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*Czechoslovak Mathematical Journal*, Vol. 44 (1994), No. 1, 47–56

Persistent URL: http://dml.cz/dmlcz/128454

# PFEFFER INTEGRABILITY DOES NOT IMPLY $M_1$-INTEGRABILITY

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(Received February 27, 1992)

In [1] the author proved that every $M_1$-integrable function (see [2]) is Pfeffer integrable (see [3]). The aim of this note is to disprove the converse inclusion by constructing a function that is Pfeffer-integrable but not $M_1$-integrable. We will do so by modifying our construction from [4] or [5].

Let us recall the relevant notation and definitions. We will work in a Euclidean space $\mathbb{R}^n$, $n > 1$ with the maximum norm $\|x\| = \max\{|x_i|; i = 1, 2, \dots, n\}$. For a set $M \subset \mathbb{R}^n$ we denote by $d(M)$, $\partial M$, $\operatorname{Int} M$, $\operatorname{Cl} M$, $m(M)$ the diameter, boundary, interior, closure and Lebesgue measure of $M$. All intervals considered, if not stated otherwise, are assumed to be compact and nondegenerate. We also write $V(t, r) = \{x \in \mathbb{R}^n ; \|x - t\| \leqslant r\}$ for $t \in \mathbb{R}^n$, $r > 0$.

Given an interval $I \subset \mathbb{R}^n$, a finite family of tagged intervals $\Delta = \{(x, J)\}$ with $x \in J \subset I$ where the intervals $J$ are nonoverlapping is called *a system in* $I$; it is called *a partition of* $I$ if, moreover, the union of $J$'s is $I$. Given $\delta \colon I \to (0, \infty)$ (*a gauge*), a tagged interval $(x, J)$ is called *$\delta$-fine* if $J \subset V\big(x, \delta(x)\big)$. Given $\alpha$, $0 < \alpha \leqslant 1$, then an interval $J$ is called *$\alpha$-regular* if $\operatorname{reg} J \geqslant \alpha$, where $\operatorname{reg} J$ (the *regularity*) stands for the ratio of the minimal and the maximal edge of $J$. A system in $I$ is called *$\delta$-fine* or *$\alpha$-regular* if all its members are $\delta$-fine or $\alpha$-regular, respectively.

Given $k \in \{0, 1, \dots, n-1\}$ then any $k$-dimensional linear manifold $E$ in $\mathbb{R}^n$ which is parallel to $k$ distinct coordinate axes will be called *a plane* (of dimension $k$). If $J = [a_1, b_1] \times \dots \times [a_n, b_n]$ and $E$ is a $k$-plane parallel to the $x_{j_i}$-axes, $i = 1, 2, \dots, k$, we define:

$$\operatorname{reg}_E J = \operatorname{reg} J \quad \text{if} \quad J \cap E = \emptyset;$$

This research was supported by grant No. 11928 GA of the Czechoslovak Academy of Sciences.

if $J \cap E \neq \emptyset$, then

$$\operatorname{reg}_E J = \min\{b_{j_i} - a_{j_i}\,;\, i = 1, 2, \ldots, k\}/d(J) \quad \text{if} \quad k \neq 0,$$
$$\operatorname{reg}_E J = 1 \quad \text{if} \quad k = 0.$$

Let $\mathcal{E}$ be a finite family of planes. Then we define

$$\operatorname{reg}_{\mathcal{E}} J = \max\{\operatorname{reg}_E J\,;\, E \in \mathcal{E}\} \quad \text{if} \quad \mathcal{E} \neq \emptyset,$$
$$\operatorname{reg}_{\emptyset} J = \operatorname{reg} J.$$

Now let $I \subset \mathbf{R}^n$ be an interval, $f\colon I \to \mathbf{R}$.

