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SYMMETRIC POROSITY OF SYMMETRIC CANTOR SETS

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1. Introduction

If A is a subset of the real line $\mathbb{R}$ and $x \in \mathbb{R}$, then the *porosity of A at x* is defined to be

$$\limsup_{r \to 0+} \frac{\lambda(A,x,r)}{r},$$

where $\lambda(A,x,r)$ is the length of the longest open interval contained in either $(x, x+r) \cap A^c$ or $(x-r, x) \cap A^c$ and A^c denotes the complement of A. A set is said to be *porous at x* if it has positive porosity at x and is called a *porous set* if it is porous at each of its points. The *symmetric porosity of A at x* is defined as

$$\limsup_{r \to 0+} \frac{\gamma(A,x,r)}{r},$$

where $\gamma(A,x,r)$ is the supremum of all positive numbers h such that there is a positive number t with $t+h \leqslant r$ such that both of the intervals $(x-t-h, x-t)$ and $(x+t, x+t+h)$ lie in A^c. A set A is *symmetrically porous* if it has positive symmetric porosity at each of its points. Porous sets and symmetrically porous sets have been contrasted in [2] and [6].

Symmetric Cantor sets are quite useful in real analysis for constructing examples of pathological behavior. Examples of such constructions can be found in [2] and [1]. In [5] and [4] necessary and sufficient conditions were established for a symmetric Cantor set to be porous. There it turned out that such a set is porous if and only if it is σ-porous, that is, a countable union of porous sets. With symmetric porosity the situation is markedly different. In [3] it was observed that a symmetric Cantor set can be σ-symmetrically porous without being symmetrically porous. Further, it was shown in [3] that a symmetric Cantor set is σ-symmetrically porous if and only if it is porous. Thus it seems appropriate now to investigate the nature of symmetrically porous symmetric Cantor sets, and this is the topic of the present work.

2. Preliminary notation

First we define the class of symmetric Cantor sets in $[0,1]$. Let Σ denote the set of all finite sequences of 0's and 1's, and let Σ^* denote the set of all infinite sequences of 0's and 1's. If $\sigma \in \Sigma$ we denote the length of σ by $|\sigma|$ and will write σ in expanded form as $\sigma(1)\sigma(2)\sigma(3)\ldots\sigma(n)$. If $\sigma \in \Sigma^*$, and $n \in \mathbb{N}$, the set of natural numbers, then $\sigma\big|_n$ will denote $\sigma(1)\sigma(2)\sigma(3)\ldots\sigma(n)$. If $0 \leqslant \alpha_n < 1$ for all $n = 0, 1, \ldots$, then $\{\alpha_n\}$ determines a symmetric Cantor set, $\mathscr{C}(\alpha_n)$, in $[0,1]$. If $\alpha_n \neq 0$, we identify the complementary intervals and the noncomplementary intervals to this Cantor set using subscripts from Σ in the usual way, i.e. $I_\emptyset = \left(\frac{1}{2} - \frac{\alpha_0}{2}, \frac{1}{2} + \frac{\alpha_0}{2}\right)$, J_0 and J_1 are the right and left hand components of the complement of $I_\emptyset$ respectively; I_0 and I_1 are the open intervals of length $\frac{1}{2}\alpha_1(1-\alpha_0)$ centered in J_0 and J_1 respectively, and so on. (Here and throughout the remainder of this paper complementation is taken relative to $[0,1]$.) If one of the $\alpha_n = 0$ we proceed as above with the exception that is $|\sigma| = n$, then I_σ is a "marking" of the center point of J_σ (not a interval) and $J_{\sigma 0}$ and $J_{\sigma 1}$ intersect in I_σ. The Cantor set defined by the sequence $\{\alpha_n\}$ is then

$$C(\alpha_n) = \bigcap_{n=1}^{\infty} \bigcup_{|\sigma|=n} J_\sigma.$$

Note that

$$|J_\sigma| = \prod_{n=0}^{|\sigma|-1} \left(\frac{1-\alpha_n}{2}\right) \quad \text{and} \quad |I_\sigma| = \alpha_{|\sigma|}|J_\sigma|,$$

where $|H|$ is used to denote the length of an interval H. In the obvious manner each $\sigma \in \Sigma^*$ determines a point in $\mathscr{C}(\alpha_n)$. We shall denote this point by x_σ.

We adopt the notation $d(x, I)$ for the distance from a point x to an interval I.

A sequence $\{\alpha_n\}$ in $[0,1)$ is called *sparse* if for each subsequence $\{\alpha_{n_k}\}$ having a positive limit inferior, the sequence $\{n_k - n_{k-1}\}$ diverges to ∞. Finally, a sequence $\{\alpha_n\}$ in $[0,1)$ is called *weakly sparse* if for each subsequence $\{\alpha_{n_k}\}$ having a positive limit inferior, the sequence $\{n_k - n_{k-1}\}$ is unbounded.

We shall prove that following theorem and then conclude with examples to show that neither implication can be reversed.

Theorem 1. *Let $\mathscr{C}\{\alpha_n\}$ be the symmetric Cantor set determined by the sequence $\{\alpha_n\}$.*

1. *If* $\limsup \alpha_n > \frac{1}{2}$ *or $\{\alpha_n\}$ is not weakly sparse, then $\mathscr{C}(\alpha_n)$ is symmetrically porous.*
2. *If $\mathscr{C}(\alpha_n)$ is symmetrically porous, then* $\limsup \alpha_n > \frac{1}{2}$ *or $\{\alpha_n\}$ is not sparse.*

This theorem will be established via the following three propositions.

