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A CHARACTERIZATION OF ARITHMETICAL VARIETIES BY TWO-ELEMENT SUBSETS

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An algebra A is *arithmetical* if the congruence lattice $\operatorname{Con} A$ is distributive and A is congruence permutable, i.e. $\theta \cdot \Phi = \Phi \cdot \theta$ for each two θ, $\Phi \in \operatorname{Con} A$. A variety $\mathcal{V}$ is *arithmetical* if every A of $\mathcal{V}$ has this property. Denote by $F_v(x_1, \ldots, x_n)$ the free algebra of a variety $\mathcal{V}$ with n free generators $x_1, \ldots, x_n$. A. F. Pixley [1] establishes the following Mal'cev type characterization of arithmetical varieties.

Theorem 1 (Pixley). *For a variety $\mathcal{V}$ the following conditions are equivalent:*
(a) *$\mathcal{V}$ is arithmetical;*
(b) *$F_v(x, y, z)$ is arithmetical;*
(c) *there exists a ternary term $p(x, y, z)$ such that*

$$p(x, x, y) = p(y, x, y) = p(y, x, x) = y.$$

A term p satisfying the identities of (c) is called the *Pixley term.* The aim of this short note is to give another characterization of arithmetical varieties based on properties of two-element subsets of algebras of $\mathcal{V}$. This enables us to characterize arithmetical varieties by free algebras with two generators only.

Definition. An n-ary algebraic function $\varphi(x_1, \ldots, x_n)$ over an algebra A is said to be *derived by* $a \in A$ if there exists an $(n+1)$-ary term $t(x_1, \ldots, x_{n+1})$ with $\varphi(x_1, \ldots, x_n) = t(x_1, \ldots, x_n, a)$.

Theorem 2. *For a variety $\mathcal{V}$, the following conditions are equivalent:*
(1) *$\mathcal{V}$ is arithmetical;*
(2) *for every A of $\mathcal{V}$ and each $a, b \in A$, there exist algebraic functions $\vee, \wedge, '$ on A such that $B = (\{a, b\}; \vee, \wedge, ', a, b)$ is a Boolean algebra and $\vee$ is derived by the*

least element a of B;

(3) *for every A of $\mathscr{V}$ and each a, $b \in A$, there exists an algebraic function $\vee$ derived by a and such that $S = (\{a, b\}; \vee)$ is a $\vee$-semilattice with the least element a;*

(4) *for $F_v(x, y)$ there exists a binary term $\vee$ such that $(\{x, y\}; \vee)$ is the $\vee$-semilattice with the least element x.*

Proof. (1) $\Rightarrow$ (2): Let $p(x, y, z)$ be the Pixley term and for $A \in \mathscr{V}$, let a, $b \in A$. Put $c \vee d = p(c, a, d)$, $c \wedge d = p(c, b, d)$, $c' = p(a, c, b)$ for each c, $d \in \{a, b\}$. It is easy to check that $(\{a, b\}; \vee, \wedge, ', a, b)$ is a two-element Boolean algebra where a is the least element. Evidently, $\vee$ is derived by a.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4): By (3), $(\{x, y\}; \vee)$ is the $\vee$-semilattice with the least element x and $\vee$ is a binary algebraic function derived by x, i.e. $z \vee v = p(z, x, v)$ for some ternary term p of $\mathscr{V}$. Since x is also a term of $\mathscr{V}$, $\vee$ is a term of $\mathscr{V}$.

(4) $\Rightarrow$ (1): If $(\{x, y\}; \vee)$ is the semilattice with the least element x and $\vee$ is derived by x, there exists a ternary term p of $\mathscr{V}$ with $z \vee v = p(z, x, v)$. For z, $v \in \{x, y\}$ we have

$$y = y \vee y = p(y, x, y),$$
$$y = y \vee x = p(y, x, x),$$
$$y = x \vee y = p(x, x, y),$$

whence p is the Pixley term. By Theorem 1, $\mathscr{V}$ is arithmetical. □

References

[1] *A. F. Pixley*: Distributivity and permutability of congruence relations in equational classes of algebras. Proc. Amer. Math. Soc. *14* (1963), 105–109.

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