**Definition 1 [2].** The function $f$ is *$M_1$-integrable* on $I$ if there is a real number $c$ such that for every $\varepsilon > 0$ there is a gauge $\delta$ on $I$ such that

$$\Big|c - \sum_{\Delta} f(x)m(J)\Big| \leqslant \varepsilon \tag{1}$$

for every $\delta$-fine partition $\Delta = \{(x, J)\}$ of $I$ such that

$$\sum_{\Delta} d(J)m_{n-1}(\partial J) \leqslant \varepsilon^{-1}$$

(the quantity on the left-hand side was called the *measure of irregularity* in [2]).

Remark. We have formally modified the definition by replacing the arbitrary constant $K$ by $\varepsilon^{-1}$ to make it more similar to Pfeffer's definition. It is easy to see that it is equivalent to the definition of $M_1$-integrability in [2] or [1].

**Definition 2 [3].** The function $f$ is *Pfeffer-integrable* on $I$ if there is a real number $c$ such that for every $\varepsilon > 0$ and any finite family $\mathcal{E}$ of planes there is a gauge $\delta$ on $I$ such that (1) holds for every $\delta$-fine partition $\Delta$ of $I$ such that

$$\operatorname{reg}_{\mathcal{E}} J \geqslant \varepsilon \quad \text{for} \quad (x, J) \in \Delta.$$

It is evident that $c$ from Definition 1 or 2, if it exists, is uniquely determined; we write $c = M_1 \int_I f$ or $c = Pf \int_I f$, respectively.

In [6] we have introduced the notion of weak Pfeffer integrability. Since we have proved that it is equivalent to the above defined Pfeffer integrability, we can formulate Definition 2 equivalently in the following form.

**Definition 2* [6].** The function $f$ is *Pfeffer integrable* on $I$ if there is a real $c$ such that for every $\varepsilon > 0$ there is a gauge $\delta$ on $I$ such that (1) holds for every $\delta$-fine partition $\Delta$ of $I$ such that

$$\operatorname{reg}_{\mathcal{F}} J \geqslant \varepsilon \quad \text{for} \quad (x, J) \in \Delta,$$

where $\mathcal{F}$ is the family of all $k$-planes which contain $2^k$ vertices of $I$, $k = 0, 1, \ldots, n-1$.

**Theorem.** *For $n > 1$ there is a function $f\colon [-1,2]^n \to \mathbf{R}$ that is Pfeffer-integrable but not $M_1$-integrable on $I$.*

To construct the function $f$ we will use the idea from [4], replacing the constant $\eta$ by a sequence $\eta_k$ as in [5]. To avoid technical details which are identical to those from [4] we will suppose $n = 2$; the generalization of our construction to $n > 2$ is straightforward, involving more or less just forming Cartesian products of some sets.

Construction. Let us choose sequences $\{r_k\}$, $\{\xi_k\}$, $\{\eta_k\}$, $k = 0, 1, 2, \ldots$ such that

$$(2) \qquad 3 \leqslant r_k \nearrow \infty, \quad \frac{1}{3} > \xi_k \searrow 0, \quad 1 < \eta_k \nearrow \infty, \quad \eta_k (r_0 r_1 \ldots r_{k-1})^{-1} \searrow 0$$

(their further properties will be specified later) and set

$$\zeta_k = \frac{1}{2}(1 - 2r_k^{-1}).$$

First, we construct a Cantor discontinuum on $[0,1]$. Set $S_0 = [0,1]$, $\lambda_0 = \frac{1}{2}, T_0 = (\lambda_0 - \zeta_0, \lambda_0 + \zeta_0)$. The set $S_0 \setminus T_0$ is the union of two compact intervals. Let us denote any of them by $S_1$ and its center by $\lambda_1$; then $S_1 = \left[\lambda_1 - \frac{1}{2}r_0^{-1}, \lambda_1 + \frac{1}{2}r_0^{-1}\right]$ and we set $T_1 = (\lambda_1 - \zeta_1 r_0^{-1}, \lambda_1 + \zeta_1 r_0^{-1})$.

