \in J(x_{h+1}, y_{h+1}) \) and \( y_{h+1} \in J(x_{h}, y_{h}) \), Axiom VII implies that \( y_{h} \in J(x_{h+1}, y_{h+1}) \). Combining this fact and (14) with Axioms IV and VI, we have \( 11 \), which contradicts the definition of \( h \).

Thus \( J(u, v) \subseteq I(u, v) \), hence \( P_a(J, I) \), which completes the proof of the theorem.

Remark. There is a connection between the interval function of \( G \) and the set of all shortest paths in \( G \). A characterization of the set of all shortest paths in \( G \) was given by the present author in Theorem 1 of [4].

References


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