Proposition 1. *If* $\limsup \alpha_n > \frac{1}{2}$, *then $\mathscr{C}(\alpha_n)$ is symmetrically porous.*

Proof. Let $0 < \varepsilon < \limsup \alpha_n - \frac{1}{2}$, and let $\{\alpha_n\}$ be a subsequence of $\{\alpha_n\}$ for which $\alpha_{n_k} \equiv \frac{1}{2} + \varepsilon_k > \frac{1}{2} + \varepsilon$. Fix a k and let σ have length n_k. We shall show that for each $x \in J_\sigma \setminus I_\sigma$ we can find positive numbers t, h such that

$$(x - t - h, x - t) \cup (x + t, x + t + h) \subseteq \left(\mathscr{C}(\alpha_n)\right)^c,$$

and

$$\frac{h}{t+h} > \varepsilon. \tag{1}$$

It will suffice to establish (1) for each $x \in J_{\sigma 0}$. Letting $s = |J_\sigma|$, we have $|I_\sigma| = \alpha_{|\sigma|} s = \alpha_{n_k} s$. Let τ be such that $|\tau| = |\sigma|$, I_τ is to the left of I_σ, and is closer than any other I_ϱ for which $|\varrho| = |\sigma|$. There is a unique I_ν which is centered between I_τ and I_σ. The right endpoint of I_ν is thus the left endpoint of $J_{\sigma 0}$. For notational convenience let $I_\tau = (a, b)$, $I_\nu(c, d)$, and $I_\sigma = (e, f)$. (Note that $c = d$ in the event that I_ν is simply a "mark.") We have $b - a = f - e = (\frac{1}{2} + \varepsilon_k)s$ and $c - b = e - d = (\frac{1}{4} - \frac{1}{2}\varepsilon_k)s$.

Now let x be any point in $J_{\sigma 0}$. How we proceed to find t and h will depend on the size of $d - c$.

First, if $d - c < \varepsilon_k s$, we choose

$$t = x - b, \text{ and } \quad h = f + b - 2x.$$

Then

$$(x + t, x + t + h) = (2x - b, f) \subseteq (e, f),$$
$$(x - t - h, x - t) = (2x - f, b) \subseteq (a, b),$$

and

$$\begin{aligned}\frac{h}{t+h} &= \frac{f-2x+b}{f-x} \geqslant \frac{f-2e+b}{f-e} = 1-\frac{e-b}{f-e}\\ &= 1-\frac{2\left(\frac{1}{4}-\frac{\varepsilon_k}{2}\right)s+d-c}{\left(\frac{1}{2}+\varepsilon_k\right)s} > 1-\frac{2\left(\frac{1}{4}-\frac{\varepsilon_k}{2}\right)s+\varepsilon_k s}{\left(\frac{1}{2}+\varepsilon_k\right)s}\\ &= \frac{2\varepsilon_k}{1+2\varepsilon_k} > \varepsilon_k > \varepsilon.\end{aligned}$$

Next, suppose that $d-c \geqslant \varepsilon_k s$ and that x lies the right half of $J_{\sigma 0}$. In this case we choose

$$t = x-d, \text{ and } \quad h = \min\{2x-c, f\} - t - x.$$

Then

$$\begin{aligned}&(x+t, x+t+h) = (2x-d, \min\{2x-c, f\}) \subseteq (e, f) \quad \text{ and}\\ &(x-t-h, x-t) = (2x-\min\{2x-c, f\}, d) \subseteq (c, d).\end{aligned}$$

Furthermore, if $\min\{2x-c, f\} = f$, then

$$\begin{aligned}\frac{h}{t+h} &= \frac{f-2x+d}{f-x} = 1-\frac{x-d}{f-x} \geqslant 1-\frac{e-d}{f-e}\\ &= 1-\frac{\left(\frac{1}{4}-\frac{\varepsilon_k}{2}\right)s}{\left(\frac{1}{2}+\varepsilon_k\right)s} = \frac{1+6\varepsilon_k}{2+4\varepsilon_k} > \frac{1+6\varepsilon_k}{4} > \varepsilon_k > \varepsilon;\end{aligned}$$

and if $\min\{2x-c, f\} = 2x-c$, then

$$\begin{aligned}\frac{h}{t+h} &= \frac{d-c}{x-c} \geqslant \frac{d-c}{f-x} \geqslant \frac{d-c}{(f-e)+\frac{1}{2}(e-d)}\\ &> \frac{\varepsilon_k s}{\left(\frac{1}{2}+\varepsilon_k\right)s+\frac{1}{2}\left(\frac{1}{4}-\frac{\varepsilon_k}{2}\right)s} = \frac{8\varepsilon_k}{5-2\varepsilon_k} > \frac{8\varepsilon_k}{5} > \varepsilon_k > \varepsilon.\end{aligned}$$