The general step of the construction is described as follows: let $i \in \mathbf{N}$ and let us have $2^{i-1}$ pairwise disjoint compact intervals

$$S_{i-1} = \left[\lambda_{i-1} - \frac{1}{2}(r_0 r_1 \ldots r_{i-2})^{-1}, \lambda_{i-1} + \frac{1}{2}(r_0 r_1 \ldots r_{i-2})^{-1}\right]$$

and the same number of open intervals

$$T_{i-1} = \left(\lambda_{i-1} - \zeta_{i-1}(r_0 r_1 \ldots r_{i-2})^{-1}, \lambda_{i-1} + \zeta_{i-1}(r_0 r_1 \ldots r_{i-2})^{-1}\right).$$

Each set $S_{i-1} \setminus T_{i-1}$ (where $S_{i-1}$, $T_{i-1}$ have the same center) is the union of two compact intervals which we can write in the form

$$S_i = \left[\lambda_i - \frac{1}{2}(r_0 r_1 \dots r_{i-1})^{-1}, \lambda_i + \frac{1}{2}(r_0 r_1 \dots r_{i-1})^{-1}\right];$$

we set

$$T_i = \left(\lambda_i - \zeta_i(r_0 r_1 \dots r_{i-1})^{-1}, \lambda_i + \zeta_i(r_0 r_1 \dots r_{i-1})^{-1}\right).$$

The intervals $S_i$ or $T_i$ with the same index $i$ will be called the intervals of the $i$-th order. The set $D = \bigcup_{i=0}^{\infty} (\bigcup S_i)$ where the union is taken over all intervals $S_i$ of the $i$-th order, is a Cantor discontinuum.

We denote

$$\begin{aligned} Q_i^+ &= T_i \times [\eta_i(r_0 \dots r_{i-1})^{-1}, (\eta_i + \xi_i)(r_0 \dots r_{i-1})^{-1}], \\ Q_i^- &= T_i \times [(\eta_i - \xi_i)(r_0 \dots r_{i-1})^{-1}, \eta_i(r_0 \dots r_{i-1})^{-1}], \\ Q_i &= Q_i^+ \cap Q_i^- . \end{aligned}$$

Now we can define the function $f$. Let us choose a sequence $\{\beta_k\}$, $k = 0, 1, 2, \dots$ such that

$$\beta_k \searrow 0, \quad \sum_{k=0}^{\infty} \beta_k = \infty. \tag{3}$$

and define

$$f(x) = \begin{cases} \beta_i\left(2^{i-1} m(Q_i)\right)^{-1} & \text{for} \quad x \in \operatorname{Int} Q_i^+, \\ -\beta_i\left(2^{i-1} m(Q_i)\right)^{-1} & \text{for} \quad x \in \operatorname{Int} Q_i^-, \\ 0 & \text{elsewhere for} \quad x \in [-1, 2]^2. \end{cases}$$

Note that in view of (2), $f$ is Lebesgue integrable over any closed set $H \subset [-1, 2]^2$ such that $H \cap ([0, 1] \times \{0\}) = \emptyset$. Thus the following proposition can be applied to $f$.

**Proposition.** *Let $I \subset \mathbb{R}^n$ be an interval, $f\colon I \to \mathbb{R}$, $L \subset I$ a closed set, $f(x) = 0$ for $x \in L$. Assume that for every closed set $H \subset I$ with $H \cap L = \emptyset$ the Lebesgue integral $\int_H f = F(H)$ exists. Let $q \in \mathbb{R}$. Then the following two assertions are equivalent:*

(a) *$M_1 \int_I f$ exists ($Pf \int_I f$ exists) and equals $q$;*

(b) *for every $\varepsilon_0 > 0$ there is a gauge $\delta_0\colon L \to (0, \infty)$ such that $\left|F\left(I \setminus \bigcup_{\Delta} J\right) - q\right| \leqslant \varepsilon_0$*

*for every* $\delta_0$*-fine system* $\Delta = \{(t, J)\}$ *such that* $\operatorname{Int} \bigcup_{\Delta} J \supset L$, $t \in L$ *for all* $(t, J) \in \Delta$ *and*

$$\sum_{\Delta} d(J) m_{n-1}(\partial J) \leqslant \varepsilon_0^{-1}$$

($\operatorname{reg}_{\mathcal{F}} J \geqslant \varepsilon_0$ *for all* $(t, J) \in \Delta$, *where* $\mathcal{F}$ *is the system of planes from Definition* 2*).

