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ON THE EXISTENCE OF SOLUTIONS AND ONE-STEP METHOD
FOR FUNCTIONAL DIFFERENTIAL EQUATIONS
WITH PARAMETERS

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INTRODUCTION

For an integer \( i \geq 0 \), let \( C^i(I, \mathbb{R}^p) \) denote the collection of functions with continuous derivatives up to the order \( i \) on \( I = [a, b] \), \( a < b \), into \( \mathbb{R}^p \) and put \( C(I, \mathbb{R}^p) = C^0(I, \mathbb{R}^p) \). We are concerned with the question of solving boundary value problems for Volterra neutral functional differential equations of the form

\[
\begin{align*}
(1) & \quad x'(t) = f(t, x(-), x'(-), \lambda), \quad t \in I, \\
(2) & \quad x(a) = x_p, \quad L(x(-), \lambda) = \theta \in \mathbb{R}^q,
\end{align*}
\]

where \( f: I \times C^1(I, \mathbb{R}^p) \times C(I, \mathbb{R}^p) \times \mathbb{R}^q \to \mathbb{R}^p, L: C^1(I, \mathbb{R}^p) \times \mathbb{R}^q \to \mathbb{R}^q, x_p \in \mathbb{R}^p \) are given. It is assumed that the mapping \( t \to f(t, x(-), x'(-), \lambda) \) is continuous on \( I \) for any \( x \in C^1(I, \mathbb{R}^p) \) and \( \lambda \in \mathbb{R}^q \). We seek \( x \in C^1(I, \mathbb{R}^p) \) and \( \lambda \in \mathbb{R}^q \) such that (1)–(2) to be satisfied. We mean that the problem (1)–(2) is solved if such \( x \) and \( \lambda \) are found.

Indeed special cases of (1) are following equations:

\[
x'(t) = f(t, x(t), x'(t), \lambda), \quad t \in I,
\]

or

\[
x'(t) = f\left(t, x(\alpha_1(t)), \ldots, x(\alpha_r(t)), x'(\beta_1(t)), \ldots, x'(\beta_s(t)), \lambda\right), \quad t \in I,
\]

where \( \alpha_i \) and \( \beta_j \) are continuous functions such that \( a \leq \alpha_i(t) \leq t, a \leq \beta_j(t) \leq \beta_j t, \)

\( 0 \leq \beta_j \leq 1 \) for \( t \in I \) and \( i = 1, 2, \ldots, r, j = 1, 2, \ldots, s \), or

\[
x'(t) = f\left(t, x(t), x'(t), \int_a^{\alpha(t)} g(t, \tau, x(\tau), x'(\tau)) \, d\tau, \lambda\right), \quad t \in I,
\]
where \( a \leq \alpha(t) \leq t \), or directly
\[
x'(t) = f(t, x(\cdot), \lambda), \quad t \in I,
\]
if the function \( f \) is independent on the derivative \( x' \) (for example see [6, 7, 10]).

By the substitution \( y(t) = x'(t), \ t \in I \), the problem (1)-(2) is equivalent to the following
\[
y(t) = f\left(t, x_p + \int_a^t y(\tau) \, d\tau, y(\cdot), \lambda \right), \ t \in I,
\]
(3)
\[
L\left(x_p + \int_a^t y(\tau) \, d\tau, \lambda \right) = \theta.
\]
(4)

To show the problem (3)-(4) has a solution \( (y, \lambda) \in C(I, \mathbb{R}^p) \times \mathbb{R}^q \) we introduce two sequences \( \{y_n\} \) and \( \{\lambda_n\} \) by formulas
\[
y_0(t) = x_p, \ t \in I,
\]
(5)
\[
y_{n+1}(t) = f\left(t, x_p + \int_a^t y_n(\tau) \, d\tau, y_n(\cdot), \lambda_n \right) = F(t, y_n, \lambda_n), \ n = 0, 1, \ldots,
\]
and
\[
\begin{align*}
\lambda_0 & \text{ is an arbitrary vector in } \mathbb{R}^q, \\
\lambda_{n+1} & = \lambda_n - B^{-1}L\left(x_p + \int_a^t y_{n+1}(\tau) \, d\tau, \lambda_n \right), \ n = 0, 1, \ldots,
\end{align*}
\]
(6)

where a nonsingular square matrix \( B \) of order \( q \) will be defined later. The general sufficient conditions by which the sequences \( \{y_n\}, \{\lambda_n\} \) have the limits \( \bar{y} \) and \( \bar{\lambda} \), respectively, and that \( (\bar{y}, \bar{\lambda}) \) is a solution of (3)-(4), are given in the first part.

To solve the problem (1)-(2) numerically we apply the one-step methods for finding \( y \) combined with the Newton method for finding \( \lambda \). Due to this fact we divide the interval \( I \) into \( N \) subintervals all of the same length \( h = (b - a)/N \). The points \( t_{hi} \) of division are defined by \( t_{hi} = a + ih, \ i = 0, 1, \ldots, N \). Now we can describe our method by
\[
\begin{align*}
y^j_h(t_{hn} + rh) & = y^j_h(t_{hn}) + h \Phi(t_{hn}, y^j_h(\cdot), z^j_h(\cdot), \lambda_{hj}, r, h), \\
z^j_h(t_{hn} + rh) & = \Psi(t_{hn}, y^j_h(\cdot), z^j_h(\cdot), \lambda_{hj}, r, h), \quad r \in (0, 1], \ n = 0, 1, \ldots, N - 1,
\end{align*}
\]
(7)
\[
\lambda_{h,j+1} = \lambda_{hj} - B^{-1}L\left(y^j_h(\cdot), \lambda_{hj} \right), \quad r \in (0, 1], \ n = 0, 1, \ldots, N - 1,
\]
for \( j = 0, 1, \ldots \). Here usually \( y^0_h(a) = x_p + \xi_{1j}(h), \lambda_{h0} = \lambda_0 \in \mathbb{R}^q \) is given and \( z^0_h(a) = \bar{x}_{pj} + \xi_{2j}(h) \) where \( \bar{x}_{pj} \) is a solution of the equation
\[
\bar{x}_{pj} = f(a, x_p, \bar{x}_{pj}, \lambda_{hj}),
\]

