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TOTALLY UMBILICAL PSEUDO-RIEMANNIAN SUBMANIFOLDS OF THE PARACOMPLEX PROJECTIVE SPACE*

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1. INTRODUCTION

Para-Kaehlerian manifolds were introduced by Rasevskii [14] and Libermann [12], and studied by several authors (see Bejan [2] and the long list of references therein). An interesting class of para-Kaehlerian manifolds is the class of para-Hermitian symmetric spaces. Kaneyuki and Kozai [10] gave the infinitesimal classification in the case of semisimple group. A particular type is given by the paracomplex projective spaces, introduced by the authors in [4]. These spaces are harmonic symmetric spaces ([1], [5], [6]), and models of spaces of constant non vanishing paraholomorphic sectional curvature, which have a rich family of para-Kaehlerian space forms ([4], [8], [9]). These spaces have also been studied in [2] and [7].

Totally umbilical submanifolds of a given manifold, provided they exist, constitute one of the most natural and useful families of submanifolds. They are known for several classes of important manifolds (see Chen [3]). In the present paper we determine all of the totally umbilical pseudo-Riemannian submanifolds of the paracomplex projective spaces. Let $P(E \oplus E^*)$ be the paracomplex projective space naturally associated to the finite dimensional real vector space $E$. We prove that its non totally geodesic, totally umbilical pseudo-Riemannian submanifolds are of constant (ordinary) sectional curvature. In fact, if $h$ is any non-degenerate symmetric bilinear form in $E$ and $S_h = \{ x \in E : h(x,x) = 1 \}$ is the corresponding sphere, then $S_h$ can be isometrically immersed as a totally geodesic submanifold of $P(E \oplus E^*)$ (cf. [7]). We prove that the parallels of $S_h$, that is its intersections with affine subspaces of $E$, are then isometrically immersed as totally umbilical submanifolds of $P(E \oplus E^*)$, and

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that every non totally geodesic, totally umbilical pseudo-Riemannian submanifold of
$P(E \oplus E^*)$ of dimension greater than 1 is part of such an immersed parallel.

2. Preliminaries

Let $E$ be an $(r + 1)$-dimensional real vector space, and $E^*$ its dual. Typically, we
shall write $x + \alpha$ to denote an element of $E \oplus E^*$. On the space $E \oplus E^*$ there exist
a natural non-degenerate bilinear form $\langle , \rangle$ given by

$$\langle x + \alpha, y + \beta \rangle = \frac{1}{2} (\alpha(y) + \beta(x)),$$

and a linear automorphism $J$ such that

$$J|_E = \text{id}_E, \quad J|_{E^*} = -\text{id}_{E^*}.$$

We introduce in

$$(E \oplus E^*)_+ = \{ x + \alpha \in E \oplus E^*: \langle x + \alpha, x + \alpha \rangle = \alpha(x) > 0 \}$$

the equivalence relation $\sim$ such that $x + \alpha \sim ax + b\alpha$ whenever $0 < a, b \in \mathbb{R}$, and
define the paracomplex projective space $P(E \oplus E^*)$ by

$$P(E \oplus E^*) = (E \oplus E^*)_+ / \sim.$$ 

Let $p$ denote the natural projection $p: (E \oplus E^*)_+ \rightarrow P(E \oplus E^*)$. We define the
vector fields $n, v$ in $E \oplus E^*$ by $n_{x+\alpha} = x + \alpha, v_{x+\alpha} = x - \alpha$, so that $Jn = v$. The
pseudosphere in $E \oplus E^*$ is defined as

$$S = \{ x + \alpha \in (E \oplus E^*)_+: \langle x + \alpha, x + \alpha \rangle = \alpha(x) = 1 \}.$$