By $F(M)$ we of course denote the (Lebesgue) integral of $f$ over $M$. Note that this proposition covers two notions of integral, namely the $M_1$-integral and the Pfeffer integral. Its analogue for the $\alpha$-regular Perron integral was proved in [4]. Since the proof of its present version is analogous (relying primarily upon the Saks-Henstock Lemma which is valid also for the integrals introduced and studied in the present paper), we omit it.

In the next two lemmas we will prove assertion (b) from Proposition in the version corresponding to the Pfeffer integral, and disprove it in the version for the $M_1$-integral. It is clear that the only "candidate" for the value $q$ of the integral is zero.

**Lemma 1.** *Let* $\varepsilon > 0$, *let* $p \in \mathbb{N}$ *be such that*

$$\eta_p - \xi_p > \varepsilon^{-1}, \quad 2\beta_{p+1} \leqslant \varepsilon \tag{4}$$

*and let* $\delta$ *be a gauge on* $L = [0, 1] \times \{0\}$ *such that*

$$\delta(x) \leqslant (\eta_p - \xi_p)(r_0 r_1 \ldots r_{p-1})^{-1} \quad \text{for} \quad x \in L. \tag{5}$$

*Let* $\Delta = \{(t, J)\}$ *be a* $\delta$*-fine system in* $[-1, 2]^2$ *such that*

$$t \in L, \quad \operatorname{reg}_{\mathcal{F}} J \geqslant \varepsilon \quad \text{for} \quad (t, J) \in \Delta,$$
$$\operatorname{Int} \bigcup_{\Delta} \supset L.$$

*Then*

$$\Big| F\Big(I \setminus \bigcup_{\Delta} J\Big)\Big| \leqslant \varepsilon.$$

Proof. If $(t, J) \in \Delta$ then $J \cap E = \emptyset$ for all $E \in \mathcal{F}$ since $\mathcal{F}$ includes only the faces of $I = [-1, 2]^2$. Hence $\operatorname{reg}_{\mathcal{F}} J = \operatorname{reg} J$ and we can proceed as in the proof of Lemma 3 in [4] with the single change that $\eta$ is replaced by $\eta_p$. If $F(Q_i \setminus J) \neq 0$ (which is the "dangerous" case) then writing $J = [u, v] \times [w, z]$ we obtain the crucial inequality in the form

$$\begin{aligned} v - u &\geqslant \varepsilon(z - w) > \varepsilon(\eta_i - \xi_i)(r_0 \ldots r_{i-1})^{-1} \\ &> \varepsilon(\eta_p - \xi_p)(r_0 \ldots r_{i-1})^{-1} > (r_0 \ldots r_{i-1})^{-1} \end{aligned}$$

(making use of the fact that reg $J \geqslant \varepsilon$ and $i \geqslant p$ because of (5)). In the same way as in [4], estimating the number of intervals $Q_l$ such that $F(Q_l \setminus J) \neq 0$ and adding the contributions over all intervals $J$ of the system $\Delta$, we arrive at the estimate

$$\Big|F\Big(I \setminus \bigcup_{\Delta} J\Big)\Big| = \Big|\sum_{Q_l} F\Big(Q_l \setminus \bigcup_{\Delta} J\Big)\Big| \leqslant 2\beta_{p+1} \leqslant \varepsilon,$$

which proves the lemma and hence also the validity of (b) from Proposition for the Pfeffer integral and $q = 0$. □