194
and
\[ \lim_{h \to 0} \xi_1(h) = \lim_{h \to 0} \xi_2(h) = 0. \]

In the second part of this paper we study the convergence of \((y_h^j, \lambda_h^j)\) to the solution \((\varphi, \lambda)\) of (1)-(2).

This paper is an extension of some results obtained in [6, 7, 8].

**PART 1**

We introduce the following

**Assumption H_1.** Suppose that

1° \( f: I \times C^1(I, \mathbb{R}^p) \times C(I, \mathbb{R}^p) \times \mathbb{R}^q \to \mathbb{R}^p, L: C^1(I, \mathbb{R}^p) \times \mathbb{R}^q \to \mathbb{R}^q \), and for any \( x \in C^1(I, \mathbb{R}^p), \lambda \in \mathbb{R}^q \) the mapping \( t \to f(t, x(\cdot), x'(\cdot), \lambda) \) is continuous on \( I \),

2° there exist a constant \( \beta \in [0, 1] \) and nondecreasing functions \( K_1, K_2, K_3 \in C(I, \mathbb{R}_+), \mathbb{R}_+ = [0, \infty) \) such that

\[
\| f(t, x_1(\cdot), x_2(\cdot), \mu_1) - f(t, \bar{x}_1(\cdot), \bar{x}_2(\cdot), \mu_2) \|
\leq K_1(t) \sup_{[a, t]} \| x_1(s) - \bar{x}_1(s) \| + K_2(t) \sup_{[a, t]} \| x_2(s) - \bar{x}_2(s) \| + K_3(t) \| \mu_1 - \mu_2 \|,
\]

for \( t \in I, x_1, \bar{x}_1 \in C^1(I, \mathbb{R}^p), x_2, \bar{x}_2 \in C(I, \mathbb{R}^p) \) and \( \mu_1, \mu_2 \in \mathbb{R}^q \),

3° there exist a nonsingular square matrix \( B \) of order \( q \) and constants \( m \geq 0, d > 0 \) such that \( md < 1, d > \| B^{-1} \| \) and

\[
\| L \left( x_p + \int_a^t F(t, x, \mu_1) \, dt, \mu_1 \right) - L \left( x_p + \int_a^t F(t, x, \mu_2) \, dt, \mu_2 \right) - B(\mu_1 - \mu_2) \|
\leq m \| \mu_1 - \mu_2 \|,
\]

for \( x \in C(I, \mathbb{R}^p), \mu_1, \mu_2 \in \mathbb{R}^q \), where the matrix norm is consistent with the vector norm,

4° for any \( x_1, x_2 \in C^1(I, \mathbb{R}^p) \) and \( \mu \in \mathbb{R}^q \) we have the inequality

\[
\| L(x_1(\cdot), \mu) - L(x_2(\cdot), \mu) \| \leq L_1 \sup_{s \in I} \| x_1(s) - x_2(s) \|,
\]

where \( L_1 \geq 0 \).

**Assumption H_2.** Suppose that

1° there exists a nondecreasing solution \( w^* \in C(I, \mathbb{R}_+) \) of the inequality

\[
Gw(t) + K_3(t) \, dL_1(1 - md)^{-1} \int_a^t Gw(s) \, ds + v(t) \leq w(t), \quad t \in I,
\]

and

\[
\lim_{h \to 0} \xi_1(h) = \lim_{h \to 0} \xi_2(h) = 0. 
\]
where
\[
Gw(t) = K_1(t) \int_a^t w(s) \, ds + K_2(t)w(\beta t),
\]
\[
v_1(t) = \sup_{s \in [a, t]} \left\| f(s, x_p + \int_a^s y_0(\tau) \, d\tau, y_0(\cdot), \lambda_0) - y_0(s) \right\|
\]
\[
v^* = \sup_{s \in I} \left\| L\left( x_p + \int_a^s F(t, y_0, \lambda_0) \, dt, \lambda_0 \right) \right\|
\]
\[
v(t) = v_1(t) + K_3(t) dv^*(1 - md)^{-1},
\]

2° in the class of functions \( w \in M(I, \mathbb{R}^+) \) satisfying the condition \( 0 \leq w(t) \leq w^*(t), t \in I, \) the function \( w(t) = 0, t \in I \) is the only solution of the equation
\[
(9) \quad Gw(t) + K_3(t) dL_1(1 - md)^{-1} \int_a^b Gw(s) \, ds = w(t), \quad t \in I,
\]

where \( M(I, \mathbb{R}^+) \) denotes the class of measurable and bounded functions defined in \( I \) with a range in \( \mathbb{R}^+ \).

