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CONTEXTS AND SUBLATTICES OF CONCEPT LATTICES

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Any context $\mathscr{J}$ can be uniquely assigned a complete concept lattice $L_{\mathscr{J}}$ (see e.g. [3]). In this paper we describe substructures in $\mathscr{J}$ such that their concept lattices are all complete sublattices in $L_{\mathscr{J}}$. As a consequence a characterization of contexts with distributive or modular concept lattices is obtained. Another characterization for distributive lattices is given in [1].

Definition 1. A *context* is a triple $\mathscr{J} = (G, M, I)$ where G and M are sets and $I \subseteq G \times M$. For $B \subseteq M$, $B \neq \emptyset$, we put $B^{\downarrow} = \{g \in G \mid g\ I\ m\ \forall m \in B\}$ and $\emptyset^{\downarrow} = G$. For $A \subseteq G$, $A \neq \emptyset$, we put $A^{\uparrow} = \{m \in M \mid g\ I\ m\ \forall g \in A\}$ and $\emptyset^{\uparrow} = M$. Let further $A^{\uparrow\downarrow} = (A^{\uparrow})^{\downarrow}$, $B^{\downarrow\uparrow} = (B^{\downarrow})^{\uparrow}$.

Remark 1. From Definition 1 we have: $B_1 \subseteq B_2 \Rightarrow B_2^{\downarrow} \subseteq B_1^{\downarrow}$ for B_1, $B_2 \subseteq M$, and $A_1 \subseteq A_2 \Rightarrow A_2^{\uparrow} \subseteq A_1^{\uparrow}$ for A_1, $A_2 \subseteq G$.

$$A \subseteq G, \quad A = A^{\uparrow\downarrow} \Leftrightarrow \exists B \subseteq M, \quad B^{\downarrow} = A,$$
$$\bigcap_{i\in I} B_i^{\downarrow} = \Big(\bigcup_{i\in I} B_i\Big)^{\downarrow}, \quad B_i \subseteq M \quad \forall i \in I.$$

Theorem 1. *If $\mathscr{J} = (G, M, I)$ is a context and $Q = \{A \subseteq G \mid A = A^{\uparrow\downarrow}\}$, then $(Q, \subseteq)$ is a partially ordered set with unit element G. If we put $\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i$, $\Big(\bigcap_{i\in I} A_i^{\uparrow}\Big)^{\downarrow} = \bigvee_{i\in I} A_i$ for $A_i \in Q\ \forall i \in I$, then $L_{\mathscr{J}} = (Q, \wedge, \vee)$ is a complete lattice.*

Remark 2. The lattice from Theorem 1 is called the *concept lattice* of $\mathscr{J}$. Suppose $A_i \in Q\ \forall i \in I$ and let $B_i \subseteq M$, $B_i^{\downarrow} = A_i\ \forall i \in I$. Then $\bigwedge_{i\in I} A_i = \bigcap_{i\in I} B_i^{\downarrow} = \Big(\bigcup_{i\in I} B_i\Big)^{\downarrow}$, $\bigvee_{i\in I} A_i = \Big(\bigcap_{i\in I} B_i^{\downarrow\uparrow}\Big)^{\downarrow}$.

Let $\mathscr{J} = (G, M, I)$ be a context. For any subset $B \subseteq M$ we put $\overline{B} = \{C \subseteq M \mid C^{\downarrow} = B^{\downarrow}\}$ and $D_B = \bigcup_{C \in \overline{B}} C$. Then $D_B^{\downarrow} = \left(\bigcup_{C \in \overline{B}} C\right)^{\downarrow} = \bigcap_{C \in \overline{B}} C^{\downarrow} = B^{\downarrow}$ holds and hence $D_B \in \overline{B}$.

Definition 2. Let $\mathscr{J} = (G, M, I)$ be a context. A non-empty set $\mathscr{M} \subseteq 2^M$ is *admissible* in $\mathscr{J}$ if for any non-empty subset $\mathscr{B} \subseteq \mathscr{M}$

(1) there exists $X \in \mathscr{M}$ such that $\bigcup_{B \in \mathscr{B}} B \in \overline{X}$,

(2) there exists $Y \in \mathscr{M}$ such that $\bigcap_{B \in \mathscr{B}} B^{\downarrow\uparrow} \in \overline{Y}$.