We are left with the situation where $d-c \geqslant \varepsilon_k s$ and x lies in the left half of $J_{\sigma 0}$. We find it necessary to break this case into two subcases. First, if $\left(\frac{1}{4}-\frac{\varepsilon_k}{4}\right)s > d-c \geqslant \varepsilon_k s$, then we set

$$t = x-b, \text{ and } \quad h = f-2x+b.$$

We then have

$$\begin{aligned}&(x+t, x+t+h) = (2x-b, f) \subseteq (e, f),\\ &(x-t-h, x-t) = (2x-f, b) \subseteq (c, d),\end{aligned}$$

and

$$\frac{h}{t+h} = \frac{f-2x+b}{f-x} = 1 - \frac{x-b}{f-x} = 1 - \frac{(x-d)+(d-c)+(c-b)}{f-x}$$

$$> 1 - \frac{\frac{1}{2}\left(\frac{1}{4} - \frac{\varepsilon_k}{2}\right)s + \left(\frac{1}{4} - \frac{\varepsilon_k}{4}\right)s + \left(\frac{1}{4} - \frac{\varepsilon_k}{2}\right)s}{\left(\frac{1}{2} + \varepsilon_k\right)s + \frac{1}{2}\left(\frac{1}{4} - \frac{\varepsilon_k}{2}\right)s} = \frac{14\varepsilon_k}{5+6\varepsilon_k}$$

$$> \frac{14\varepsilon_k}{5+\frac{6}{5}} \quad (\text{since } \varepsilon_k < \frac{1}{5} \text{ in this case})$$

$$> \varepsilon_k > \varepsilon.$$

The final situation is where $d-c \geqslant \max\left\{\varepsilon_k s, \left(\frac{1}{4} - \frac{\varepsilon_k}{4}\right)s\right\}$ and x lies in the left half of $J_{\sigma 0}$. Here we set

$$t = e - x, \text{ and } \quad h = \min\{2x = c, f\} - t - x.$$

Then

$$(x+t, x+t+h) = (e, \min\{2x-c, f\}) \subseteq (e,f) \quad \text{and}$$
$$(x-t-h, x-t) = (2x - \min\{2x-c, f\}, 2x-e) \subseteq (c,d).$$

Furthermore, if $\min\{2x-c, f\} = f$, then

$$\frac{h}{t+h} = \frac{f-c}{f-x} \geqslant \frac{f-e}{f-d} > \frac{1}{2} + \varepsilon_k > \varepsilon_k > \varepsilon;$$

whereas, if $\min\{2x-c, f\} = 2x-c$, then

$$\frac{h}{t+h} = \frac{2x-c-e}{x-c} = 1 - \frac{e-x}{x-c} \geqslant 1 - \frac{e-d}{d-c}$$

$$\geqslant 1 - \frac{\left(\frac{1}{4} - \frac{\varepsilon_k}{2}\right)s}{\left(\frac{1}{4} - \frac{\varepsilon_k}{4}\right)s} = \frac{\varepsilon_k}{1-\varepsilon_k} > \varepsilon_k > \varepsilon.$$

Thus we have established (1) and, consequently, the proposition. □

Proposition 2. *If $\{\alpha_n\}$ is not weakly sparse, then $\mathscr{C}(\alpha_n)$ is symmetrically porous.*

P r o o f. Let $\{\alpha_n\}$ be a sequence which is not weakly sparse. If $\limsup \alpha_n = 1$, then it follows from Theorem 5 in [2] that $\mathscr{C}(\alpha_n)$ is strongly symmetrically porous. Hence, we assume there is an $0 < s_0 < 1$ such that $\alpha_n < s_0$ for every n. As $\{\alpha_n\}$ is not weakly sparse there exists a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$, an α with $\alpha_{n_k} > \alpha > 0$ for every k, and an M with $n_{k+1} - n_k < M$ for every k. For each k let $w_k = \prod_{n=0}^{n_k-1} \frac{1-\alpha_n}{2}$, and $\varepsilon_k = \alpha_{n_k} w_k$. Define $k^* \equiv k^*(k) = \min\{k'\colon \varepsilon_k > 3w_{k'}\}$. Note that if σ' has length $n_k + l$, then $|J_{\sigma'}| \leqslant \frac{w_k}{2^l}$. If we let $l_0 = \min\{l\colon 2^{-l} < \frac{1}{3}\alpha\}$, then it readily follows that the number $s_1 \equiv \sup\{n_{k^*} - n_k\colon k = 1, 2, \ldots\}$ is finite; indeed, $s_1 \leqslant l_0 + M$. □

We now wish to establish the following:

Claim 1. *If k is fixed and I is any interval in $[0,1]$ of length ε_k, then I contains a subinterval I^* in $(\mathscr{C}(\alpha_n))^c$ such that*

$$\frac{|I^*|}{|I|} \geqslant \alpha\Big(\frac{1-s_0}{2}\Big)^{s_1}.$$

Proof of Claim 1. Let I be as in the hypothesis. If I intersects two complementary intervals I_σ and I_τ $(\sigma \neq \tau)$ with $|\sigma|$, $|\tau| < n_{k^*}$, then I contains a "J-interval," J_ν of the n_{k^*} stage; that is, $|\nu| = n_{k^*}$. Let $I^* = I_\nu$. Then

$$\frac{|I^*|}{|I|} = \frac{\varepsilon_{k^*}}{\varepsilon_k} = \frac{\alpha_{n_{k^*}}}{\alpha_{n_k}} \prod_{i=n_k}^{n_{k^*}-1} \Big(\frac{1-\alpha_i}{2}\Big) \geqslant \alpha\Big(\frac{1-s_0}{2}\Big)^{s_1}.$$

On the other hand if I does not intersect two such complementary intervals, then as $\varepsilon_k > 3w_{k^*}$, which is three times the length of any "J-interval" of the n_{k^*} stage, I must contain an interval lying entirely in $(\mathscr{C}(\alpha_n))^c$ at least as long as w_{k^*}. But

$$\frac{w_{k^*}}{\varepsilon_k} \geqslant \frac{\varepsilon_{k^*}}{\varepsilon_k} \geqslant \alpha\Big(\frac{1-s_0}{2}\Big)^{s_1},$$

completing the proof of claim.