Remark 1. Instead of (8) and (9) we can take (10) and (11), respectively, where
\[
(10) \quad Gw(t) + K(t) \int_a^b w(s) \, ds + V(t) \leq w(t), \quad t \in I,
\]
\[
(11) \quad Gw(t) + K(t) \int_a^b w(s) \, ds = w(t), \quad t \in I,
\]
and
\[
r = 1 - md + dL_1 \int_a^b K_3(s) \, ds,
\]
\[
K(t) = K_3(t) dL_1 / r,
\]
\[
V(t) = v(t) - K(t) \int_a^b v(s) \, ds.
\]

Now we are in a position to establish the existence of the solution of (3)–(4). We have

**Theorem 1.** If Assumptions \( H_1 \) and \( H_2 \) are satisfied then there exists a solution \((\bar{\lambda}, \bar{y}) \in \mathbb{R}^q \times C(I, \mathbb{R}^p)\) of the problem (3)–(4). This solution is the limit of the
sequences \{\lambda_n\}, \{y_n\} and the following estimations

\begin{align}
(12) & \quad \|\lambda - \lambda_n\| \leq u_n, \quad n = 0, 1, \ldots, \\
(13) & \quad \sup_{[a,t]} \|\gamma(s) - y_n(s)\| \leq w_n(t), \quad t \in I, \quad n = 0, 1, \ldots,
\end{align}

hold with

\[
\begin{align*}
u_0 &= u^* = d(1 - md)^{-1} \left[ L_1 \int_a^b Gw^*(s) \, ds + v^* \right], \\
u_{n+1} &= d \left[ m u_n + L_1 \int_a^b Gw_n(s) \, ds \right], \quad n = 0, 1, \ldots,
\end{align*}
\]

and

\[
\begin{align*}
w_0(t) &= w^*(t), \quad t \in I, \\
w_{n+1}(t) &= Gw_n(t) + K_3(t)u_n, \quad t \in I, \quad n = 0, 1, \ldots.
\end{align*}
\]

Moreover, this solution \((\lambda, \bar{y})\) is unique in the class satisfying the conditions

\begin{align}
(14) & \quad \|\lambda - \lambda_0\| \leq u^*, \quad \sup_{[a,t]} \|\gamma(s) - y_0(s)\| \leq w^*(t), \quad t \in I.
\end{align}

\textbf{Proof.} Using the following relations

\[
\begin{align*}
\|y_{n+1}(t) - y_0(t)\| &\leq \|F(t, y_n, \lambda_n) - F(t, y_0, \lambda_0)\| + \|F(t, y_0, \lambda_0) - y_0(t)\| \\
&\leq K_1(t) \int_a^t \|y_n(\tau) - y_0(\tau)\| \, d\tau + K_2(t) \sup_{[a,\beta t]} \|y_n(s) - y_0(s)\| \\
&\quad + K_3(t) \|\lambda_n - \lambda_0\| + v_1(t),
\end{align*}
\]

and

\[
\begin{align*}
\|\lambda_{n+1} - \lambda_0\| &\leq B^{-1} \left[ L \left( x_p + \int_a^t F(t, y_n, \lambda_0) \, dt, \lambda_0 \right) - L \left( x_p + \int_a^t F(t, y_n, \lambda_n) \, dt, \lambda_n \right) \\
&\quad - B(\lambda_0 - \lambda_n) - L \left( x_p + \int_a^t F(t, y_n, \lambda_0) \, dt, \lambda_0 \right) \\
&\quad + L \left( x_p + \int_a^t F(t, y_0, \lambda_0) \, dt, \lambda_0 \right) - L \left( x_p + \int_a^t F(t, y_0, \lambda_0) \, dt, \lambda_0 \right) \right] \\
&\leq d \left\{ m \|\lambda_n - \lambda_0\| + L_1 \int_a^b \left[ K_1(t) \int_a^t \|y_n(\tau) - y_0(\tau)\| \, d\tau \\
&\quad + K_2(t) \sup_{[a,\beta t]} \|y_n(s) - y_0(s)\| \right] \, dt + v^* \right\},
\end{align*}
\]
we can prove
\[ \|\lambda_n - \lambda_0\| \leq u^*, \quad n = 0, 1, \ldots, \]
\[ \sup_{[a,t]} \|y_n(s) - y_0(s)\| \leq w^*(t), \quad t \in I, \quad n = 0, 1, \ldots, \]
by induction. Similarly we obtain
\[ \|\lambda_{n+j} - \lambda_n\| \leq u_n, \quad n = 0, 1, \ldots, \]
\[ \sup_{[a,t]} \|y_{n+j}(s) - y_n(s)\| \leq w_n(t), \quad t \in I, \quad n = 0, 1, \ldots, \]
by induction.

Indeed the sequences \( \{u_n\}, \{w_n\} \) are nondecreasing and bounded on \( I \), so they are convergent. In view of Assumption H2 we have
\[ u_n \to 0, \quad w_n(t) \Rightarrow 0, \quad t \in I, \]
where the sign \( \Rightarrow \) denotes the uniform convergence on \( I \). Hence \( \lambda_n \to \bar{\lambda}, y_n(t) \Rightarrow \bar{y}(t), \)
\[ t \in I, \quad \bar{y} \in C(I, \mathbb{R}^p) \] and \( (\bar{\lambda}, \bar{y}) \) is a solution of (3)–(4). There is no problem to prove that this solution is unique in the class satisfying the relations (14). The estimations (12)–(13) follow from (15)–(16) if \( j \to \infty \). Now the proof of Theorem 1 is completed. \( \Box \)

We give the conditions by which the problem (3)–(4) has at most one solution. They do not guarantee the existence of the solution. We have

**Theorem 2.** If Assumption H1 is satisfied and in the class \( u \in M(I, \mathbb{R}_+) \), the function \( u(t) = 0, t \in I \) is the only solution of the inequality
\[ u(t) \leq dL_1(1 - md)^{-1}K_3(t) \int_a^b Gu(s) \, ds + Gu(t), \quad t \in I, \]
then the problem (3)–(4) has at most one solution.