**Lemma 2.** *There exists $\varepsilon > 0$ such that for every gauge $\delta$ on $L$ there are $\delta$-fine systems $\Delta_1$, $\Delta_2$ in $I$ satisfying*

$$t \in L \quad \textit{for} \quad (t, J) \in \Delta_i,$$
$$\sum_{\Delta_i} d(J) m_{n-1}(\partial J) \leqslant \varepsilon^{-1}, \ \operatorname{Int} \bigcup_{\Delta_i} J \supset L, \quad i = 1, 2,$$

*and*

$$F\Big(I \setminus \bigcup_{\Delta_1} J\Big) = 0,$$
$$F\Big(I \setminus \bigcup_{\Delta_2} J\Big) > \varepsilon.$$

Proof. First, let $\{(\tau_l, [\sigma_{l-1}, \sigma_l]) ; l = 1, \ldots, s\}$ be a $\delta_1$-fine partition of $[-\gamma, 1 + \gamma]$ where $0 < \gamma < 1$ and $\delta_1(\tau) = \delta(t)$ with $t = (\tau, 0)$. Set

$$\Delta_1 = \big\{ \big((\tau_l, 0), [\sigma_{l-1}, \sigma_l] \times [-d_l, -d_l + \sigma_l - \sigma_{l-1}]\big) ; l = 1, \ldots, s \big\}$$

with $0 < d_l < \sigma_l - \sigma_{l-1}$ chosen in such a way that

$$-d_l + \sigma_l - \sigma_{l-1} \notin \big((\eta_i - \xi_i)(r_0 \ldots r_{i-1})^{-1}, (\eta_i + \xi_i)(r_0 \ldots r_{i-1})^{-1}\big)$$

for $i \in \mathbb{N}$ (such a choice is evidently possible provided the conditions (2) are suitably specified, e.g. by assuming $(\eta_i - \xi_i)(r_0 \ldots r_{i-1})^{-1} > (\eta_{i+1} + \xi_{i+1})(r_0 \ldots r_i)^{-1}$). Then $\Delta_1$ is a system in $I$ satisfying

$$\sum_{\Delta_1} d(J) m_{n-1}(\partial J) = \sum_{\Delta_1} (\sigma_l - \sigma_{l-1}) 4(\sigma_l - \sigma_{l-1} \leqslant 4(1 + 2\gamma)$$

(provided we assume — without loss of generality — $\sigma_l - \sigma_{l-1} \leqslant 1$). Choosing $\varepsilon < \frac{1}{4}(1 + 2\gamma)^{-1}$ we find that $\Delta_1$ satisfies all conditions of Lemma 2 including $F(I \setminus \bigcup_{\Delta_1} J) = 0$ (since no $J$ cuts any $Q_i$ "horizontally").

R e m a r k. This part of the proof requires a comment concerning the case $n > 2$. Then the first step of the proof consists in constructing a partition of $[-\gamma, 1+\gamma]^{n-1}$ into $(n-1)$-dimensional cubes (then the estimate of $\sum_{\Delta_l} d(J)m_{n-1}(\partial J)$ is obtained similarly as above). Such a partition exists by virtue of the Cousin Lemma. It is even inessential that the interval to be partitioned is an $(n-1)$-dimensional cube since by a strong version of the Cousin Lemma (see Appendix) any interval can be partitioned into intervals arbitrarily close to cubes, i.e. with $\operatorname{reg} J \geqslant \alpha$, where $0 < \alpha < 1$ is arbitrary. The above mentioned estimate than follows in the same way as above.

P r o o f—c o n t i n u e d Let us now construct the partition $\Delta_2$ from Lemma 2. We proceed again similarly as in the proof of Lemma 4 [4].