Then $n$ is the unit normal to $S$. We have a principal bundle $p: S \rightarrow P(E \oplus E^*)$
with group $\mathbb{R}^+$. This group acts on the right upon $S$ by $(x + \alpha)a = ax + a^{-1}\alpha$, for
$a \in \mathbb{R}^+$. If $S$ is given the pseudo-Riemannian metric induced by that of $E \oplus E^*$,
then $\mathbb{R}^+$ acts on $S$ by isometries. Thus, it induces a pseudo-Riemannian metric $g$ on
$P(E \oplus E^*)$ so that $p$ is a pseudo-Riemannian submersion. The vector field $v$, when
restricted to $S$ is parallel to the fibres of $p$. Therefore, a vector tangent to $S$ is $p$-
horizontal iff it is orthogonal to $v$. Also, $J$ passes to the quotient and gives an almost
product structure $J$ on $P(E \oplus E^*)$ such that $J^2 = 1$ and $g(JX, Y) = -g(X, JY)$. If $\nabla$ is the Levi-Civita connection on $P(E \oplus E^*)$, then $\nabla J = 0$. Thus $P(E \oplus E^*)$ is a
para-Kaehlerian manifold, and if $r > 1$ it is simply connected. Also, it has constant
para-holomorphic sectional curvature (equal to 4) [4], that is the Riemann-Christoffel tensor field is given by

\[
\tilde{R}(X, Y, Z, W) = g(X, Z)g(Y, W) - g(X, W)g(Y, Z) - g(X, JZ)g(Y, JW) \\
+ g(X, JW)g(Y, JZ) - 2g(X, JY)g(Z, JW).
\]

where we define the Riemann-Christoffel tensor field by

\[ R(X, Y, Z, W) = g(R(X, Y)Z, W) \]

and the curvature operator by \[ R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]. \]

We shall study (regular) pseudo-Riemannian submanifolds of \( P(E \oplus E^*) \), that is imbedded submanifolds \( i: \mathcal{V} \rightarrow P(E \oplus E^*) \) such that \( i^*g \) is non-degenerate. Let \( 1 < s = \dim \mathcal{V} \). If \( m \in \mathcal{V} \) then we shall put

\[ \mathcal{N}_m = (T_m \mathcal{V})^\perp, \quad \mathcal{N} = \bigcup_{m \in \mathcal{V}} \mathcal{N}_m. \]

Thus \( T_m P(E \oplus E^*) = T_m \mathcal{V} \perp \mathcal{N}_m \), and we shall denote by \( \tau \) and \( \nu \) the corresponding projectors to \( T_m \mathcal{V} \) and \( \mathcal{N}_m \). Let \( P = \tau \circ J, \ Q = \nu \circ J \). Then if \( X, Y \in \mathcal{X}(\mathcal{V}) \) and \( \eta, \mu \in \Gamma(\mathcal{N}) \) we have \( g(X, PY) = -g(PX, Y), \ g(Q\eta, \mu) = -g(\eta, Q\mu), \) and if \( \nabla \) denotes the Levi-Civita connection on \( \mathcal{V} \) we put

\[
\nabla_X Y = \tau \tilde{\nabla}_X Y, \quad \alpha(X, Y) = \nu \tilde{\nabla}_X Y, \\
A_\eta X = -\tau \tilde{\nabla}_X \eta, \quad D_X \eta = \nu \tilde{\nabla}_X \eta.
\]

We have

\[
g(A_\eta X, Y) = g(\alpha(X, Y), \eta).
\]

We say that \( \mathcal{V} \) is totally umbilical iff there exists \( \xi \in \Gamma(\mathcal{N}) \) such that

\[
\alpha(X, Y) = g(X, Y)\xi
\]

for every \( X, Y \in \mathcal{X}(\mathcal{V}) \). Then, \( \xi \) is called the normal curvature vector field.
3. **Totally umbilical submanifolds of** $P(E \oplus E^*)$

**either are totally geodesic or have constant curvature**

In the following, $V$ will be a totally umbilical pseudo-Riemannian submanifold of $P(E \oplus E^*)$ with normal curvature vector field $\xi$. Let $X,Y,Z \in \mathcal{X}(V)$. Codazzi’s equation [11, Vol. II, p. 25] reads

$$-\nu \tilde{R}(X,Y)Z = (\tilde{\nabla}_X \alpha)(Y,Z) - (\tilde{\nabla}_Y \alpha)(X,Z),$$

where $\tilde{\nabla} \alpha$ is defined by

$$(\tilde{\nabla}_X \alpha)(Y,Z) = D_X (\alpha(Y,Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z).$$

Having in mind (2), that is

$$(\tilde{\nabla}_X \alpha)(Y,Z) = D_X (g(Y,Z)\xi) - g(\nabla_X Y, Z)\xi - g(Y, \nabla_X Z)\xi = g(Y,Z)D_X \xi.$$ 

Then, Codazzi’s equation is

$$
\begin{align*}
(g(X,PZ)g(Y,P\eta) - g(Y,PZ)g(X,P\eta) + 2g(X,PY)g(Z,P\eta) & = g(Y,Z)g(D_X \xi, \eta) - g(X,Z)g(D_Y \xi, \eta),
\end{align*}
$$

where $\eta \in \Gamma(\mathcal{N})$.