Now, consider any $x \in (0,1) \cap \mathscr{C}(\alpha_n)$. Choose any k sufficiently large so that if $|\sigma| = n_k$ and $x \in J_\sigma$, then $2x - I_\sigma \subseteq [0,1]$. For each such k and σ we apply the claim to the interval $I = 2x - I_\sigma$, obtaining the associated subinterval $I^* \subset (\mathscr{C}(\alpha_n))^c$. Then

$$\begin{aligned}\frac{|I^*|}{|I^*| + d(x, I^*)} &\geqslant \frac{|I^*|}{|I| + 2|J_{\sigma 0}|} = \frac{|I^*|}{|I| + \frac{1-\alpha_{n_k}}{\alpha_{n_k}}|I|} \\ &> \frac{|I^*|}{|I| + \frac{1-\alpha}{\alpha}|I|} = \alpha\frac{|I^*|}{|I|} \leqslant \alpha^2\Big(\frac{1-s_0}{2}\Big)^{s_1}.\end{aligned}$$

Thus, $\mathscr{C}(\alpha_n)$ is symmetrically porous at x. Furthermore, it is clear that $\mathscr{C}(\alpha_n)$ has symmetric porosity at least $2\alpha/(1+\alpha)$ at both 0 and 1 and the proposition is proved. □

Prior to proving the next result, we state (sans proof) the following obvious lemma, which will be useful in obtaining upper bounds on the porosity of $\mathscr{C}(\alpha_n)$ at various points.

Lemma 1. *Let I_1 be left of x and I_2 be to the right of x with $|I_1| = |I_2| = L$. Let d denote the difference in distances between x and the two intervals. Then*

$$r(I_1, I_2, x) \equiv \sup_{h>0} \left\{ \frac{|\{t\colon 0 < t < h, x+t \in I_2, x-t \in I_1\}|}{h} \right\} \leqslant \frac{L-d}{L}.$$

Proposition 3. *If $\{\alpha_n\}$ is a sparse sequence with $\limsup \alpha_n \leqslant \frac{1}{2}$ then $\mathscr{C}(\alpha_n)$ is not symmetrically porous.*

Proof. If $\lim \alpha_n = 0$, then Corollary 1 of [2] guarantees that $\mathscr{C}(\alpha_n)$ is not symmetrically porous. So we shall assume that $\limsup \alpha_n > 0$. Let M be a positive integer such that $10^{-M} < \limsup \alpha_n$. For each integer $m \geqslant M$, let $n(m,k)$ denote the position (subscript) of the k^{th} term of $\{\alpha_n\}$ which exceeds 10^{-m}. Then $\{n(m,k)\colon k = 1, \dots\}$ is a subsequence of $\{n(m+1,k)\colon k = 1, \dots\}$ for each $m \geqslant M$. As $\{\alpha_n\}$ is sparse, for each $m \geqslant M$ there is a $k^*(m)$ such that if $k \geqslant k^*(m)$ then $n(m,k) - n(m,k-1) > 10^{m+1}$. We also assume $\{k^*(m)\}$ is strictly increasing and that there is a $k'(m) > k^*(m)$ with $n\big(m+1, k^*(m+1)\big) = n\big(m, k'(m)\big)$, whenever $m \geqslant M$. If n is such that $\alpha_n > 0$ there is a smallest integer $M(n)$ for which $10^{-M(n)} < \alpha_n$. If $\alpha_n = 0$ we let $M(n) = +\infty$. Let $m(k)$ denote that value of m for which $k^*(m) \leqslant k < k^*(m+1)$ unless $k < k^*(M)$ in which case we let $m(k) = 0$. For $m \geqslant M$ define

$$S_m = \{n(m,k)\colon k^*(m) \leqslant k < k'(m)\}.$$

Note that $\min(S_m) - \max(S_{m-1}) > 10^m$ for every $m \geqslant M$. Define $T_m = S_m - m$ and note that $\min(T_m) - \max(S_{m-1}) > 10^m - m$ for every $m \geqslant M$. Finally, let

$$S = \bigcup_{m=M}^{\infty} S_m \quad \text{and} \quad T = \bigcup_{m=M}^{\infty} T_m.$$

We use these sets to define a sequence $\sigma \in \Sigma^*$ for which x_σ is not a point of symmetric porosity of $\mathscr{C}(\alpha_n)$. Specifically, define

$$\sigma(n) = \begin{cases} 1 & \text{if } n-1 \in S \cup T, \\ 0 & \text{otherwise.} \end{cases}$$

□

We begin by establishing some properties of S.

Claim 1. *The following are equivalent:*
1. $n \in S$.

