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FILIPPOV'S OPERATION AND SOME ATTRIBUTES

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Let $f\colon \mathbb{R} \to \mathbb{R}$ be given. The *Filippov* of f is defined as follows:

$$\mathscr{F}[f](x) = \bigcap_{\varepsilon>0} \bigcap_{Z:\mu(Z)=0} \overline{\operatorname{conv}} f(B_\varepsilon(x) \setminus Z),$$

where μ denotes Lebesgue measure, $\overline{\operatorname{conv}} A$ represents the closure of the convex hull of the set A and $B_\varepsilon(x)$ represents the open ball of radius ε about the point x. The Filippov is used in defining a generalized solution of the ordinary differential equation $x' = f(x)$, particularly in the case in which f is discontinuous. Information concerning the Filippov can be found in many references, including [1] through [10]. (Actually, Filippov's operation and the notion of a Filippov solution were defined for nonautonomous differential equations. However, for the present paper, consideration of the nonautonomous case essentially has only the effect of introducing an unnecessary parameter t into our results.) In this paper, we treat $\mathscr{F}$ as a function, mapping real-valued functions into set-valued functions, and investigate the properties of $\mathscr{F}$. Such results add to our understanding of this operation. We note that there is an alternate definition of the Filippov (for $f \in L^\infty$), equivalent [5] to the previous one, that we will frequently use:

$$\mathscr{F}[f](x) = \{y\colon \lim_{\varepsilon\to 0} \operatorname*{ess\,inf}_{B_\varepsilon(x)} f \leqslant y \leqslant \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f\}.$$

We first consider choosing an appropriate domain for $\mathscr{F}$. Certainly, there are a number of possibilities, but we require that the domain be restricted to f's which are useful for differential equations in the following sense. It can be shown that the functions in L^∞ are precisely the ones which satisfy the classical local existence theorem for Filippov solutions in the case of $x' = f(x)$, namely Theorem 4 in [5]. Hence, we choose L^∞ as the domain for $\mathscr{F}$, using $\|\cdot\|$ to denote the usual norm on L^∞.

We now discuss the selection of a codomain for the function $\mathscr{F}$. We recall two standard definitions (see [1]). We shall say $F\colon \mathbb{R} \to \mathscr{P}(\mathbb{R})$ (= power set of $\mathbb{R}$) is *bounded* if and only if $\sup_{x\in\mathbb{R}}\{\sup\{|y|\colon y \in F(x)\}\} < \infty$. Also F is said to be *upper semicontinuous* if and only if for each $x \in \mathbb{R}$ and for each open set N containing $F(x)$ there exists an open set M containing x such that $F(M) \subseteq N$. We choose for the codomain the set $\mathscr{B} \equiv \{F\colon \mathbb{R} \to \mathscr{P}(\mathbb{R}) \mid F$ is upper semicontinuous, F is closed-interval valued and F is bounded$\}$. $\mathscr{B}$ can be made into a metric space by defining

$$D(F, G) = \sup_{x\in\mathbb{R}}\{h(F(x), G(x))\},$$

where $F, G \in \mathscr{B}$ and h represents the Hausdorff distance between the two sets $F(x)$ and $G(x)$. It follows easily that $\mathscr{F}[L^\infty] \subseteq \mathscr{B}$ using well-known facts such as for $f \in L^\infty$, $\mathscr{F}[f]\colon \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is upper semicontinuous. We note that $D(F, G)$ is alternatively given by

$$D(F, G) = \sup_{x\in\mathbb{R}}\{\max\{|\min F(x) - \min G(x)|, |\max F(x) - \max G(x)|\}\},$$

where, for example, $\max F(x) \equiv \max\{y\colon y \in F(x)\}$.

We now consider questions involving the range of $\mathscr{F}$. In the results which follow, we shall make use of the following definition. Let $F\colon \mathbb{R} \to \mathscr{P}(\mathbb{R})$. Then the *Filippov* of F is defined by

$$\mathscr{F}[F](x) = \bigcap_{\varepsilon>0}\ \bigcap_{Z:\mu(Z)=0} \overline{\operatorname{conv}} \bigcup_{y\in B_\varepsilon(x)\setminus Z} F(y).$$

(Note that the purpose of this is to extend the Filippov so that it can be applied to set-valued functions. We emphasize that in Corollary 1 and Theorems 2, 6, 7 and 9 that the domain of $\mathscr{F}$, as mentioned earlier, is L^∞.) Results from Jarnik's paper [8] allow us to completely characterize the range of $\mathscr{F}$.