Let $R_D$ be the curvature of the connection $D$ in $\mathcal{N}$. Then Ricci’s equation [15, Vol. 4, p. 60] is

$$\nu \tilde{R}(X,Y)\eta = R_D(X,Y)\eta - \alpha(A_\eta X, Y) + \alpha(A_\eta Y, X).$$

Since $g(A_\eta X, Y) = g(\alpha(X,Y), \eta) = g(X,Y)g(\xi, \eta)$, we have $A_\eta X = g(\xi, \eta)X$ and $\alpha(A_\eta X, Y) = g(\xi, \eta)g(X,Y)\xi$. Ricci’s equation reduces thus to

$$\nu \tilde{R}(X,Y)\eta = R_D(X,Y)\eta.$$

We take the trace of (3) in the arguments $X, Z$. Let $\{e_i\}$ be a $g$-orthonormal local reference for $V$, in the sense that $e_i \in \mathcal{X}(U)$, $U \subset V$, $g(e_i, e_j) = \varepsilon_i \delta_{ij}$, $\varepsilon_i = \pm 1$. Then

$$0 = \sum_{i=1}^{s} \varepsilon_i \left( g(e_i, Pe_i)g(Y,P\eta) - g(Y, Pe_i)g(e_i, P\eta) + 2g(e_i, PY)g(e_i, P\eta) 
\right.
\left. - g(Y, e_i)g(D_{e_i} \xi, \eta) + g(e_i, e_i)g(D_\xi \eta, \eta) \right)
= (s-1)g(D_Y \xi, \eta) - 3g(QPY, \eta).$$

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Since \( g \) is non-degenerate and \( \eta \in \Gamma(\mathcal{N}) \) is arbitrary, we conclude that

\[
D\gamma \xi = \frac{3}{s - 1} QPY.
\]

If we bring (5) to (3), we get

\[
g(X, PZ)g(Y, P\eta) - g(Y, PZ)g(X, P\eta) + 2g(X, PY)g(Z, P\eta) + \frac{3}{s - 1} (g(Y, Z)g(PX, P\eta) - g(X, Z)g(PY, P\eta)) = 0.
\]

If we put \( Y = Z \), then

\[
g(X, PZ)g(Z, P\eta) + \frac{1}{s - 1} (g(Z, Z)g(PX, P\eta) - g(X, Z)g(PZ, P\eta)) = 0.
\]

Since \( X \) is arbitrary and \( i^*g \) is non-degenerate, we have

\[
g(Z, P\eta)PZ - \frac{1}{s - 1} g(Z, Z)P^2\eta - \frac{1}{s - 1} g(PZ, P\eta)Z = 0.
\]

Finally, we put \( Z = P\eta \), and have

\[
(s - 2)g(P\eta, P\eta)P^2\eta = 0
\]

for any \( \eta \in \Gamma(\mathcal{N}) \). Thus, it is clear that we must separate the case \( s = 2 \) from the others. Assume first that \( s > 2 \). Then, (8) reads \( g(P\eta, P\eta)P^2\eta = 0 \) for any \( \eta \in \Gamma(\mathcal{N}) \). Assume that we have chosen such a field \( \eta \) and that in some open subset \( U \) of the submanifold \( V \) we have \( P^2\eta \neq 0 \). Then \( g(P\eta, P\eta) = 0 \) in \( U \). Putting \( Y = P\eta \) in (6) we obtain

\[
g(P^2\eta, Z)g(P\eta, X) + \frac{2s - 5}{s - 1} g(P\eta, Z)g(P^2\eta, X) = 0.
\]

Since \( X, Z \) are arbitrary, we conclude that

\[
P^2\eta \otimes P\eta + \frac{2s - 5}{s - 1} P\eta \otimes P^2\eta = 0.
\]