2. *There is an* $m_0 \geqslant M$ *such that* $n(m_0, k^*(m_0)) \leqslant n$.
3. $n \geqslant n(M(n), k^*(M(n)))$.
4. *There is an* $m_0 \geqslant M(n)$ *such that* $n(m_0, k^*(m_0)) \leqslant n < n(m_0, k'(m_0))$.

Proof of Claim 1. Let $n \in S$. Then there exists a unique m_0 such that $n \in S_{m_0}$. Hence there is a k_0 with $k^*(m_0) \leqslant k_0 < k'(m_0)$ for which $n = n(m_0, k_0)$. Then $m(k_0) = m_0$ and so $n = n(m(k_0), k_0)$. As $\alpha_n \geqslant 10^{-m(k_0)}$ and $M(n)$ is minimal, $m_0 = m(k_0) \geqslant M(n)$. Hence 2. and 4. follow. Further, as $m_0 \geqslant M(n)$ and $n(M(n), k^*(M(n))) = \min S_{M(n)}$, follows that $n \geqslant n(M(n), k^*(M(n)))$.

It remains to prove that 4. implies 1. If there exists $m_0 \geqslant M(n)$ with $n(m_0, k^*(m_0)) \leqslant n < n(m_0, k'(m_0))$, then as $m_0 \geqslant M(n)$, $\alpha_n > 10^{-m_0}$. Since $n \geqslant n(m_0, k^*(m_0))$, there exists a $k_0 \geqslant k^*(m_0)$ with $n(m_0, k_0) = n$. As $n < n(m_0, k'(m_0))$ it also follows that $k_0 < k'(m_0)$. Hence, $n = n(m_0, k_0) \in S_{m_0}$, completing the proof of Claim 1.

Claim 2. *If* $\{n_i\}$ *is an increasing sequence in* $\mathbb{N} \setminus S$, *then* $\{\alpha_{n_i}\} \to 0$.

Proof of Claim 2. Fix m_0 and suppose $n_i \notin S$ but $\alpha_{n_i} > 10^{-m_0}$ for every i. As $\alpha_{n_i} > 10^{-m_0}$, there is a k such that $n_i = n(m_0, k)$. But $M(n_i) \leqslant m_0$, and if $k \geqslant k^*(m_0)$, then by part 2 of Claim 1 $n_i \in S$, which is false. Consequently, $k < k^*(m_0)$. This can occur for at most $k^*(m_0) - 1$ values of i. This completes the proof of Claim 2.

Some properties of k^* and m we will make use of are:

- $k^*(m(k_0)) \leqslant k_0 < k^*(m(k_0) + 1)$
- $m(k^*(m_0)) = m_0$
- If there exist m, k such that $n(m, k) < n < n(m, k+1)$ then $\alpha_n \leqslant 10^{-m}$.

We now show that $\mathscr{C}(\alpha_n)$ is not symmetrically porous at x_σ. Assume to the contrary that $\mathscr{C}(\alpha_n)$ is symmetrically porous at x_σ. Then there are sequences $\{\tau_n\}$ and $\{\tau_n^*\}$ in Σ and an $\varepsilon > 0$ such that I_{τ_n} is left of x_σ, $I_{\tau_n^*}$ is right of x_σ and

$$r(n) \equiv \frac{\min\{|I_{\tau_n}|, |I_{\tau_n^*}|\}}{\max\{d(x, I_{\tau_n}), d(x, I_{\tau_n^*})\}} \geqslant \varepsilon \quad \text{for every } n.$$

We shall find it convenient later on to assume that this ε is less than $\frac{2}{5}$. Clearly, this causes no loss of generality.

Claim 3. *Both* $|\tau_n|$ *and* $|\tau_n^*| \in S$ *for all but finitely many* n.

Proof of Claim 3. Suppose that $|\tau_n^*| \notin S$ for infinitely many n. As x_σ is left of $I_{\tau_n^*}$, x_σ is either left of or in $J_{\tau_n^*0}$. Now, x_σ could be in $J_{\tau_n^*01}$, but is not in $J_{\tau_n^*011}$ since σ contains no consecutive 1's for sufficiently large n. Hence,

$$r(n) \leqslant \frac{|I_{\tau_n^*}|}{d(x, I_{\tau_n^*})} \leqslant \frac{|I_{\tau_n^*}|}{|J_{\tau_n^*011}|} \leqslant \frac{\alpha_{|\tau_n^*|}|J_{\tau_n^*}|}{8^{-3} \cdot |J_{\tau_n^*}|}.$$

The last inequality takes advantage of the assumption that $\limsup \alpha_n \leqslant \frac{1}{2}$, implying that we may assume that $a_n < \frac{3}{4}$ for all sufficiently large n. This, in turn, implies that if ν and μ are in Σ with $|\nu| = |\mu| + 1$, then $|J_\nu| > \frac{1}{8}|J_\mu|$. Hence, by Claim 2 $r(n) \to 0$.

Now suppose $|\tau_n| \notin S$ for infinitely many n. If $|\tau_n| \notin T$ then $\sigma(|\tau_n| + 1) = 0$ so that $x_\sigma \in J_{\sigma|_{|\tau_n|}0}$. As I_{τ_n} is left of x_σ and $J_{\sigma|_{|\tau_n|}0} \cap J_{\tau_n 1} = \emptyset$, it follows that $d(x_\sigma, I_{\tau_n}) \geqslant |J_{\tau_n 1}| \geqslant \frac{|J_{\tau_n}|}{8}$. Consequently,

$$r(n) \leqslant \frac{|I_{\tau_n}|}{d(x_\sigma, I_{\tau_n})} < \frac{\alpha_{|\tau_n|}|J_{\tau_n}|}{8^{-1}|J_{\tau_n}|} < 8 \cdot \alpha_{|\tau_n|},$$

which by Claim 2 can only happen for finitely many n.