**Proof.** Assuming that the problem (3)–(4) has two solutions \( (\bar{\lambda}_i, \bar{y}_i) \in \mathbb{R}^q \times C(I, \mathbb{R}^p) \) we are able to get the estimations
\[ \|\bar{y}_1(t) - \bar{y}_2(t)\| = \|F(t, \bar{y}_1, \bar{\lambda}_1) - F(t, \bar{y}_2, \bar{\lambda}_2)\| \]
\[ \leq K_1(t) \int_a^t \|\bar{y}_1(\tau) - \bar{y}_2(\tau)\| \, d\tau + K_2(t) \sup_{[a,b]} \|\bar{y}_1(s) - \bar{y}_2(s)\| + K_3(t)\|\bar{\lambda}_1 - \bar{\lambda}_2\| \]
Now combining these inequalities we have the assertion of our theorem. \hfill \Box

Let for \( t \in I \)
\[
K_1(t) = k_1, \quad K_2(t) = k_2.
\]

Now we can cite the following

**Lemma 1** (see [7]). If
\begin{itemize}
  \item[1°] \( v, K_3 \in C(I, \mathbb{R}_+) \) and are nondecreasing,
  \item[2°] \( 0 \leq \beta \leq 1 \),
  \item[3°] \( 0 \leq k_2 \beta < 1 \),
  \item[4°] \( T_1(t) = \sum_{n=0}^{\infty} k_2^n v(t \beta^n) < \infty \) and \( T_1 \in C(I, \mathbb{R}_+) \),
  \item[5°] \( T_2(t) = \sum_{n=0}^{\infty} k_2^n K_3(t \beta^n) < \infty \) and \( T_2 \in C(I, \mathbb{R}_+) \),
  \item[6°] there exists a unique nondecreasing solution \( \tilde{u} \in C(I, \mathbb{R}_+) \) of the equation
\end{itemize}

\begin{equation}
(18) \quad u(t) = k_1 \sum_{n=0}^{\infty} k_2^n \int_a^{t \beta^n} u(\tau) \, d\tau \\
+ dL_1(1 - md)^{-1} T_2(t) \int_a^b Gu(\tau) \, d\tau + T_1(t), \quad t \in I,
\end{equation}

where \( k_1, m, L_1 \geq 0, \, d > 0 \) and \( md < 1 \), then
\begin{itemize}
  \item[i°] in the class of functions \( u \in M(I, \mathbb{R}_+) \) satisfying the condition \( 0 \leq u(t) \leq \tilde{u}(t) \), \( t \in I \), the function \( \tilde{u} \) is the unique, continuous and nondecreasing solution of the equation (8),
  \item[ii°] in the class of functions \( u \in M(I, \mathbb{R}_+) \) satisfying the condition \( 0 \leq u(t) \leq \tilde{u}(t) \), \( t \in I \), the function \( u(t) = 0, \, t \in I \) is the unique solution of the inequality (17).
Lemma 3 (see [7]). If the assumptions (1)-(4) of Lemma 1 are satisfied, \( \alpha \geq 0 \) and the function \( K_3 \) is such that for some constants \( L_2 \geq 0 \) and \( \phi > 0 \) satisfying the condition

\[
\phi > (1 - k_2 \beta)^{-1} \{ k_1 + L_2 dL_1 (1 - md)^{-1} [k_1 (b - a) + k_2] \},
\]

the inequality

\[
K_3(t) \leq L_2 \left( \exp(\phi t) - \exp(\phi a) \right) \left( \exp(\phi b) - \exp(\phi a) \right)^{-1}, \quad t \in I,
\]

holds then there exists a unique nondecreasing solution \( \tilde{u} \in C(I, \mathbb{R}_+) \) of (18).

**PART 2**

Now we are concerned with the numerical solution of the problem (1)-(2). As it was mentioned earlier we apply the method defined by (7). At first we introduce the following

**Assumption H_3.** Suppose that

1° \( \Phi, \Psi : I \times C(I, \mathbb{R}^p) \times \tilde{C}(I, \mathbb{R}^p) \times \mathbb{R}^q \times [0, 1] \times H \to \mathbb{R}^p, L : C(I, \mathbb{R}^p) \times \mathbb{R}^q \to \mathbb{R}^q, H = [0, h_0], h_0 > 0 \) and \( \tilde{C}(I, \mathbb{R}^p) \) denotes the space of piecewise continuous functions from \( I \) into \( \mathbb{R}^p, \Phi(t, y(\cdot), z(\cdot), \lambda, \cdot, h) \) and \( \Psi(t, y(\cdot), z(\cdot), \lambda, \cdot, h) \) are continuous for fixed \( t, y, z, \lambda, h, \) and \( \Phi(t, y(\cdot), z(\cdot), \lambda, 0, h) = 0, \)

2° there exist constants \( M_1, M_2, M_3 \geq 0 \) and a function \( \delta : I \times [0, 1] \times H \to \mathbb{R}_+ \) such that the conditions

\[
\left\| \Phi(t, y_1(\cdot), z_1(\cdot), \mu_1, r, h) - \Phi(t, y_2(\cdot), z_2(\cdot), \mu_2, r, h) \right\|
\leq M_1 \sup_{[a, t]} \| y_1(s) - y_2(s) \| + M_2 \sup_{[a, t]} \| z_1(s) - z_2(s) \| + M_3 \| \mu_1 - \mu_2 \| + \delta_1(t, r, h),
\]

\[
\lim_{N \to \infty} h \sum_{i=0}^{N-1} \sup_{r \in [0, 1]} \delta_1(t_{hi}, r, h) = 0,
\]

hold for \( t \in I, h \in H, y_1, y_2 \in C(I, \mathbb{R}^p), z_1, z_2 \in \tilde{C}(I, \mathbb{R}^p), \mu_1, \mu_2 \in \mathbb{R}^q, \)