This implies that \( P\eta \) and \( P^2\eta \) are linearly dependent, but this is absurd because

\[
1 + \frac{(2s - 5)}{(s - 1)} = 3\frac{(s - 2)}{(s - 1)} \neq 0 \quad \text{and} \quad P^2\eta \neq 0.
\]

Therefore we have proved that \( P^2\eta = 0 \) for every \( \eta \in \Gamma(\mathcal{N}) \). Then, by (7) we have \( g(P\eta, Z)g(PX, Z) = 0 \), and by polarization \( g(P\eta, Y)g(PX, Z) + g(P\eta, Z)g(PX, Y) = 0 \), from which

\[
P\eta \otimes PX + PX \otimes P\eta = 0.
\]
Lemma 1. Let $V$ be a totally umbilical pseudo-Riemannian submanifold of $P(E \oplus E^*)$ with $s = \dim V > 2$ and let $\xi$ be its normal curvature vector field. Let $X, Y, Z \in \mathfrak{X}(V)$ and $\eta \in \Gamma(\mathcal{N})$. Then:

(i) $\nu \tilde{R}(X, Y)Z = 0$;
(ii) $D_X \xi = 0$;
(iii) $\tilde{R}(X, Y, \eta, \xi) = 0$.

Proof. From (9) we see that at each point $m \in V$ we have that either $P(T_m V) = 0$ or $P(\mathcal{N}_m) = 0$. Then if we multiply (5) by $\eta$ we have

$$g(D_Y \xi, \eta) = -\frac{3}{s-1}g(PY, P\eta) = 0,$$

and (ii) follows. Then the right hand side of Codazzi’s equation vanishes identically and this is (i). From (ii) we have $R_D(X, Y)\xi = 0$. Hence, by (4) we have (iii). \qed

Assume now that $s = \dim V = 2$. Let $m \in V$ and let $v_m, w_m$ be an orthonormal base of $T_m V$, that is $g(v_m, v_m) = a$, $g(w_m, w_m) = b$, $g(v_m, w_m) = 0$, $a^2 = b^2 = 1$. For $u$ in a neighborhood of 0, let $\gamma(u)$ be the geodesic in $V$ with initial condition $(m, w_m)$. Let $v(u)$ be the $V$-parallel displacement of $v_m$ along $\gamma$. Let $t \mapsto \varphi(t, u)$ be the geodesic in $V$ with initial condition $(\gamma(u), v(u))$. We thus have a local chart $(t, u) \mapsto \varphi(t, u)$ of $V$ defined in a neighborhood of $0 \in \mathbb{R}^2$. We define two local vector fields $v, w$ as follows: if $m_1 = \varphi(t_1, u_1)$, then we put

$$v_{m_1} = \frac{\partial \varphi}{\partial t} \bigg|_{(t_1, u_1)}$$

and $w_{m_1}$ is defined as the $V$-parallel displacement of $\gamma(u_1)$ along the curve $t \mapsto \varphi(t, u_1)$ up to the point $m_1$. By this construction, it is clear that $g(v, v) = a$, $g(w, w) = b$, $g(v, w) = 0$, and that

$$\nabla_v v = 0, \quad \nabla_v w = 0, \quad (\nabla_w v) \circ \gamma = 0, \quad (\nabla_w w) \circ \gamma = 0.$$

Let us call $f = g(v, Jw)$. Then

$$QPv = Q(v Jv) = Q(af(v, Jv)v + bg(w, Jv)w)$$

$$= -bfQw = -bf(Jw - ag(v, Jw)v) = af(v, Jw - Jw),$$

$$QPw = af(Jv + bfw),$$

$$\tilde{\nabla}_v v = \nabla_v v + \alpha(v, v) = a\xi, \quad \tilde{\nabla}_v w = g(v, w)\xi = 0,$$

$$\tilde{\nabla}_v \xi = -A\xi v + D_v \xi = -g(\xi, \xi)v + 3QPv = -g(\xi, \xi)v + 3bf(afv - Jw),$$

$$\tilde{\nabla}_w \xi = -g(\xi, \xi)w + 3af(Jv + bfw),$$

$$(\tilde{\nabla}_w w) \circ \gamma = b\xi \circ \gamma, \quad (\tilde{\nabla}_w v) \circ \gamma = 0,$$

