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A CHARACTERIZATION OF ARITHMETICAL VARIETIES
BY TWO-ELEMENT SUBSETS

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An algebra $A$ is arithmetical if the congruence lattice $\text{Con} A$ is distributive and $A$

is congruence permutable, i.e. $\theta \cdot \Phi = \Phi \cdot \theta$ for each two $\theta, \Phi \in \text{Con} A$. A variety $\mathcal{V}$
is arithmetical if every $A$ of $\mathcal{V}$ has this property. Denote by $F_v(x_1, \ldots, x_n)$ the free
algebra of a variety $\mathcal{V}$ with $n$ free generators $x_1, \ldots, x_n$. A. F. Pixley [1] establishes
the following Mal’cev type characterization of arithmetical varieties.

Theorem 1 (Pixley). For a variety $\mathcal{V}$ the following conditions are equivalent:

(a) $\mathcal{V}$ is arithmetical;
(b) $F_v(x, y, z)$ is arithmetical;
(c) there exists a ternary term $p(x, y, z)$ such that

$$p(x, x, y) = p(y, x, y) = p(y, x, x) = y.$$ 

A term $p$ satisfying the identities of (c) is called the Pixley term. The aim of
this short note is to give another characterization of arithmetical varieties based on
properties of two-element subsets of algebras of $\mathcal{V}$. This enables us to characterize
arithmetical varieties by free algebras with two generators only.

Definition. An $n$-ary algebraic function $\varphi(x_1, \ldots, x_n)$ over an algebra $A$ is said
to be derived by $a \in A$ if there exists an $(n + 1)$-ary term $t(x_1, \ldots, x_{n+1})$ with

$\varphi(x_1, \ldots, x_n) = t(x_1, \ldots, x_n, a)$.

Theorem 2. For a variety $\mathcal{V}$, the following conditions are equivalent:

(1) $\mathcal{V}$ is arithmetical;
(2) for every $A$ of $\mathcal{V}$ and each $a, b \in A$, there exist algebraic functions $\vee, \wedge, ' \,$ on
\na such that $B = (\{a, b\}; \vee, \wedge, ' , a, b)$ is a Boolean algebra and $\vee$ is derived by the
least element \( a \) of \( B \);

(3) for every \( A \) of \( \mathcal{V} \) and each \( a, b \in A \), there exists an algebraic function \( \vee \) derived by \( a \) and such that \( S = (\{a,b\}; \vee) \) is a \( \vee \)-semilattice with the least element \( a \);

(4) for \( F_v(x,y) \) there exists a binary term \( \vee \) such that \( (\{x,y\}; \vee) \) is the \( \vee \)-semilattice with the least element \( x \).

Proof. (1) \( \Rightarrow \) (2): Let \( p(x,y,z) \) be the Pixley term and for \( A \in \mathcal{V} \), let \( a, b \in A \). Put \( c \vee d = p(c,a,d), c \wedge d = p(c,b,d), c' = p(a,c,b) \) for each \( c, d \in \{a,b\} \). It is easy to check that \( (\{a,b\}; \vee, \wedge, ' , a, b) \) is a two-element Boolean algebra where \( a \) is the least element. Evidently, \( \vee \) is derived by \( a \).

(2) \( \Rightarrow \) (3) is trivial.

(3) \( \Rightarrow \) (4): By (3), \( (\{x,y\}; \vee) \) is the \( \vee \)-semilattice with the least element \( x \) and \( \vee \) is a binary algebraic function derived by \( x \), i.e. \( z \vee v = p(z,x,v) \) for some ternary term \( p \) of \( \mathcal{V} \). Since \( x \) is also a term of \( \mathcal{V} \), \( \vee \) is a term of \( \mathcal{V} \).

(4) \( \Rightarrow \) (1): If \( (\{x,y\}; \vee) \) is the semilattice with the least element \( x \) and \( \vee \) is derived by \( x \), there exists a ternary term \( p \) of \( \mathcal{V} \) with \( z \vee v = p(z,x,v) \). For \( z, v \in \{x,y\} \) we have

\[
\begin{align*}
y &= y \vee y = p(y,x,y), \\
y &= y \vee x = p(y,x,x), \\
y &= x \vee y = p(x,x,y),
\end{align*}
\]

whence \( p \) is the Pixley term. By Theorem 1, \( \mathcal{V} \) is arithmetical. \( \square \)

References


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