Remark 3. Examples of admissible sets in a context $\mathscr{J}$:

a) $\mathscr{M} = 2^M$,

b) $\mathscr{M} = \{B\}$, where $B \subseteq M$.

c) If $\mathscr{M}$ is an admissible set in $\mathscr{J}$, then $\overline{\mathscr{M}} = \bigcup_{B \in \mathscr{M}} \overline{B}$, $D_{\mathscr{M}} = \{D_B \mid B \in \mathscr{M}\}$ are also admissible sets in $\mathscr{J}$, and $\mathscr{M}$, $D_{\mathscr{M}} \subseteq \overline{\mathscr{M}}$.

Theorem 2. *Let $\mathscr{J} = (G, M, I)$ be a context.*

1. *If $\mathscr{M}$ is an admissible subset in $\mathscr{J}$, then $L_1 = \{B^{\downarrow} \mid B \in \mathscr{M}\}$ is a complete sublattice of $L_{\mathscr{J}}$.*

2. *Let L_1 be a complete sublattice of the lattice $L_{\mathscr{J}}$. Let us consider a subset $\mathscr{M} \subseteq 2^M$ such that $B^{\downarrow} \in L_1$ $\forall B \in \mathscr{M}$ and for any $x \in L_1$ there exists $X \in \mathscr{M}$ such that $X^{\downarrow} = x$. Then $\mathscr{M}$ is an admissible subset in $\mathscr{J}$.*

Proof. 1. Evidently $L_1 \subset L_{\mathscr{J}}$. Let $\mathscr{A} \subseteq L_1$, $\mathscr{A} \neq \emptyset$. Then there exists a (non-empty) subset $\mathscr{B} \subseteq \mathscr{M}$ such that $\mathscr{A} = \{B^{\downarrow} \mid B \in \mathscr{B}\}$. We get $\wedge \mathscr{A} = \bigwedge_{B \in \mathscr{B}} B^{\downarrow} = \left(\bigcup_{B \in \mathscr{B}} B\right)^{\downarrow}$. By (1), there is $X \in \mathscr{M}$ such that $\bigcup_{B \in \mathscr{B}} B \in \overline{X}$, i.e. $\wedge \mathscr{A} \in L_1$. Further, $\bigvee \mathscr{A} = \left(\bigcap_{B \in \mathscr{B}} B^{\downarrow\uparrow}\right)^{\downarrow}$ and by (2) there is $Y \in \mathscr{M}$ with $\bigcap_{B \in \mathscr{B}} B^{\downarrow\uparrow} \in \overline{Y}$, which means $\bigvee \mathscr{A} \in L_1$.

2. Consider $\mathscr{B} \subseteq \mathscr{M}$, $\mathscr{B} \neq \emptyset$. We have $\bigwedge_{B \in \mathscr{B}} B^{\downarrow}\left(\bigcup_{B \in \mathscr{B}} B\right)^{\downarrow} \in L_1$. Hence there exists $X \in \mathscr{M}$ such that $X^{\downarrow} = \left(\bigcup_{B \in \mathscr{B}} B\right)^{\downarrow}$ and thus $\bigcup_{B \in \mathscr{B}} B \in \overline{X}$. Simultaneously we have $\bigvee_{B \in \mathscr{B}} B^{\downarrow} = \left(\bigcap_{B \in \mathscr{B}} B^{\downarrow\uparrow}\right)^{\downarrow} \in L_1$. Similarly as in the previous case there exists $Y \in \mathscr{M}$ such that $\bigcap_{B \in \mathscr{B}} B^{\downarrow\uparrow} \in \overline{Y}$. □

Remark 4. If $\mathscr{M}_1$ is an admissible set in a context $\mathscr{J}$, L_1 the sublattice in $L_{\mathscr{J}}$ corresponding to $\mathscr{M}_1$ by 1 of Theorem 2, and $\mathscr{M}_2$ the admissible set in $\mathscr{J}$ by 2 of Theorem 2, then $\overline{\mathscr{M}}_1 = \overline{\mathscr{M}}_2$ (see Remark 3c)). If L_1 is a complete sublattice in $L_{\mathscr{J}}$, $\mathscr{M}$ the admissible set in $\mathscr{J}$ corresponding to L_1 by 2 of Theorem 2 and L_2 the lattice corresponding to $\mathscr{M}$ by 1 of Theorem 2, then $L_1 = L_2$.

Definition 3. Let $\mathscr{J} = (G, M, I)$ be a context, G_1, M_1 subsets of G, M and $I_1 \subseteq G_1 \times M_1$. If $I_1 \subseteq I$, then the context $\mathscr{J}_1 = (G_1, M_1, I_1)$ is *embedded* in $\mathscr{J}$. If $I_1 = I \cap (G_1 \times M_1)$, then $\mathscr{J}_1$ is a *subcontext* of the context $\mathscr{J}$.