$$v(f) = \tilde{\nabla}_v g(v, Jw) = ag(\xi, Jw), \quad w(f) \circ \gamma = bg(v, J\xi) \circ \gamma.$$
Thus, as computation shows,

\[
(\tilde{R}(v, w)\xi) \circ \gamma = \left(-3g(v, J\xi)Jw + 3g(w, J\xi)Jv - 6g(v, Jw)J\xi + 12f\left(ag(J\xi,v)v + bg(J\xi,w)w\right)\right) \circ \gamma,
\]

whereas by (1) we have

\[
\tilde{R}(v, w)\xi = g(v, J\xi)Jw - g(w, J\xi)Jv + 2g(v, Jw)J\xi.
\]

Therefore

\[
\left(g(J\xi, w)Jv - g(J\xi, v)Jw - 2g(v, Jw)J\xi + 3f\left(ag(J\xi, v)v + bg(J\xi, w)w\right)\right) \circ \gamma = 0.
\]

If we apply J and then make the inner product by v we have along γ:

\[
ag(J\xi, w) + 3bf g(J\xi, w)g(v, Jw) = g(J\xi, w)(a + 3bf^2) = 0.
\]

Assume that \(g(J\xi, w)_m \neq 0\). Then, \(f \circ \gamma\) is constant in a neighborhood of 0 and equal to \(\sqrt{-\frac{1}{3}ab}\). But then, by the preceding formulae, we would have \(d(f \circ \gamma)/du = w(f) \circ \gamma = bg(v, J\xi) \circ \gamma = 0\) in that neighborhood. In particular, \(g(J\xi, v)_m = 0\). Then \(P\xi_m = bg(J\xi, w)_m w_m\). Since f is real we have that \(-ab\) is positive, so that \(a = -b\). Let c be an arbitrary real number and put \(v'_m = v_m \cosh c + w_m \sinh c\), \(w'_m = v_m \sinh c + w_m \cosh c\). Then \(g(v'_m, v'_m) = a, g(w'_m, w'_m) = b, g(v'_m, w'_m) = 0\), so that we have another orthonormal base of \(T_m V\).

Then \(P\xi_m = ag(J\xi_m, v'_m)v'_m + bg(J\xi_m, w'_m)w'_m = g(J\xi_m, w)_m (v'_ma \sinh c + w'_mb \cosh c)\).

If \(c \neq 0\) we have an orthonormal base of \(T_m V\) on which both components of \(P\xi_m\) are non-zero. Since the whole construction could have been done starting from the new base, we have reached a contradiction. We conclude that \(g(J\xi, w)_m = g(J\xi, v)_m = 0\) and as a consequence, if \(\xi_m \neq 0\) one has moreover \(g(v, Jw)_m = 0\). Since \(m\) is arbitrary, the same holds in the whole \(V\). Then, if \(\xi \neq 0\), we have \(f = 0, D\xi = 0, J(TV) \subset \mathcal{N}, J\xi \in \Gamma(\mathcal{N}), \nu\tilde{R}(X,Y)Z = 0, \tilde{R}(X,Y,\eta,\xi) = 0\) and \(g(\xi, \xi)\) is constant.

**Theorem 2.** Let \(V\) be a connected totally umbilical pseudo-Riemannian submanifold of \(P(E \oplus E^*)\) with \(\dim V > 1\) and let \(\mathcal{N}\) be the bundle orthogonal to \(TV\). Then, either \(V\) is totally geodesic or \(J(TV) \subset \mathcal{N}\) and in this case \(V\) is a pseudo-Riemannian manifold with constant sectional curvature.