Theorem 1. *Let $F\colon \mathbb{R} \to \mathscr{P}(\mathbb{R})$. Then, there exists $f \in L^\infty$ such that $\mathscr{F}[f] = F$ if and only if F satisfies the following conditions:*

1) *F is upper semicontinuous,*
2) *There exists $M > 0$ such that $F(x) \subseteq [-M, M]$ for all $x \in \mathbb{R}$, and*
3) *$\mathscr{F}[F] = F$.*

Proof. Suppose there exists $f \in L^\infty$ such that $\mathscr{F}[f] = F$. As noted above, $\mathscr{F}[L^\infty] \subseteq \mathscr{B}$, thus F satisfies 1) and 2), while F satisfies property 3) by (7) in [8]. Now assuming that F satisfies 1), 2) and 3), the existence of $f \in L^\infty$ such that $\mathscr{F}[f] = F$ follows from the main result in [8] using a simple, but tedious, change of scale, which we omit for brevity. □

Corollary 1. *$\mathscr{F}$ is not onto $\mathscr{B}$.*

P r o o f. Define $F \in \mathscr{B}$ by

$$F(x) = \begin{cases} \{0\} & \text{for } x \neq 0, \\ [0,1] & \text{for } x = 0. \end{cases}$$

Clearly, $\mathscr{F}[F] \equiv \{0\}$, so $\mathscr{F}[F] \neq F$. Thus, by property 3) of Theorem 1 F is not the Filippov of an L^∞ function. □

For our next result concerning the range of $\mathscr{F}$, we shall need the following.

Lemma 1. *For all $F, G \in \mathscr{B}$, we have $D(\mathscr{F}[F], \mathscr{F}[G]) \leqslant D(F,G)$ and hence $\mathscr{F}$, thought of as mapping $\mathscr{B} \to \mathscr{B}$, is continuous.*

P r o o f. It can easily be shown that $\mathscr{F}[\mathscr{B}] \subseteq \mathscr{B}$. Define $a(x) = \lim\limits_{\varepsilon \to 0} \operatorname{ess\,inf}\limits_{B_\varepsilon(x)} F$, $b(x) = \lim\limits_{\varepsilon \to 0} \operatorname{ess\,sup}\limits_{B_\varepsilon(x)} F$, $c(x) = \lim\limits_{\varepsilon \to 0} \operatorname{ess\,inf}_{B_\varepsilon(x)} G$ and $d(x) = \lim\limits_{\varepsilon \to 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} G$. Then, $D(\mathscr{F}[F], \mathscr{F}[G]) = \sup\limits_{\mathbb{R}} \{\max\{|a-c|, |b-d|\}\}$. Similarly, define $e(x) = \min F(x)$, $f(x) = \max F(x)$, $g(x) = \min G(x)$ and $h(x) = \max G(x)$. Then, $D(F,G) = \sup\limits_{\mathbb{R}} \{\max\{|e-g|, |f-h|\}\}$. In the proof of the following claim, we shall use the fact that for $A \subseteq \mathbb{R}$,

$$|\operatorname{ess\,sup}_A f - \operatorname{ess\,sup}_A h| \leqslant \operatorname{ess\,sup}_A |f-h|.$$

This is easily verified, so we omit the proof.
Claim 1: For all $x \in \mathbb{R}$, $|b(x) - d(x)| \leqslant \sup\limits_{\mathbb{R}} |f-h|$.

P r o o f of Claim 1

$$\begin{aligned} |b(x) - d(x)| &= |\lim_{\varepsilon \to 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} F - \lim_{\varepsilon \to 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} G| \\ &= |\lim_{\varepsilon \to 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} f - \lim_{\varepsilon \to 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} h| \\ &\leqslant \lim_{\varepsilon \to 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} |f-h| \\ &\leqslant \operatorname{ess\,sup}_{\mathbb{R}} |f-h| \leqslant \sup_{\mathbb{R}} |f-h|. \end{aligned}$$

Claim 2: For all $x \in \mathbb{R}$, $|a(x) - c(x)| \leqslant \sup\limits_{\mathbb{R}} |e-g|$.