Definition 4. Let $\mathscr{J} = (G, M, I)$ be a context and $\mathscr{M} \subseteq 2^M$ an admissible set in $\mathscr{J}$. Put $M_1 = \bigcup\limits_{B\in\mathscr{M}} B$, $G_1 = \bigcup\limits_{B\in\mathscr{M}} B^{\downarrow}$ and let for $g \in G_1$, $m \in M_1 \colon g\ I_1\ m \Leftrightarrow \exists B \in \mathscr{M}$, $m \in B$, $g \in B^{\downarrow}$. The context $\mathscr{J}_{\mathscr{M}} = (G_1, M_1, I_1)$ is *$\mathscr{M}$-embedded* in $\mathscr{J}$.

Remark 5. In Definition 4 we have $M_1 \subseteq M$, $G_1 \subseteq G$ and $g\ I_1\ m \Rightarrow g\ I\ m$. Hence the context $\mathscr{J}_{\mathscr{M}}$ is embedded in $\mathscr{J}$ by Definition 3.

Remark 6. Let $\mathscr{J}_{\mathscr{M}} = (G_1, M_1, I_1)$ be a context $\mathscr{M}$-embedded in a context $\mathscr{J} = (G, M, I)$. By (2), there exists $X \in \mathscr{M}$ such that $\bigcap\limits_{B\in\mathscr{M}} B^{\downarrow\uparrow} \in \overline{X}$. Moreover, $X \in M_1$, $G_1 = X^{\downarrow}$ holds.

Theorem 3. *If $\mathscr{J}_{\mathscr{M}} = (G_1, M_1, I_1)$ is a context $\mathscr{M}$-embedded in a context $\mathscr{J} = (G, M, I)$, then the lattice $L_{\mathscr{J}_{\mathscr{M}}}$ is a complete sublattice of the lattice $L_{\mathscr{J}}$ and $L_{\mathscr{J}_{\mathscr{M}}} = \{B^{\downarrow} \mid B \in \mathscr{M}\}$.*

Proof. The symbol $\downarrow$ from Definition 1 will be written in $\mathscr{J}_{\mathscr{M}}$ on the left and in $\mathscr{J}$ on the right (as usual). Hence for any sets $C \subseteq M_1 \colon {}^{\downarrow}C = \{g \in G_1 \mid g\ I_1\ c\ \forall c \in C\}$ and $L_{\mathscr{J}_{\mathscr{M}}} = \{{}^{\downarrow}C \mid C \subseteq M_1\}$. By Remark 5, ${}^{\downarrow}C \subseteq G_1 \cap C^{\downarrow}$. Let $\mathscr{B} \subseteq \mathscr{M}$, $\mathscr{B} \neq \emptyset$, and put $D = \bigcup\limits_{B\in\mathscr{B}} B$. If $g \in D^{\downarrow}$, then $g\ I\ d\ \forall d \in D$. Moreover, for any $d \in D$ there is $B \in \mathscr{B}$, i.e. $B \in \mathscr{M}$, such that $d \in B$ and since $g\ I\ d$, we have $g \in B^{\downarrow}$. Consequently, $g\ I_1\ d$ and $g \in {}^{\downarrow}D$. Hence $D^{\downarrow} \subseteq {}^{\downarrow}D$ and therefore $D^{\downarrow} = {}^{\downarrow}D$.

Consider any set $C \subseteq M_1$ and let $g \in {}^{\downarrow}C$. Then $g\ I_1\ m\ \forall m \in C$. For any $m \in C$ there exists $B_m \in \mathscr{M}$ such that $m \in B_m$ and $g \in B_m^{\downarrow}$. If we put $D = \bigcup\limits_{m\in C} B_m$, then $C \subseteq D$. Because $D \subseteq M_1$, we get, by Remark 1, ${}^{\downarrow}D \subseteq {}^{\downarrow}C$ and hence $D^{\downarrow} \subseteq {}^{\downarrow}C$. Moreover, $g \in \bigcap\limits_{m\in C} B_m^{\downarrow} = \Big(\bigcup\limits_{m\in C} B_m\Big)^{\downarrow} = D^{\downarrow}$ and ${}^{\downarrow}C \subseteq D^{\downarrow}$. That means ${}^{\downarrow}C = D^{\downarrow}$. Since $\mathscr{M}$ is an admissible set in $\mathscr{J}$, by (1) there exists a set $X \in \mathscr{M}$ such that $D = \bigcup\limits_{m\in C} B_m \in \overline{X}$, so $D^{\downarrow} = X^{\downarrow}$ and ${}^{\downarrow}C = X^{\downarrow}$.