**Proof.** Let \(s > 2\). Then, we put

\[
\mathcal{A} = \left\{ m \in V : (P \circ \nu)|_{T_m P(E \oplus E^*)} = 0 \right\}, \quad \mathcal{B} = \left\{ m \in V : (P \circ \tau)|_{T_m P(E \oplus E^*)} = 0 \right\}.
\]

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Clearly, these subsets are closed in $V$. By (9), $\mathcal{A} \cup \mathcal{B} = V$. If $m \in \mathcal{A} \cap \mathcal{B}$, then 
\[ P = \tau \circ J = 0 \text{ on } T_m P(E \oplus E^*), \] and this is absurd because $J$ is an isomorphism.
Then $\mathcal{A} \cap \mathcal{B} = \emptyset$, and therefore either $\mathcal{A} = V$ or $\mathcal{B} = V$. Assume that $\mathcal{A} = V$.
Then, by (1) and Lemma 1, (iii) we have

\[ \tilde{R}(X,Y,\eta,\xi) = -2g(X, JY)g(\eta, J\xi) = 2g(X, JY)g(J\eta, \xi) = 0. \]

Now $g(J\eta, X) = g(P\eta, X) = 0$, whence $J(\mathcal{N}) \subset \mathcal{N}$. Then, applying (10) to $J\eta$ instead of $\eta$, and having in mind that $X, Y$ are arbitrary, we conclude that $g(\eta, \xi) = 0$, that is $\xi = 0$, and so $V$ is totally geodesic.

Thus, assume that $\mathcal{B} = V$. Then $J(TV) \subset \mathcal{N}$. By Gauss' equation we have directly

\[
R(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) + g(\alpha(X,Z), \alpha(Y,W)) - g(\alpha(Y,Z), \alpha(X,W)) \]
\[
= (1 + l)(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)),
\]

where $l = g(\xi, \xi)$, which by Lemma 1, (ii), is a constant. The same results hold obviously when $s = 2$. 

4. **Parallels as Totally Umbilical Submanifolds of $P(E \oplus E^*)$**

Let $F$, $\Lambda$ be subspaces of $E$ and $E^*$, respectively, such that the pairing $F \times \Lambda \to \mathbb{R}$ given by $(x, \alpha) \mapsto \alpha(x)$ is non-degenerate. Let $f: F \to \Lambda$ be an isomorphism such that $f(x, y) \equiv f(x)(y) = f(y, x)$ for any $x, y \in F$. We shall use the following notation

\[ F^\perp = \{ \alpha \in E^*: \alpha(x) = 0, \text{ if } x \in F \}, \quad \Lambda^\perp = \{ x \in E: \alpha(x) = 0, \text{ if } \alpha \in \Lambda \}. \]

We put

\[ \Sigma = \{ x \in F: f(x,x) = a \}, \quad 0 \neq a \in \mathbb{R}, \]

and consider it as a pseudo-Riemannian sphere defined by the pseudo-Riemannian metric $f$ on $F$. Let $x_0 + \alpha_0$ be some fixed element of $E \oplus E^*$ such that

\[ \alpha_0 \in F^\perp, \quad x_0 \in \Lambda^\perp, \quad \alpha_0(x_0) + a = 1. \]

We map $F$ into $E \oplus E^*$ by means of $j: F \to E \oplus E^*$ defined by

\[ j(x) = x + x_0 + f(x) + \alpha_0. \]
It is clear that since \( j_*(X) = X + f(X) \), \( j \) is an isometry. Let \( x \in \Sigma \); then 
\[
\langle j(x), j(x) \rangle = f(x, x + x_0) + \alpha_0(x + x_0) = a + \alpha_0(x_0) = 1.
\]
Thus, \( j(\Sigma) \subset \mathcal{S} \). Also, if \( X \in T_x \Sigma \) we have 
\[
\langle j_*(X), v_{j(x)} \rangle = \langle X + f(X), x + x_0 - f(x) - \alpha_0 \rangle
\]
\[
= \frac{1}{2} (f(X, x + x_0) - f(x, X) - \alpha_0(X)) = 0
\]
because \( X \in F \). Therefore, \( j_*(X) \) is \( p \)-horizontal and, as a consequence, \( p \circ j : \Sigma \to P(E \oplus E^*) \) is an isometry. Let us prove that \( V = p(j(\Sigma)) \) is a totally umbilical submanifold of \( P(E \oplus E^*) \).