We omit the proof of Claim 2 since it is similar to that of Claim 1. We now have, applying Claims 1 and 2, for all $x \in \mathbb{R}$,

$$\begin{aligned}\max\{|a(x)-c(x)|,|b(x)-d(x)|\} &\leqslant \max\{\sup_{\mathbb{R}}|e-g|,\sup_{\mathbb{R}}|f-h|\}\\ &= \sup_{\mathbb{R}}\max\{|e-g|,|f-h|\}\end{aligned}$$

and thus

$$\sup_{\mathbb{R}}\max\{|a-c|,|b-d|\} \leqslant \sup_{\mathbb{R}}\max\{|e-g|,|f-h|\},$$

i.e., $D(\mathscr{F}[F],\mathscr{F}[G]) \leqslant D(F,G)$. □

Theorem 2. *The range of $\mathscr{F}: L^\infty \to \mathscr{B}$ is closed and unbounded in $(\mathscr{B}, D)$.*

Proof. Let $\{F_n\}_{n=1}^\infty \subseteq \mathscr{F}[L^\infty]$ and $F_n \to F$ in $(\mathscr{B}, D)$. Lemma 1 implies that $\mathscr{F}[F_n] \to \mathscr{F}[F]$ in $(\mathscr{B}, D)$. By Theorem 1, for each $n \in \mathbb{N}$, $\mathscr{F}[F_n] = F_n$, hence $F_n \to \mathscr{F}[F]$. Since limits are unique in $(\mathscr{B}, D)$, we have $F = \mathscr{F}[F]$. Also, since $F \in \mathscr{B}$, we have that i) F is upper semicontinuous and ii) there exists $M > 0$ such that $F(x) \subseteq [-M, M]$ for all $x \in \mathbb{R}$. Thus, applying Theorem 1 to F, we have $F \in \mathscr{F}[L^\infty]$ and hence $\mathscr{F}$ has closed range. Now, for each $n \in \mathbb{N}$, define $f_n \in L^\infty$ by $f_n(x) \equiv n$, and also define $f_\infty \in L^\infty$ by $f_\infty(x) \equiv 0$. It follows that $\sup_{n\in\mathbb{N}} D(\mathscr{F}[f_n], \mathscr{F}[f_\infty]) = \infty$, and hence the range is unbounded. □

We now consider the question of whether or not $\mathscr{F}$ is one-to-one. In [2], the following appeared.

Theorem 3. *Let $f, g \in \mathscr{C} \equiv \{h \in L^\infty$: there exists a set A_h of full measure such that $h|_{A_h}$ is continuous$\}$. If $\mathscr{F}[f] = \mathscr{F}[g]$, then $f = g$ (in L^∞).*

To complete the one-to-one question, we shall need the following lemmas, the first of which is proven, for example, in [8].

Lemma 2. *Let $A \subseteq \mathbb{R}$ be Lebesgue measurable with $\mu A > 0$. Then, there exist Lebesgue measurable sets D and E such that $D \cap E = \emptyset$, $D \cup E = A$ and for all $\varepsilon > 0$, for all $x \in A$ with $\mu(B_\varepsilon(x) \cap A) > 0$, we have both $\mu(D \cap B_\varepsilon(x)) > 0$ and $\mu(E \cap B_\varepsilon(x)) > 0$. ($D$ and E are known as "metrically dense" subsets of A.)*

Lemma 3. *$f \in \mathscr{C}$ if and only if $\mathscr{F}[f]$ is a singleton a.e.*

The proof is given in [2]. We are now able to prove the following, which, in a sense, tells us that the set $\mathscr{C}$ in Theorem 3 is the largest subset of L^∞ on which $\mathscr{F}$ is one-to-one.

Theorem 4. *Let $f \in L^\infty \setminus \mathscr{C}$. Then, there exists some $g \in L^\infty$ such that $\mathscr{F}[f] = \mathscr{F}[g]$ but $f \neq g$ (in L^∞).*

Proof. Define $\overline{f}, \underline{f}\colon \mathbb{R} \to \mathbb{R}$ by

$$\overline{f}(x) = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f \text{ and } \underline{f}(x) = \lim_{\varepsilon\to 0} \operatorname*{ess\,inf}_{B_\varepsilon(x)} f.$$

Since $f \notin \mathscr{C}$, by Lemma 3 there exists a set $D \subseteq \mathbb{R}$ of positive measure such that for all $x \in D$, $\underline{f}(x) < \overline{f}(x)$. It follows from Lusin's Theorem that there exists a set F with the following properties:

1) $F \subseteq D$,
2) F is closed,
3) $\mu(F) > 0$, and
4) $\overline{f}$ and $\underline{f}$ are both continuous relative to F.