Consequently, we are left to consider the case where $|\tau_n| \notin S$ and $|\tau_n| \in T$ for infinitely many n. In this case $\sigma(|\tau_n| + 1) = 1$. As $|\tau_n| \in T$ there is an m such that $|\tau_n| + m \in S$. Thus, $x_\sigma \in J_{\sigma|_{|\tau_n|}10(m-1)1}$ where $0(m-1)$ denotes a string of $m-1$ 0's. It follows that $d(x_\sigma, I_{\tau_n}) \geqslant d(x_\sigma, I_{\sigma|_{|\tau_n|}}) \geqslant |J_{\tau_n 10(m-1)0}|$, and hence

$$r(n) \leqslant \frac{|I_{\tau_n}|}{d(x_\sigma, I_{\tau_n})} < \frac{\alpha_{|\tau_n|}|J_{\tau_n}|}{8^{-m-1}|J_{\tau_n}|} < 8^{m+1} \cdot \alpha_{|\tau_n|}.$$

But, $|\tau_n| \notin S$ since $|\tau_n| + m \in S$ and consecutive terms of S_m are at least 10^m apart. Consequently, $\alpha_{|\tau_n|} \leqslant 10^{-m}$ and hence

$$r(n) \leqslant 8 \cdot \left(\frac{4}{5}\right)^m.$$

Hence, we may assume both $|\tau_n|$ and $|\tau_n^*|$ are in the S for every n.

Claim 4. *For all but finitely many n, $|\tau_n| = |\tau_n^*|$.*

Proof of Claim 4. As both $|\tau_n|$ and $|\tau_n^*| \in S$ there are m_0, k_0, m_1, k_1 with $|\tau_n| = n(m_0, k_0)$ and $|\tau_n^*| = n(m_1, k_1)$.

Assume $|\tau_n| < |\tau_n^*|$. Then either $m_0 < m_1$, or $m_0 = m_1$ and $k_0 < k_1$. We use $|J_{\tau_n^*}|$ to estimate $r(n)$. As $|\tau_n| \in S$, $\sigma(|\tau_n| + 1) = 1$ and the next value of t for which $\sigma(t) = 1$ is $t = n(m_0, k_0 + 1) - m_0$ which is in T_{m_0}. As $|\tau_n^*| \in S$, $|\tau_n^*|$ must be at least as large as $n(m_0, k_0+1)$ and so $x_\sigma \in J_{\sigma|_{|\tau_n|}10(10^{m_0}-m_0)1}$ and hence lies to the right of $J_{\tau_n 10(10^{m_0}-m_0)0}$. Since $|\tau_n^*| - |J_{\tau_n 10(10^{m_0}-m_0)0}| \geqslant m_0 - 1$,

$$|J_{\tau_n 10(10^{m_0}-m_0)0}| \geqslant 2^{m_0-1}|J_{\tau_n^*}|.$$

Hence,

$$r(n) \leqslant \frac{|I_{\tau_n^*}|}{d(x_\sigma, I_{\tau_n})} < \frac{|J_{\tau_n^*}|}{|J_{\tau_n 10(10^{m_0}-m_0)0}|} \leqslant 2^{-m_0+1},$$

which according to Claim 2 can happen but for finitely many n.

If $|\tau_n| > |\tau_n^*|$, then as $\sigma(|\tau_n^*|+1) = 1$, x_σ must be to the left of the interval $J_{\tau_n^* 0}$. Hence,

$$r(n) \leqslant \frac{|I_{\tau_n}|}{d(x_\sigma, I_{\tau_n^*})} \leqslant \frac{|I_{\tau_n}|}{|J_{\tau_n^* 0}|} < \frac{|J_{\tau_n}|}{8^{-1}|J_{\tau_n^*}|} \leqslant 8 \cdot 2^{-10^{m_0}}$$

since $|\tau_n| - |\tau_n^*| \geqslant 10^{m_0}$. This completes the proof of Claim 4.

Hence, we may assume $|\tau_n| = |\tau_n^*| \equiv \ell_n$ and that $\ell_n \in S_n$ for all n. As $r(I_{\tau_n}, I_{\tau_n^*}, x_\sigma) \geqslant \varepsilon$ there is a positive real number ξ such that $\alpha_{\ell_n} > 10^{-\xi} > \frac{\varepsilon}{10}$ for every n, and as each S_m is finite, $\{m_n\} \to \infty$, where $m_n \equiv m(\ell_n)$. As $\limsup \alpha_n \leqslant \frac{1}{2}$, there is an N_1 such that if $n \geqslant N_1$, then $\alpha_n < 1/(2 - \frac{1}{2}\varepsilon)$; that is, $\frac{2\alpha_n - 1}{\alpha_n} < \frac{\varepsilon}{2}$ for $n \geqslant N_1$. Note further that if $n \geqslant N_1$ and J_σ is a "J-interval" of the n^{th} stage and $J_{\sigma'}$ is a subinterval of J_σ and is a "J-interval" of the $(n+1)^{th}$ stage, then

$$\frac{|J_\sigma|}{|J_{\sigma'}|} = \frac{2}{1-\alpha_n} < \frac{8}{2-\varepsilon} < 5,$$

where the final inequality uses the fact that $\varepsilon < \frac{2}{5}$. Next, there is an N_2 such that if $n \geqslant N_2$, then $10^\xi \cdot 2^{2-m_n} < \varepsilon$. Fix $n \geqslant \max\{N_1, N_2\}$. For notational convenience, we let $\ell = \ell_n$, $m = m_n$ and set $\gamma \equiv \sigma\big|_{\ell - m}$. Then $\sigma(\ell+1) = 1 = \sigma(\ell - m + 1)$ and $\sigma(i) = 0$ for $\ell - m + 1 < i \leqslant \ell$.