By Theorem 2, $L_1 = \{B^{\downarrow} \mid B \in \mathscr{M}\}$ is a complete sublattice in $L_{\mathscr{J}}$. By the preceding, $L_1 = L_{\mathscr{J}}$. The lattice operations in L_1 and $L_{\mathscr{J}_{\mathscr{M}}}$ are the same, and hence the lattices L_1 and $L_{\mathscr{J}_{\mathscr{M}}}$ coincide. □

Theorem 4. *Let $\mathscr{J} = (G, M, I)$ be a context and L_1 a complete sublattice of the lattice $L_{\mathscr{J}}$. Let us consider a set $\mathscr{M} \subseteq 2^M$ such that $B^{\downarrow} \in L_1$ $\forall B \in \mathscr{M}$ and for any element $x \in L_1$ there exists $B \in \mathscr{M}$ such that $B^{\downarrow} = x$. If $\mathscr{J}_{\mathscr{M}}$ is a context $\mathscr{M}$-embedded in $\mathscr{J}$, then the lattices L_1 and $L_{\mathscr{J}_{\mathscr{M}}}$ coincide.*

P r o o f. By Theorem 2, $\mathscr{M}$ is an admissible set in $\mathscr{J}$. If we consider a context $\mathscr{J}_{\mathscr{M}}$ $\mathscr{M}$-embedded in $\mathscr{J}$, then, by Theorem 3, $L_{\mathscr{J}_{\mathscr{M}}} = \{B^{\downarrow} \mid B \in \mathscr{M}\} = L_1$. □

R e m a r k 7. The equality $L_{\mathscr{J}_{\mathscr{M}_1}} = L_{\mathscr{J}_{\mathscr{M}_2}}$ may hold for different sets $\mathscr{M}_1$, $\mathscr{M}_2$ in $\mathscr{J}$ (see Remark 4 and e.g. [2]).

Theorem 5. *Let $\mathscr{J}_{\mathscr{M}} = (G_1, M_1, I_1)$ be a context $\mathscr{M}$-embedded in a context $\mathscr{J} = (G, M, I)$. The following conditions are equivalent (the mapping $\downarrow$ is denoted in the context $\mathscr{J}$ on the right and in the context $\mathscr{J}_{\mathscr{M}}$ on the left):*

1. *$\mathscr{J}_{\mathscr{M}}$ is a subcontext in $\mathscr{J}$.*
2. *For $g \in G_1$, $m \in M_1$ we have $g\ I\ m \Rightarrow g\ I_1\ m$.*
3. *For any $m \in M_1$ we have ${}^{\downarrow}\{m\} = \{m\}^{\downarrow} \cap G_1$.*
4. *For any sets $B \subseteq M_1$ we have ${}^{\downarrow}B = B^{\downarrow} \cap G_1$.*

P r o o f of this theorem is easy. □

By using well-known theorems of the lattice theory we get

Theorem 6. *Consider a context $\mathscr{J}$. The concept lattice $L_{\mathscr{J}}$ is distributive (modular) if and only if there is no context $\mathscr{M}$-embedded in $\mathscr{J}$ with the concept lattice isomorphic to the lattices in Figs. 1, 2 (Fig. 2).*

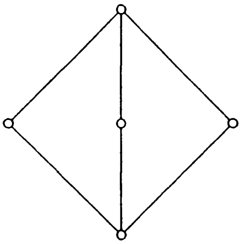

Fig. 1

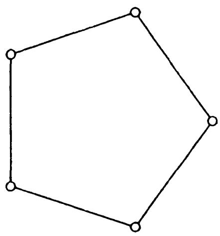

Fig. 2

E x a m p l e s. 1. Consider the context $\mathscr{J} = (G, M, I)$ in Fig. 3, where G and M are sets of points and the relation I is denoted by segments connecting the corresponding points. Then $\mathscr{M} = \{\{m_2\}, \{m_2, m_3\}, \{m_2, m_6\}, \{m_2, m_3, m_5\}, \{m_2, m_3, m_6\}, \{m_2, m_3, m_5, m_6\}\}$ is an admissible set in $\mathscr{J}$. Fig. 4 shows the $\mathscr{M}$-embedded context $\mathscr{J}_{\mathscr{M}}$ while in Fig. 5 we see the lattice $L_{\mathscr{J}_{\mathscr{M}}}$ ($\mathscr{J}_{\mathscr{M}}$ is a subcontext in $\mathscr{J}$). By Theorem 6 the lattice $L_{\mathscr{J}}$ is not modular. Fig. 6 shows the lattice $L_{\mathscr{J}}$ and the sublattice $L_{\mathscr{J}_{\mathscr{M}}}$ (marked).