Let A and B be disjoint metrically dense subsets of F such that $A \cup B = F$. (Such sets exist by Lemma 2.) Define $k_1, k_2\colon \mathbb{R} \to \mathbf{R}$ by

$$k_1(x) = \begin{cases} f(x) & \text{for } x \notin F,\\ \overline{f}(x) & \text{for } x \in A,\\ \underline{f}(x) & \text{for } x \in B, \end{cases} \qquad k_2(x) = \begin{cases} f(x) & \text{for } x \notin F,\\ \underline{f}(x) & \text{for } x \in A,\\ \overline{f}(x) & \text{for } x \in B. \end{cases}$$

Clearly, k_1 and k_2 disagree on F, a set of positive measure. Without loss of generality, assume f and k_1 disagree on a set of positive measure, and let $g(x) = k_1(x)$ for all $x \in \mathbb{R}$.

Claim: $\mathscr{F}[f] = \mathscr{F}[g]$.

Case 1: $x \notin F$.

Since F is closed, there exists an open interval $I \subseteq \mathbb{R} \setminus F$ containing x. We note that $f(y) = g(y)$ for all $y \in I$, thus $\mathscr{F}[f](x) = \mathscr{F}[g](x)$.

Case 2: $x \in F$ and x is a point of density of F.

We want to show

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} g = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f.$$

Note that $\lim\limits_{\varepsilon\to 0} \operatorname*{ess\,sup}\limits_{B_\varepsilon(x)\cap F} \overline{f} = \overline{f}(x)$ since $\overline{f}$ is continuous on F, and

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F^c} f \leqslant \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f = \overline{f}(x).$$

We then have

$$\begin{aligned}\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} g &= \lim_{\varepsilon\to 0} \max\{\operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F} g, \operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F^c} g\} \\ &\leqslant \lim_{\varepsilon\to 0} \max\{\operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F} \overline{f}, \operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F^c} f\} \\ &= \max\{\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F} \overline{f}, \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F^c} f\} \\ &= \overline{f}(x) = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f.\end{aligned}$$

For the opposite inequality, we note that for all $\varepsilon > 0$ and for all $Z \subseteq \mathbb{R}$ with $\mu(Z) = 0$, we have

$$\sup_{B_\varepsilon(x)\setminus Z} g \geqslant \sup_{(B_\varepsilon(x)\setminus Z)\cap A} g = \sup_{(B_\varepsilon(x)\setminus Z)\cap A} \overline{f} \geqslant \overline{f}(x),$$

since $\overline{f}$ is continuous on A and x is a point of density of F. Thus, for all $\varepsilon > 0$,

$$\operatorname*{ess\,sup}_{B_\varepsilon(x)} g \geqslant \overline{f}(x)$$

and so

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} g \geqslant \overline{f}(x) = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f.$$

The fact that $\lim_{\varepsilon\to 0} \operatorname*{ess\,inf}_{B_\varepsilon(x)} g = \lim_{\varepsilon\to 0} \operatorname*{ess\,inf}_{B_\varepsilon(x)} f$ is handled analogously.

We note that we need not handle Case 3, in which $x \in F$ but x is not a point of density of F since these form a set of measure zero. The result follows since it can be shown that if $\mathscr{F}[f]$ and $\mathscr{F}[g]$ agree on a set of full measure, then $\mathscr{F}[f](x) = \mathscr{F}[g](x)$ for all $x \in \mathbb{R}$. □

We now investigate whether or not $\mathscr{F}$ is continuous. It not only turns out to be continuous, but is "Lipschitz," in the following sense.

Theorem 5. *For all $f, g \in L^\infty$, we have $D(\mathscr{F}[f], \mathscr{F}[g]) \leqslant \|f - g\|$.*

The proof is similar to that of Lemma 1, so for brevity, we omit it. Examples can be given to show that the Lipschitz constant of 1 is sharp, and also that, in general, the inequality cannot be replaced with equality.

We now investigate other topological properties of $\mathscr{F}$.