For each integer a satisfying $0 \leqslant a < 2^{m_n - 1}$ we let $\overline{a}$ be the base two representation of a proceeded by $m - 2 - \operatorname{int}(\log_2(a))$ zeros, where we let $\log_2 0 \equiv 0$; eq. $\overline{2} = 00\ldots010$. As $\sigma\big|_\ell = \gamma 1\overline{0}$, and $\sigma(\ell+1) = 1$, $x_\sigma \in J_{\gamma 1\overline{0}1}$, so it follows that $I_{\gamma 1\overline{0}}$ and $I_{\gamma 1\overline{1}}$ are the complementary intervals of the ℓ^{th} stage immediately left and right of x_σ respectively. For $0 \leqslant a \leqslant m$ let $I_a \equiv I_{\gamma 1\overline{a}}$ and $I_{-a} \equiv I_{\gamma 0\overline{a^*}}$ where $a^* = 2^{m-1} - a$. As there are at least $10^m - m$ zeros following $\sigma(\ell+1)$ in σ, x_σ is within

$$\Delta \equiv \frac{|J_{\sigma|\ell}|}{2^{10^m - m}}$$

of the right endpoint of I_0. (Again, $I_0 = I_{\gamma 1\overline{0}}$.)

Claim 5. *If $-m \leqslant a, b \leqslant m$ then $r(I_a, I_b, x_\sigma) < \varepsilon$.*

Proof. For $b = 1$ an elementary computation shows that $\{t\colon x_\sigma - t \in I_a$ and $x_\sigma + t \in I_1\} = \emptyset$ (and hence $r(I_a, I_b, x_\sigma) = 0$) unless $a = 0$ or $a = -1$. If $a = -1$ the difference

$$d(x_\sigma, I_{-1}) - d(x_\sigma, I_1) \geqslant |I_0| + \Delta + |I_\gamma| - |I_{\sigma|_{\ell-1}}| \geqslant |I_0| - |I_{\sigma|_{\ell-1}}|.$$

Using Lemma 1 we compute

$$\begin{aligned} r(I_{-1}, I_1, x_\sigma) &\leqslant \frac{|I_0| - (|I_0| - |I_{\sigma|_{\ell-1}}|)}{|I_0|} = \frac{|I_{\sigma|_{\ell-1}}|}{|I_0|} \\ &\leqslant \frac{10^{-m}|J_{\sigma|_{\ell-1}}|}{\alpha_\ell \cdot |J_{\sigma|_\ell}|} \\ &< \frac{5 \cdot 10^{-m}}{\alpha_\ell} < 5 \cdot 10^{\xi - m} < \varepsilon. \end{aligned}$$

If $a = 0$, then

$$\begin{aligned} d(x_\sigma, I_1) - d(x_\sigma, I_0) &\geqslant d(I_1, I_0) - 2\Delta \geqslant 2|J_{\sigma|_{\ell+1}}| - 2\Delta \\ &= |J_{\sigma|_\ell}|(1 - \alpha_\ell) - 2\Delta. \end{aligned}$$

Using Lemma 1 we compute

$$\begin{aligned} r(I_0, I_1, x_\sigma) &\leqslant \frac{|I_0| - |J_{\sigma|_\ell}|(1 - \alpha_\ell) + 2\Delta}{|I_0|} \\ &= \frac{\alpha_\ell - (1 - \alpha_\ell) + 2^{-(10^m - m)}}{\alpha_\ell} \\ &= \frac{2_{\alpha_\ell} - 1}{\alpha_\ell} + \frac{2^{-(10^m - m)}}{\alpha_\ell} \\ &< \frac{\varepsilon}{2} + 10^\xi \cdot 2^{-(10^m - m)} < \varepsilon. \end{aligned}$$

For general values of $1 < b < m$, as in the $b = 1$ case, it is again easy to see that $r(I_a, I_b, x_\sigma) = 0$ unless $a = -b$ or $a = -b + 1$. If $a = -b$,

$$\begin{aligned} d(x_\sigma, I_{-b}) - d(x_\sigma, I_b) &\geqslant |I_0| + \Delta + |I_\gamma| - |I_{\sigma|_{\ell-b}}| \\ &\geqslant |I_0| - |I_{\sigma|_{\ell-b}}|. \end{aligned}$$