3° there exist constants \( D_1, D_3 \geq 0, 0 \leq D_2 < 1 \) and a function \( \delta_2 : I \times [0, 1] \times H \to \mathbb{R}_+ \) such that for \( t \in I, h \in H, r \in (0, 1], y_1, y_2 \in C(I, \mathbb{R}^p), z_1, z_2 \in \tilde{C}(I, \mathbb{R}^p), \mu_1, \mu_2 \in \mathbb{R}^q \) we have

\[
\left\| \Psi(t, y_1(\cdot), z_1(\cdot), \mu_1, r, h) - \Psi(t, y_2(\cdot), z_2(\cdot), \mu_2, r, h) \right\|
\leq D_1 \sup_{[a, t+h]} \| y_1(s) - y_2(s) \| + D_2 \sup_{[a, t+h]} \| z_1(s) - z_2(s) \| + D_3 \| \mu_1 - \mu_2 \| + \delta_2(t, r, h),
\]

\[
\lim_{N \to \infty} \sum_{i=0}^{N-1} \sup_{r \in [0, 1]} \delta_2(t_{hi}, r, h) = 0,
\]

200
4° there exist a nonsingular matrix $B_{q \times q}$ and constants $d > 0$, $m_1 \geq 0$ such that $d \geq \|B^{-1}\|$, $m_1 d < 1$ and

$$(19) \quad \|L(y_1(\cdot), \mu_1) - L(y_1(\cdot), \mu_2) - B(\mu_1 - \mu_2)\| \leq m_1\|\mu_1 - \mu_2\|,$$

for $y_1 \in C(I, \mathbb{R}^p)$ and $\mu_1, \mu_2 \in \mathbb{R}^q$, where the matrix norm is consistent with the vector norm.

5° for $y_1, y_2 \in C(I, \mathbb{R}^p)$ and $\mu \in \mathbb{R}^q$ we have

$$\|L(y_1(\cdot), \mu) - L(y_2(\cdot), \mu)\| \leq m_2 \sup_{s \in I} \|y_1(s) - y_2(s)\|,$$

where $m_2 \geq 0$.

We introduce the standard definitions of convergence and consistency.

**Definition 1.** The method (7) is said to be convergent to the solution $(\varphi, \lambda)$ of (1)--(2) if

$$\lim_{h \to 0} \sup_{j \to \infty} \|y_h(t) - \varphi(t)\| = 0, \quad \lim_{h \to 0} \|\lambda_j - \lambda\| = 0.$$

**Definition 2.** The method (7) is consistent with (1)--(2) on $(\varphi, \lambda)$ if for $(t, r, h) \in I \times [0,1] \times H$ the following conditions are satisfied.

$$\|\varphi(t + rh) - \varphi(t) - h\Phi(t, \varphi(\cdot), \varphi'(\cdot), \lambda, r, h)\| \leq \varepsilon_1(t, r, h),$$

$$\|\varphi'(t + rh) - \Psi(t, \varphi(\cdot), \varphi'(\cdot), \lambda, r, h)\| \leq \varepsilon_2(t, r, h),$$

$$\lim_{N \to \infty} \sum_{i=0}^{N-1} \sup_{r \in [0,1]} \varepsilon_1(t_{hi}, r, h) = 0, \quad \varepsilon_1(t, 0, h) \equiv 0,$$

$$\lim_{N \to \infty} \sum_{i=0}^{N-1} \sup_{r \in [0,1]} \varepsilon_2(t_{hi}, r, h) = 0,$$

are satisfied.
Put

\[ \tilde{e}_2(t, r, h) = \varepsilon_2(t, r, h) + \delta_2(t, r, h), \]
\[ \tilde{e}_2(t, h) = \sup_{r \in [0,1]} \tilde{e}_2(t, r, h), \]
\[ \tilde{e}_1(t, r, h) = \varepsilon_1(t, r, h) + h\delta_1(t, r, h), \]
\[ \tilde{e}_1(t, h) = \sup_{r \in [0,1]} \tilde{e}_1(t, r, h), \]

\[
\tilde{M}_i = M_i + M_2 D_i (1 - D_2)^{-1}, \quad i = 1, 3, \\
\eta(h) = M_2 \left[ \| z^i_h(a) - \varphi'(a) \| + \sum_{t=0}^{N-1} \tilde{e}_2(t_h, h) \right] (1 - D_2)^{-1}, \\
\beta(h) = (b - a) \left[ \tilde{M}_3 \| \lambda_{h,j} - \lambda \| + \eta(h) \right] + \sum_{t=0}^{N-1} \tilde{e}_1(t_h, h), \\
W(h) = dm_2 \left[ \| y^j_h(a) - \varphi(a) \| + (b - a) \eta(h) + \sum_{t=0}^{N-1} \tilde{e}_1(t_h, h) \right] \exp \left((b - a)\tilde{M}_1\right).
\]

Now we can formulate the main theorem of this part.