Let \( \tilde{X} \in \mathcal{X}(j(\Sigma)) \). Then \( \tilde{X} \) is \( p \)-horizontal and there are fields \( X \in \mathcal{X}'(V) \), \( \tilde{X} \in \mathcal{X}'(\Sigma) \) such that
\[
j_0 \circ \tilde{X} = \tilde{X} \circ j,
\]
\[
p_0 \circ \tilde{X} = X \circ p,
\]
\[
j_0(\tilde{X}, \tilde{X}) = (\tilde{X}, \tilde{X}) \circ j = g(X, X) \circ p \circ j.
\]

We shall also consider fields \( \hat{Y}, \tilde{Y}, Y \) with the analogous properties. We denote by \( \hat{X}(\tilde{Y}) \) and \( \tilde{X}(\hat{Y}) \) the canonical covariant derivative in \( E \) and in \( E \oplus E^* \). Let \( \nabla^\Sigma \), \( \nabla^S \), \( \nabla \), \( \nabla \) be the Levi-Civita connections in \( \Sigma \), \( S \), \( P(E \oplus E^*) \) and \( V \), respectively. We have
\[
\nabla^\Sigma^\mathcal{X} \tilde{Y} = \tilde{X}(\hat{Y}) + (\tilde{X}, \hat{Y})n.
\]

Also, \( \langle \tilde{X}(\hat{Y}), v \rangle \circ j = -\langle \hat{Y}, \tilde{X}(v) \rangle \circ j = -\langle \hat{Y}, J\tilde{X} \rangle \circ j = -\langle \hat{Y} + f(\hat{Y}), \tilde{X} - f(\tilde{X}) \rangle = -\frac{1}{2} (f(\hat{Y}, \tilde{X}) - f(\tilde{X}, \hat{Y})) = 0 \). Since \( n \) is also orthogonal to \( v \), we have that \( \nabla^\Sigma^\mathcal{X} \tilde{Y} \) is \( p \)-horizontal. Let \( x(t) \in \Sigma \) be an integral curve of \( \tilde{X} \); then, \( j(x(t)) \) is an integral curve of \( \tilde{X} \). If \( x = x(0) \), then
\[
(\tilde{X}(\hat{Y}))_{j(x(t))} = \frac{d}{dt} \bigg|_{t=0} \tilde{X}(\hat{Y}(x(t))) = \frac{d}{dt} \bigg|_{t=0} j_0(\tilde{X}(\hat{Y}(x(t)))) = (\hat{X}(\tilde{Y}) + f(\hat{X}(\tilde{Y})))_{x(t)}.
\]

Therefore, if \( H \hat{U} \) denotes the \( p \)-horizontal part of \( \tilde{U} \in \mathcal{X}'(S) \), we have
\[
(H \nabla^\Sigma^\mathcal{X} \tilde{Y}) \circ j = \hat{X}(\tilde{Y}) + f(\hat{X}(\tilde{Y})) + f(\hat{X}, \tilde{Y})(n \circ j).
\]

Since \( p : S \to P(E \oplus E^*) \) is a pseudo-Riemannian submersion, we know [13, p. 212] that
\[
p_0 \circ (H \nabla^\Sigma^\mathcal{X} \tilde{Y}) = (\nabla^\mathcal{X} Y) \circ p.
\]

Therefore
\[
(\nabla^\mathcal{X} Y) \circ p \circ j = p_0 \circ (\hat{X}(\tilde{Y}) + f(\hat{X}(\tilde{Y})) + f(\hat{X}, \tilde{Y})(n \circ j)).
\]
On the other hand, since \( p \circ j \): \( \Sigma \to V \) is an isometry, we have
\[
(p \circ j)_* \nabla^\Sigma_X \hat{Y} = \nabla^X_Y \circ p \circ j.
\]