Then, as in the $a = -1$, $b = 1$ case we have

$$\begin{aligned} r(I_{-b}, I_b, x_\sigma) &\leqslant \frac{|I_b| - (|I_0| - |I_{\sigma|_{\ell-b}}|)}{|I_b|} = \frac{|I_{\sigma|_{\ell-b}}|}{|I_0|} \\ &< \frac{10^{-m}|J_{\sigma|_{\ell-b}}|}{\alpha_\ell \cdot |J_{\sigma|_\ell}|} < \frac{5^b \cdot 10^{-m}}{\alpha_\ell} \\ &< 5^m \cdot 10^{\xi - m} = 2^{-m} \cdot 10^\xi < \varepsilon. \end{aligned}$$

Finally, if $a = -b + 1$, then there is an $i = 1, 2, \ldots, m$ such that

$$\begin{aligned} d(x_\sigma, I_b) - d(x_\sigma, I_{-b+1}) &= \big(d(x_\sigma, I_1) - d(x_\sigma, I_0)\big) + \big(|I_{\sigma|_{\ell-i}}| - |I_\gamma|\big) \\ &> \big(d(x_\sigma, I_1) - d(x_\sigma, I_0)\big) - |I_\gamma|. \end{aligned}$$

Hence,

$$\begin{aligned} r(I_{-b+1}, I_b, x_\sigma) &< \frac{|I_b| - \big(d(x_\sigma, I_1) - d(x_\sigma, I_0)\big) + |I_\gamma|}{|I_b|} \\ &= \frac{|I_0| - \big(d(x_\sigma, I_1) - d(x_\sigma, I_0)\big)}{|I_b|} + \frac{|I_\gamma|}{|I_b|} \end{aligned}$$

and using the inequality from the $a = 0$, $b = 1$ case,

$$\begin{aligned} &< 10^\xi \cdot 2^{-(10^m - m)} + \frac{\varepsilon}{2} + \frac{10^{-m}|J_{\sigma|_{\ell-m}}|}{10^{-\xi}|J_{\sigma|_\ell}|} \\ &< 10^\xi \cdot 2^{-(10^m - m)} + 10^{\xi - m} \cdot 5^m + \frac{\varepsilon}{2} \\ &= 10^\xi (2^{-(10^m - m)} + 2^{-m}) + \frac{\varepsilon}{2} \\ &< 10^\xi \cdot 2^{1-m} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This then completes the proof of Claim 5.

In order to complete the proof of Proposition 3, and hence of Theorem 1, we note that if $\tau_n^* = \gamma 1 \bar{b}$ where $b > m_n$ then the right porosity at x_σ due to $I_{\tau_n^*}$ cannot exceed $\frac{1}{m_n}$; that is,

$$\limsup_{n\to\infty} \frac{|I_{\tau_n^*}|}{d(x_\sigma, I_{\tau_n^*}) + |I_{\tau_n^*}|} \leqslant \frac{1}{m_n},$$

and this expression clearly dominates $r(n)$ and an analogous statement can be made for the τ_n's not previously considered. This and the previous work, contradicts the existence of the sequence $\{\tau_n, \tau_n^*\}$. □

We conclude this paper by providing two examples to show that the converses of the implications 1 and 2 in Theorem 1 are not true.

Example 1. *There is a sequence* $\{\alpha_n\}$ *such that* $\mathscr{C}(\alpha_n)$ *is symmetrically porous, yet* $\limsup \alpha_n = \frac{1}{2}$ *and* $\{\alpha_n\}$ *is weakly sparse.*

Proof of Example 1. Let $S = \left\{\frac{1}{2}k^2 + \frac{9}{2}k \colon k = 0, 1, 2, \ldots\right\}$ and for each $n = 0, 1, 2, \ldots$ set

$$\alpha_n = \begin{cases} \frac{1}{2} & \text{if } n \in S \text{ or } n - 1 \in S \\ 0 & \text{otherwise;} \end{cases}$$

that is,

$$\{\alpha_n\} = \left\{\frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \dots\right\}.$$

Clearly, $\limsup \alpha_n = \frac{1}{2}$ and $\{\alpha_n\}$ is weakly sparse. Further, it is an easy exercise to show that $\mathscr{C}(\alpha_n)$ has symmetric porosity at least $\frac{1}{2}$ at each of its points. □

Example 2. *There is a nonsparse sequence $\{\alpha_n\}$ for which $\mathscr{C}(\alpha_n)$ is not symmetrically porous.*

Proof of Example 2. Let S be as in Example 1. For each $n = 0, 1, 2, \dots$ set

$$\alpha_n = \begin{cases} \frac{1}{10} & \text{if } n \in S \text{ or } n-1 \in S \\ 0 & \text{otherwise;} \end{cases}$$

that is,

$$\{\alpha_n\} = \left\{\frac{1}{10}, \frac{1}{10}, 0, 0, 0, \frac{1}{10}, \frac{1}{10}, 0, 0, 0, 0, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, 0, 0, 0, 0, 0, \frac{1}{10}, \frac{1}{10}, \dots\right\}.$$

Although $\{\alpha_n\}$ is weakly sparse, it is not sparse. Furthermore, $\mathscr{C}(\alpha_n)$ is not symmetrically porous. Specifically, by employing the same strategy as that used by the present authors in Example 1 of [3], it is a straightforward matter to show that $\mathscr{C}(\alpha_n)$ is not symmetrically porous at the point $x_\sigma \in \mathscr{C}(\alpha_n)$ where for each natural number n

$$\sigma(n) = \begin{cases} 1 & \text{if } n \in S \text{ or } n-3 \in S \\ 0 & \text{otherwise.} \end{cases}$$

□

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