**Theorem 3.** If Assumption H₃ is satisfied and if

1° there exists the solution \((\varphi, \lambda) \in C^1(I, \mathbb{R}^p) \times \mathbb{R}^q\) of (1)-(2),

2° \(A = d[m_1 + m_2 \tilde{M}_3 (b - a) \exp \left((b - a)\tilde{M}_1\right)] < 1,\)

3° \(\lim_{h \to 0} \| y^j_h(a) - x_p \| = \lim_{h \to 0} \| z^j_h(a) - \varphi'(a) \| = 0,\) then the method (7) is convergent to the solution \((\varphi, \lambda)\) of (1)-(2). Furthermore, the following estimations

\[ \| \lambda_{h,j} - \lambda \| \leq u_j(h), \quad j = 0, 1, \ldots, \]
\[ \sup_{t \in I} \| y^j_h(t) - \varphi(t) \| \leq w_j(h), \quad j = 0, 1, \ldots, \]
\[ \sup_{t \in I} \| z^j_h(t) - \varphi'(t) \| \leq v_j(h), \quad j = 0, 1, \ldots, \]

hold, where

\[ u_j(h) = A^j \| \lambda_{h,0} - \lambda \| + W(h) \frac{1 - A^j}{1 - A}, \]
\[ w_j(h) = \left[ \| y^j_h(a) - x_p \| + (b - a)\tilde{M}_3 u_j(h) + (b - a)\eta(h) \right. \]
\[ + \left. \sum_{t=0}^{N-1} \tilde{e}_1(t_h, h) \right] \exp \left((b - a)\tilde{M}_1\right), \]
\[ v_j(h) = (1 - D_2)^{-1} \left[ \| z^j_h(a) - \varphi'(t) \| + D_1 w_j(h) + D_3 u_j(h) + \sum_{t=0}^{N-1} \tilde{e}_2(t_h, h) \right]. \]
for \( j = 0, 1, \ldots \).

**Proof.** Put

\[
e^j_h(t) = \|y^j_h(t) - \varphi(t)\|, \quad E^j_{hn} = \sup_{[a, t_{hn}]} e^j_h(t),
\]

\[
g^j_h(t) = \|z^j_h(t) - \varphi'(t)\|, \quad G^j_{hn} = \sup_{[a, t_{hn}]} g^j_h(t),
\]

\[
z_{hj} = \|\lambda_{hj} - \lambda\|,
\]

for \( n = 0, 1, \ldots, N, \ j = 0, 1, \ldots \). Using the assumptions we get

\[
g^j_h(t_{hn} + rh) = \|\Psi(t_{hn}, y^j_h(\cdot), z^j_h(\cdot), \lambda_{hj}, r, h) - \Psi(t_{hn}, \varphi(\cdot), \varphi'(\cdot), \lambda, r, h)
+ \Psi(t_{hn}, \varphi(\cdot), \varphi'(\cdot), \lambda, r, h) - \varphi'(t_{hn} + rh)\|
\leq D_1 E^j_{hn, n+1} + D_2 G^j_{hn, n+1} + D_3 z_{hj} + \tilde{\varepsilon}_2(t_{hn}, r, h),
\]

\( n = 0, 1, \ldots, N - 1 \)

and hence by induction we see that

\[
G^j_{hn} \leq G^j_{h0} + D_1 E^j_{hn} + D_2 G^j_{hn} + D_3 z_{hj} + \sum_{i=0}^{n-1} \tilde{\varepsilon}_2(t_{h\iota}, h), \quad n = 1, 2, \ldots, N - 1,
\]

or

(23) \( G^j_{hn} \leq (1 - D_2)^{-1} \left[ G^j_{h0} + D_1 E^j_{hn} + D_3 z_{hj} + \sum_{i=0}^{n-1} \tilde{\varepsilon}_2(t_{h\iota}, h) \right], \quad n = 0, 1, \ldots, N. \)

Similarly, for the error \( e^j_h \) we get

\[
e^j_h(t_{hn} + rh) = \|y^j_h(t_{hn}) + h\Phi(t_{hn}, y^j_h(\cdot), z^j_h(\cdot), \lambda_{hj}, r, h) - \varphi(t_{hn})
- h\Phi(t_{hn}, \varphi(\cdot), \varphi'(\cdot), \lambda, r, h) + \varphi(t_{hn})
+ h\Phi(t_{hn}, \varphi(\cdot), \varphi'(\cdot), \lambda, r, h) - \varphi(t_{hn} + rh)\|
\leq e^j_h(t_{hn}) + hM_1 E^j_{hn} + hM_2 G^j_{hn} + hM_3 z_{hj} + \tilde{\varepsilon}_1(t_{hn}, r, h),
\]

\( n = 0, 1, \ldots, N - 1 \)

and hence and (23)

\[
e^j_h(t_{hn} + rh) \leq e^j_h(t_{hn}) + h\tilde{M}_1 E^j_{hn} + h\tilde{M}_3 z_{hj} + h\eta(h) + \tilde{\varepsilon}_1(t_{hn}, r, h), \quad r \in (0, 1), \ n = 0, 1, \ldots, N - 1.
\]
Now by induction we see
\[ E_{hn}^j \leq E_{h0}^j + h\tilde{M}_1 \sum_{i=0}^{n-1} E_{hi}^j + nh[\tilde{M}_3z_{hj} + \eta(h)] + \sum_{i=0}^{n-1} \tilde{\varepsilon}_i(t_{hi}, h), \quad n = 0, 1, \ldots, N \]
or
\[ E_{hn}^j \leq d_n = E_{h0}^j + h\tilde{M}_1 \sum_{i=0}^{n-1} E_{hi}^j + \beta(h), \quad n = 0, 1, \ldots, N. \]

Indeed, we have
\[ d_{n+1} \leq (1 + h\tilde{M}_1)d_n, \quad n = 0, 1, \ldots, N - 1, \]
so
\[ E_{hn}^j \leq [E_{h0}^j + \beta(h)] \exp \left((b - a)\tilde{M}_1 \right), \quad n = 0, 1, \ldots, N. \]