Let $\delta$ be a gauge on $L$ and let us denote

$$W_k = \{w_1 \in D;\, \delta((w_1, 0)) > k^{-1}\}, \quad k = 1, 2, \ldots.$$

By Baire's theorem on complete spaces there is $p \in \mathbb{N}$ such that $W_p$ is not nowhere dense, i.e. there are $z \in D$ and $\omega > 0$ such that

(i) $$D \cap [z-\omega, z+\omega] \subset \operatorname{Cl} W_p.$$

Since $D = \bigcap_{i=0}^{\infty} (\bigcap S_i)$, there is $q \in \mathbb{N}$ and an interval $S_q$ of $q$-th order such that

(ii) $$S_q \subset [z-\omega, z+\omega];$$

without loss of generality we may and will assume that $q$ is chosen such that (see (2))

(iii) $$\eta_q(r_0 r_1 \ldots r_{q-1})^{-1} < \frac{1}{p}.$$

Finally, since $\beta_0 + \beta_1 + \ldots = \infty$, there is $m \in \mathbb{N}$ such that

(iv) $$\beta_q + \beta_{q+1} + \cdots + \beta_{q+m} \geqslant 2^q.$$

Now, there is one interval $T_q$ of order $q$ such that $T_q \subset S_q$, two intervals $T_{q+1}$ such that $T_{q+1} \subset S_q$, generally $2^j$ intervals $T_{q+j}$ such that $T_{q+j} \subset S_q$ for $j = 0, 1, \ldots, m$.

Let us find numbers $\varphi_{q+j}$,

(v) $$\frac{1}{2}(r_0 \ldots r_{q+j-1})^{-1} > \varphi_{q+j} > \zeta_{q+j}(r_0 \ldots r_{q+j-1})^{-1}$$

such that all intervals

$$\tilde{T}_{q+j} = \left(\lambda_{q+j} - \zeta_{q+j}(r_0 \dots r_{q+j-1})^{-1}, \lambda_{q+j} + \varphi_{q+j}\right)$$

with $j = 0, 1, \dots, m$ are pairwise disjoint. Thus their closures are nonoverlapping and $H_{q+j} = \mathrm{Cl}\,\tilde{T}_{q+j} \subset S_{q+j} \subset S_q$. By virtue of (i), (ii) and (v) we find in each $H_{q+j}$ a point $\tau_{q+j} \in H_{q+j} \cap W_p$, set

$$\begin{aligned} (6) \qquad & J = H_{q+j} \times [-\psi(r_0 \dots r_{q+j-1})^{-1}, \eta_{q+j}(r_0 \dots r_{q+j-1})^{-1}], \\ & t = (\tau_{q+j}, 0) \end{aligned}$$

with $\psi$ a (sufficiently small) positive number, and include that pair $(t, J)$ in the system $\Delta_2$.

Now the complement of the union of the intervals $\tilde{T}_{q+j}$ in $[-\gamma, 1+\gamma]$, i.e. the set $[-\gamma, 1+\gamma] \setminus \bigcup \tilde{T}_{q+j}$ consists of a finite number of intervals. Applying to each of them the procedure analogous to the construction of $\Delta_1$ in the preceding part of the proof, we complete the system $\Delta_2$ to a system covering the interval $[-\gamma, 1+\gamma]$ as required, and we evidently have

$$F\Big(I \setminus \bigcup_{\Delta_2} J\Big) = \sum_{i=0}^{m} \sum F(Q^+_{q+i}),$$

where the inner sum is taken over all intervals corresponding to the intervals $H_{q+j}$ constructed above. Taking into account the definition of $f$ and the inequality (iv) we obtain

$$F\Big(I \setminus \bigcup_{\Delta_2} J\Big) = \beta_q 2^{-q} + 2\beta_{q+1} 2^{-(q+1)} + \dots + 2^m \beta_{q+m} 2^{-(q+m)} \geqslant 1.$$