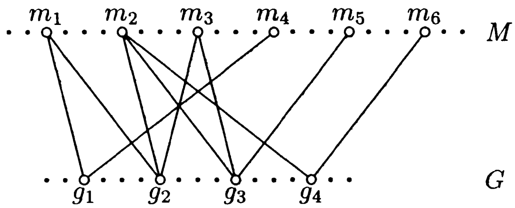

Fig. 3

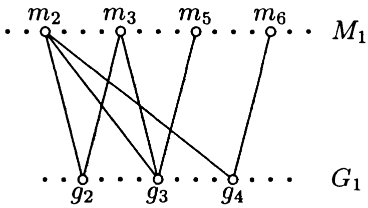

Fig. 4

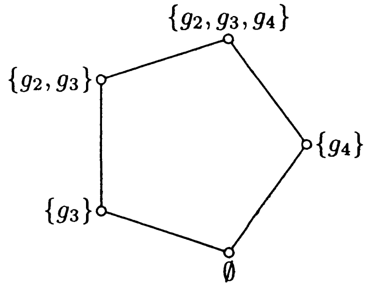

Fig. 5

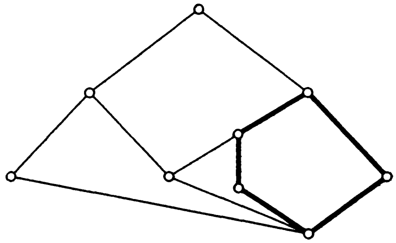

Fig. 6

2. Let us consider the context $\mathcal{J} = (G, M, I)$, where M is the set of planes of the extended three-dimensional Euclidean space E_3, G is the set of points of this space and I the usual incidence relation. Hence for $m \in M$, $\{m\}^{\downarrow}$ is the set of points of the plane m. In Fig. 7 a sublattice L_1 of the lattice $L_{\mathcal{J}}$ is shown. The unit element is

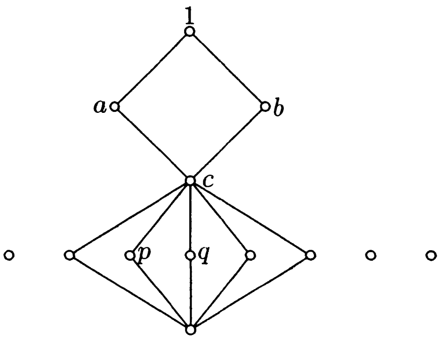

Fig. 7

the set of all points of E_3, i.e. $1 = G = \emptyset^{\downarrow}$. The elements a, b are the sets of points of some planes m_1, m_2, i.e. $a = \{m_1\}^{\downarrow}$, $b = \{m_2\}^{\downarrow}$. An element c is a set of all points of the line r which is a meet of planes m_1, m_2, i.e. $c = \{m_1, m_2\}^{\downarrow}$. The elements p, q, ... are one-point sets of c (points of the line r) We have $\{p\} = \{m_1, m_2, m\}^{\downarrow}$, where m is a plane such that $p \in \{m\}^{\downarrow}$, $r \not\subseteq \{m\}^{\downarrow}$. Evidently $\{m_1, m_2, m, n\}^{\downarrow} = \emptyset$ where m,

n are the planes defined by the preceding two different points. By Theorem 2 the set $\mathscr{M} = \{B \subseteq M \mid B^{\downarrow} \in L_1\}$ is admissible in $\mathscr{I}$ and determines the $\mathscr{M}$-embedded context $\mathscr{I}_{\mathscr{M}} = (G_1, M_1, I_1)$ in $\mathscr{I}$. Then the relation I_1 satisfies

$$g \in \{m_1\}^{\downarrow} \Rightarrow g\ I_1\ m_1,$$
$$g \in \{m_2\}^{\downarrow} \Rightarrow g\ I_1\ m_2,$$
$$g \in \{m_1, m_2\}^{\downarrow} \wedge g \in \{m\}^{\downarrow}, \quad m \in M_1 \Rightarrow g\ I_1\ m.$$

For other points $g \in G$ and planes $m \in M$ the relation I_1 is not defined. By Theorem 4 we obtain $L_{\mathscr{I}_{\mathscr{M}}} = L_1$. The context $\mathscr{I}_{\mathscr{M}}$ is embedded in $\mathscr{I}$, but it is not a subcontext in $\mathscr{I}$ with a concept lattice L_1.

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