We next note that
\[ z_{h,j+1} = \left\| B^{-1}[L(y_h^j(\cdot), \lambda) - L(y_h^j(\cdot), \lambda_hj) - B(\lambda - \lambda_hj) \right\|
\[ + L(\varphi(\cdot), \lambda) - L(y_h^j(\cdot), \lambda)] \right\|
\[ \leq d[m_1z_{hj} + m_2E_{hN}^j] \leq Az_{hj} + W(h), \quad j = 0, 1, \ldots. \]

From this we obtain the estimation (20) and then (21)-(22). According to our assumptions we see that
\[ \lim_{h \to 0} u_j(h) = \lim_{h \to 0} w_j(h) = \lim_{h \to 0} v_j(h) = 0. \]

It means that our method \((y_h^j, \lambda_hj)\) is convergent to the solution \((\varphi, \lambda)\) of (1)-(2) and the proof of this theorem is completed. \(\Box\)

Remark 2 (see [9]). The condition (19) is satisfied provided that
\[ \|D_\mu L(y(\cdot), \mu) - B\| \leq m_1 \quad \text{for all} \quad y \in C(I, \mathbb{R}^p), \quad v \in \mathbb{R}^q, \]
where
\[ D_\mu L(y(\cdot), \mu) = \left[ \frac{\partial L_i(y(\cdot), \mu)}{\partial \mu_j} \right]. \]

Now if
\[ L(y(\cdot), \mu) = \tilde{M}y(b) + \tilde{N}\mu + \tilde{K}, \]

204
then the condition (25) takes the form

\[(27) \quad \|\vec{\nabla} - B\| \leq m_1,\]

where \(\vec{M}_{q \times p}, \vec{N}_{q \times q}, \vec{K}_{q \times 1}\). And if \(p = q\) we may choose \(B = \vec{M} + \vec{N}\) and (25) leads us to \(\|\vec{M}\| \leq m_1\), and

\[(28) \quad \| (\vec{M} + \vec{N})^{-1} \| m_1 < 1,\]

provided that \(\vec{M} + \vec{N}\) is nonsingular.

Remark 3 (see [9]). Assume that there exist matrices \(Q_{q \times q}, Z_{q \times q}\) such that for all \(y \in C(I, \mathbb{R}^p), \mu \in \mathbb{R}^q\) the matrix

\[P(y(\cdot), \mu) = D_{\mu}L(y(\cdot), \mu) + Q(y(\cdot), \mu),\]

has a representation of the form

\[P(y(\cdot), \mu) = P_0(I + Z(y(\cdot), \mu))\]

with a constant nonsingular matrix \(P_0\). Moreover, we assume that

\[\|P_0Z(y(\cdot), \mu)\| \leq \nu_1, \quad \|Q(y(\cdot), \mu)\| \leq \nu_2, \quad \text{for all } y \in C(I, \mathbb{R}^p), \mu \in \mathbb{R}^q.\]

Now taking \(B = P_0\), the condition (19) is satisfied with \(m_1 = \nu_1 + \nu_2\) and \(\|P_0^{-1}\|(\nu_1 + \nu_2) < 1\).

Moreover if \(p = q\) and the function \(L\) is linear of the form (26) than we may put

\[Q(y(\cdot), \mu) = D_yL(y(\cdot), \mu) = \vec{M}.\]

Choosing \(B = \vec{M} + \vec{N}\) we have

\[P(y(\cdot), \mu) = \vec{M} + \vec{N}, \quad \nu_1 = 0, \quad \|\vec{M}\| \leq \nu_2 = m_1,\]

which lead to the condition (28).

Now we are interested in the construction of the method (7). The increment functions \(\Phi\) and \(\Psi\) can be created in the analogous way as for ordinary differential equations. To adopt these methods we need an interpolation scheme to compute the numerical solution \(y_{i}^{h}\) (and \(z_{i}^{h}\)) on the interval \(I\). It requires storing all previous values \(y_{i}^{h}(t_{hi})\) as well as \(z_{i}^{h}(t_{hi})\) for \(i = 0, 1, \ldots, n\) because they may be needed to compute \(y_{i}^{h}\) and \(z_{i}^{h}\) on the interval \((t_{hn}, t_{h,n+1}]\).
i° The Euler method is defined by

\[
y_h^i(t_{hn} + rh) = y_h^i(t_{hn}) + rhf(t_{hn}, y_h^i(\cdot), z_h^i(\cdot), \lambda_{hj}), \quad r \in [0, 1],
\]

\[
z_h^i(t_{hn} + rh) = \frac{d}{dr} y_h^i(t_{hn} + rh), \quad r \in (0, 1),
\]

\[
z_h^i(t_{h,n+1}) = f(t_{h,n+1}, y_h^i(\cdot), z_h^i(\cdot), \lambda_{hj}),
\]

for \( n = 0, 1, \ldots, N - 1, j = 0, 1, \ldots. \)

ii° The improved Euler method is

\[
y_h^i(t_{hn} + rh) = y_h^i(t_{hn}) + h((r - \frac{1}{2} r^3) f(t_{hn}, y_h^i(\cdot), z_h^i(\cdot), \lambda_{hj})
\]

\[
+ \frac{1}{2} r^2 f(t_{hn+1}, y_h^i(\cdot), z_h^i(\cdot))),
\]

\[ r \in [0, 1], \quad n = 0, 1, \ldots, N - 1, \quad j = 0, 1, \ldots, \]