It remains to estimate the value of $\sum_{\Delta_2} d(J) m_{n-1}(\partial J)$. We split the sum into two parts, one corresponding to the tagged intervals of the form (6) and the other corresponding to the intervals filling the gaps between the former. Since the latter are squares we have

$$\begin{aligned} \sum_{\Delta_2} d(J) m_{n-1}(\partial J) &= \sum_1 + \sum_2 \leqslant \sum_{j=0}^{m} 2^j (\eta_{q+j} + \psi)(r_0 \dots r_{q+j-1})^{-1} \\ &\quad \times 4(\eta_{q+j} + \psi)(r_0 \dots r_{q+j-1})^{-1} + \sum_2 d(J) \cdot 4d(J) \\ &\leqslant \sum_{j=0}^{\infty} 2^{j+2} (\eta_{q+j} + \psi)^2 (r_0 \dots r_{q+j-1})^{-2} + 4 \sum_2 d^2(J). \end{aligned}$$

Specifying the conditions (2) we certainly can make the first sum converge (e.g. by assuming $\eta_k \leqslant r_{k-1}$ and $r_k \geqslant r_{k-1} + 1$). The second sum is obviously bounded by a constant independent of the particular form of the system $\Delta_2$ (similarly as in the first part of the proof when $\Delta_1$ was constructed). Hence

$$\sum_{\Delta_2} d(J) m_{n-1}(\partial J) \leqslant C$$

and choosing $\varepsilon \leqslant \min(1, C^{-1})$ we complete the proof. □

Remark. In [5] we have defined the $\varrho$-integral for $\varrho\colon (0,\infty) \to [0,1)$: A function $f\colon I \to \mathbf{R}$ is $\varrho$-integrable on $I$ if there is a real number $c$ such that for every $\varepsilon > 0$ there is a gauge $\delta$ on $I$ such that (1) holds for every $\delta$-fine partition $\Delta$ of $I$ satisfying

$$\operatorname{reg} J \geqslant \varrho\big(d(J)\big) \quad \text{for} \quad (t, J) \in \Delta.$$

Modifying our method accordingly, namely, starting the proof of Lemma 1 with the inequality $v - u \geqslant \varrho(z - w)(z - w)$ and making a suitable choice of the values of $\eta_k$, we can prove the following result:

*For every $\varrho\colon (0,\infty) \to [0,1)$ such that $\lim_{\sigma \to 0+} \varrho(\sigma) = 0$ and every $n > 1$ there is a function $f\colon [-1,2]^n \to \mathbf{R}$ that is $\varrho$-integrable but not $M_1$-integrable on $I$.*

## Appendix

**Strong Cousin Lemma.** *Let $K \subset \mathbf{R}^n$ be a compact interval, $0 < \alpha < 1$, $\delta\colon K \to (0,\infty)$. Then there exists a $\delta$-fine $\alpha$-regular partition of $K$.*

Proof. Denote $K = [u_1, v_1] \times \ldots \times [u_n, v_n]$, $d_i - v_i - u_i$, and assume $d_1 \leqslant d_i$ for $i = 1, 2, \ldots, n$. We first construct an $\alpha$-regular division of $K$, i.e. a finite family of $\alpha$-regular nonoverlapping intervals whose union is $K$.

For $i = 1, 2, \ldots, n$ find positive integers $k_i$, $l_i$ such that

$$\frac{d_i}{d_1} \leqslant \frac{k_i}{l_i} < \alpha^{-1} \frac{d_i}{d_1} \tag{7}$$

(we will assume $k_1 = l_1 = 1$). Denote $l = l_1 l_2 \ldots l_n$ and cut each interval $[u_i, v_i]$ into $k_i l / l_i$ congruent subintervals. Forming all possible Cartesian products of $n$ such intervals, in which the $i$-th factor is a subinterval of $[u_i, v_i]$, we obtain a division of $K$, and by routine application of (7) we find that it is $\alpha$-regular.

Now, applying the classical Cousin Lemma to each of the intervals of this division, we obtain its $\delta$-fine partition into similar intervals (i.e., with the same regularity) by halving the edges and using the standard compactness argument. The set whose elements are all tagged intervals thus obtained forms the desired $\delta$-fine $\alpha$-regular partition of $K$. □

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