Theorem 6. *$\mathscr{F}$ is not an open map.*

Proof. Let $U \subseteq L^\infty$ be an open ball containing the zero function in L^∞ (call it f). Define $F \in \mathscr{B}$ by $F(x) = \{0\}$ for all $x \in \mathbb{R}$. Note that $\mathscr{F}[f] = F$ so $F \in \mathscr{F}[U]$. Let $\varepsilon > 0$. Define $G \in \mathscr{B}$ by

$$G(x) = \begin{cases} \{0\} & \text{for } x \neq 0 \\ [0, \frac{\varepsilon}{2}] & \text{for } x = 0. \end{cases}$$

Clearly, G lies in the ε-ball centered at F. G cannot be in $\mathscr{F}[U]$ since Filippovs which agree almost everywhere agree everywhere. Thus, F is not an interior point of $\mathscr{F}[U]$, so $\mathscr{F}[U]$ is not open. □

Theorem 7. *$\mathscr{F}$ is not a closed map. (Here, closed map is intended to mean images of closed sets are closed.)*

Proof. Let A and $\mathbb{R}\backslash A$ be metrically dense in $\mathbb{R}$ and let A_n and B_n, $n = 1, 2, \ldots$, be metrically dense in $[n, n+1] \setminus A$ with $A_n \cup B_n = [n, n+1] \setminus A$ and $A_n \cap B_n = \emptyset$. For each $n \in \mathbb{N}$, define $f_n \in L^\infty$ by

$$f_n(x) = \begin{cases} (1 + \frac{1}{n})\chi_A(x) & \text{for } x \notin [n, n+1] \\ 1 + \frac{1}{n} & \text{for } x \in [n, n+1] \cap A \\ \frac{1}{2} & \text{for } x \in A_n \\ 0 & \text{for } x \in B_n \end{cases}$$

Then, $\mathscr{F}[f_n](x) \equiv [0, 1 + \frac{1}{n}]$. Define $F \in \mathscr{B}$ by $F(x) = [0, 1]$ for all $x \in \mathbb{R}$. Clearly, $\mathscr{F}[f_n] \to F$ in $(\mathscr{B}, D)$, but $\mathscr{F}[f_n] \neq F$ for each $n \in \mathbb{N}$. Let $K \equiv \{f_n\}_{n=1}^\infty \subseteq L^\infty$. If $n \neq m$, $\|f_n - f_m\| \geqslant \frac{1}{2}$. Thus, there can be no Cauchy sequences and hence no convergent sequences in K (except, of course, those which are eventually constant), so K is closed. However, $F \in \overline{\mathscr{F}[K]} \setminus \mathscr{F}[K]$. Therefore, $\mathscr{F}$ is not a closed map. □

Another property often considered concerning functions is monotonicity. The notion of order depends on the particular application. To study monotonicity in this more abstract context, we can define a partial order $\preceq$ on the domain L^∞ by:

$$f \preceq g \text{ iff } f(x) \leqslant g(x) \text{ for almost all } x \in \mathbb{R}$$

where $f, g \in L^\infty$. We next define a partial order on the codomain $\mathscr{B}$, which we also denote by $\preceq$:

$$F \preceq G \text{ iff for all } x \in \mathbb{R}, \text{ we have both}$$
$$\min F(x) \leqslant \min G(x) \text{ and } \max F(x) \leqslant \max G(x)$$

where $F, G \in \mathscr{B}$.

Theorem 8. *Let $f, g \in L^\infty$. If $f \preceq g$, then $\mathscr{F}[f] \preceq \mathscr{F}[g]$.*

The proof is trivial so we omit it. Although the condition $f \preceq g$ is sufficient for $\mathscr{F}[f] \preceq \mathscr{F}[g]$, it is not necessary. Such an example is provided by k_1 and k_2 in the proof of Theorem 4.

Theorem 9. *$\mathscr{F}$ is not strictly monotone.*

Proof. Let A and $\mathbb{R} \setminus A$ be metrically dense in $\mathbb{R}$ and let E and $A \setminus E$ be metrically dense in A. Define $f \in L^\infty$ by

$$f(x) = \begin{cases} 0 & \text{for } x \notin A, \\ \frac{1}{2} & \text{for } x \in E, \\ 1 & \text{for } x \in A \setminus E. \end{cases}$$

Clearly, f is strictly less than $g \equiv \chi_A$. We note that $\mathscr{F}[g] \equiv [0,1]$ by the metric density of A and A^c. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Metric density implies that the sets $(\mathbb{R} \setminus A) \cap B_\varepsilon(x)$ and $(A \setminus E) \cap B_\varepsilon(x)$ each have positive measure. Thus, $\mathscr{F}[f] \equiv [0,1]$. Therefore, $\mathscr{F}[f] = \mathscr{F}[g]$. □

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