where \( z_h^i(t_{hn} + rh), r \in (0, 1) \) is defined as in (i) and

\[
\bar{y}_h^i(s) = \begin{cases} y_h^i(s), & a \leq s \leq t, \\ y_h^i(t) + (s-t)f(t, y_h^i(\cdot), z_h^i(\cdot), \lambda_{hj}), & t < s \leq t + h, \end{cases}
\]

\[
\bar{z}_h^i(s) = \begin{cases} z_h^i(s), & a \leq s \leq t, \\ f(t, y_h^i(\cdot), z_h^i(\cdot), \lambda_{hj}), & t < s \leq t + h. \end{cases}
\]

iii° The one-step method defined by

\[
y_h^i(t_{hn} + rh) = y_h^i(t_{hn}) + h((r - \frac{3}{2} r^2 + \frac{3}{4} r^3) f(t_{hn}, y_h^i(\cdot), z_h^i(\cdot), \lambda_{hj})
\]

\[
+ (2r^2 - \frac{4}{3} r^3) f(t_{hn} + .5h, \bar{y}_h^i(\cdot), \bar{z}_h^i(\cdot), \lambda_{hj})
\]

\[
+ (-\frac{1}{2} r^2 + \frac{2}{3} r^3) f(t_{hn+1}, y_h^i(\cdot), z_h^i(\cdot), \lambda_{hj})), \quad r \in [0, 1],
\]

where

\[
\bar{y}_h^i(s) = \begin{cases} y_h^i(s), & a < s \leq t, \\ y_h^i(t) + (s-t)f(t, y_h^i(\cdot), z_h^i(\cdot), \lambda_{hj}) + (s-t)^2/(2h)(f(t + h, \bar{y}_h^i(\cdot),
\]

\[
\bar{z}_h^i(s), & a \leq s \leq t, \\ f(t, y_h^i(\cdot), z_h^i(\cdot), \lambda_{hj}) + ((s-t)/h)(f(t + h, \bar{y}_h^i(\cdot), \bar{z}_h^i(\cdot), \lambda_{hj})
\]

\[
- f(t, y_h^i(\cdot), z_h^i(\cdot), \lambda_{hj})), & t < s \leq t + h. \end{cases}
\]
where $y_h^j$ and $z_h^j$ are given in (ii) while $z_h^j(t_{hn} + rh), r \in (0, 1)$ is determined in (i). Such methods were described in [1, 4, 12].

Usually in the above mentioned methods $y_h^j(a) = x_p$, whereas $z_h^j(a)$ is determined from the equation

$$z_h^j(a) = f(a, x_p, z_h^j(a), \lambda_{hj}), \quad j = 0, 1, \ldots$$

Indeed $\lambda_{h0} = \lambda_0 \in \mathbb{R}^q$ is given and

$$\lambda_{h,j+1} = \lambda_{hj} - B^{-1}L(y_h^j(\cdot), \lambda_{hj}), \quad j = 0, 1, \ldots \quad .$$

This procedure works as follows: for $y_h^0(a)$ and $\lambda_0$ find $z_h^0(a)$ and determine $y_h^0(t)$, $z_h^1(t)$ for $t \in I$, then find the new value for $\lambda_{h1}$ and $y_h^1(a), z_h^1(a)$ to determine $y_h^1(t)$, $z_h^1(t)$ for $t \in I$ and so on.

We may also consider approximations $y_h^j$ and $z_h^j$ of $\varphi$ and $\varphi'$ only on the grid points $t_{hn}$. Indeed, we have then the sets of discrete values for $y_h^j$ and $z_h^j$. We define them by

$$y_{h}^{j}(t_{hn+1}) = y_{h}^{j}(t_{hn}) + h\Phi_0(t_{hn}, y_{h}^{j}(-), z_{h}^{j}(-), \lambda_{hj}, h), \quad n = 0, 1, \ldots, N - 1,$$

where

$$y_{h}^{j}(\cdot) = y_{h}^{j}(t_{hs}) \quad \text{if} \quad t_{hs} \leq \cdot < t_{hs+1},$$

$$z_{h}^{j}(\cdot) = z_{h}^{j}(t_{hs}) = (y_{h}^{j}(t_{hs}) - y_{h}^{j}(t_{hs-1}))/h \quad \text{if} \quad t_{hs} \leq \cdot < t_{hs+1} \text{ and } s \geq 1.$$  

For example, for the problem

$$y'(t) = f(t, y(a_1(t)), \ldots, y(a_r(t)), y'(\beta_1(t)), \ldots, y'(\beta_s(t)), \lambda),$$

we have now

$$y_{h}^{j}(t_{hn+1}) = y_{h}^{j}(t_{hn}) + h\Phi_1(t_{hn}, y_{h}^{j}(t_{hn}, c_1^p(n)), \ldots, y_{h}^{j}(t_{hn}, c_1^p(n)), \ldots, y_{h}^{j}(t_{hn}, c_s^p(n)), \ldots, z_{h}^{j}(t_{hn}, c_s^p(n)), \lambda_{hj}, h), \quad n = 0, 1, \ldots, N - 1,$$

where

$$c_s^p(n) = E\left(\frac{\alpha_i(t_{hn}) - a}{h}\right), \quad E \text{ denotes integer part},$$

$$z_{h}^{j}(t_{hq}) = (y_{h}^{j}(t_{hq}) - y_{h}^{j}(t_{hq}, q-1))/h.$$  

Here $\Phi_0$ and $\Phi_1$ are increment functions. Such schemes were discussed in [2, 8] for special cases of our problems. Indeed, we may also consider more complicated algorithms to approximate $y(a_i(t))$ and $y'(\beta_q(t))$.
References


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