Since \((\nabla_X Y - \nabla_X Y) \circ p \circ j = (\nu \nabla_X Y) \circ p \circ j = \alpha(X, Y) \circ p \circ j\) defines the second fundamental form of \( V \), we need only to calculate \( \nabla^\Sigma_X \hat{Y} \). But as it is well known about pseudo-spheres, we have
\[
\nabla^\Sigma_X \hat{Y} = \hat{X}(\hat{Y}) - \frac{1}{a} f(\hat{X}(\hat{Y}), x)x = \hat{X}(\hat{Y}) + \frac{1}{a} f(\hat{X}, \hat{Y})x,
\]
where \( x \) denotes the vector field whose value at \( x \) is \( x \). Thus
\[
\alpha(X, Y) \circ p \circ j = p_* \circ \left( \hat{X}(\hat{Y}) + f(\hat{X}(\hat{Y})) + f(\hat{X}, \hat{Y})(n \circ j) - \hat{X}(\hat{Y}) \right.
\]
\[
- f(\hat{X}(\hat{Y})) - f(\hat{X}, \hat{Y}) \frac{x + f(x)}{a}
\)
\[
= (g(X, Y) \circ p \circ j) p_* \circ \left( \frac{a - 1}{a} (x + f(x)) + x_0 + \alpha_0 \right),
\]
and this proves that \( V = p(j(\Sigma)) \) is a totally umbilical submanifold of \( P(E \oplus E^*) \) with normal curvature vector field \( \xi \) given by \( \xi \circ p \circ j = p_* \left( \frac{a - 1}{a} (x + f(x)) + x_0 + \alpha_0 \right) \).

We have
\[
\langle \frac{a - 1}{a} (x + f(x)) + x_0 + \alpha_0, \nu \rangle \circ j = \left( \frac{a - 1}{a} (x + f(x)) + x_0 + \alpha_0, x - f(x) + x_0 - \alpha_0 \right) = 0
\]
because of (11). Thus, \( \frac{a - 1}{a} (x + f(x)) + x_0 + \alpha_0 \) is \( p \)-horizontal. Hence
\[
l = g(\xi, \xi) = \left( \frac{a - 1}{a} (x + f(x)) + x_0 + \alpha_0, \left( \frac{a - 1}{a} (x + f(x)) + x_0 + \alpha_0 \right) = \frac{1 - a}{a}.
\]

Let us suppose that \( l = 0 \). Then \( a = 1 \) and by (11) we have \( \alpha_0(x_0) = 0 \). Hence, if \( \dim F^\perp = \codim F = 1 \), the vector field \( \xi \), which in this case would be given by \( \xi \circ p = p_* (x_0 + \alpha_0) \), must be an eigen-vector field of \( J \). In fact, the assumption \( x_0 \neq 0 \) would imply then that \( F \oplus \mathbb{R}x_0 = E \). Since \( \alpha_0 \in F^\perp \) and \( \alpha_0(x_0) = 0 \), we conclude \( \alpha_0 = 0 \) and \( J\xi = \xi \). If \( x_0 = 0 \), then \( J\xi = \xi \).

Note that \( a = 1/(1 + l) \). Therefore, this construction cannot yield the case \( l = -1 \). To deal with it, let \( 0 \neq z \in F \) be such that \( f(z, z) = 0 \) and put \( \mu = f(z) \). We put
\[
\Sigma = \{ x \in F: f(x, x) = 1, \mu(x) = 1 \}.
\]
If \( x \in \Sigma \) and \( v \in T_x F \), then \( v \in T_x \Sigma \) iff \( f(x, v) = \mu(v) = 0 \), so that \( x, z \) span the orthogonal space to \( T_x \Sigma \) in \( F \). The orthogonal projection of a vector \( v \in T_x F \) upon \( T_x \Sigma \) is given by \( v \mapsto v + (\mu(v) - f(x, v))z - \mu(v)x \). Then, if \( \hat{X}, \hat{Y} \in \mathfrak{X}(\Sigma) \), we have
\[
\nabla^\Sigma_X \hat{Y} = \hat{X}(\hat{Y}) + \left( \mu(\hat{X}(\hat{Y})) - f(x, \hat{X})(\hat{Y}) \right) z - \mu(\hat{X}(\hat{Y}))x = \hat{X}(\hat{Y}) + f(\hat{X}, \hat{Y})z,
\]
because
\[
f(x, \hat{X}(\hat{Y})) = \hat{X}(f(x, \hat{Y})) - f(\hat{X}(x), \hat{Y}) = -f(\hat{X}, \hat{Y}),
\]
\[
\mu(\hat{X}(\hat{Y})) = \hat{X}(\mu(\hat{Y})) = 0.
\]

We map \( \Sigma \) into \( S \) by
\[
j(x) = x + f(x).
\]
As in the other case, this is an isometry and \( j(\Sigma) \) is \( p \)-horizontal, so that \( p \circ j \) is an isometry. The only change in the computations lies in the connection \( \nabla^\Sigma \). By using its new formula, we have immediately with the same notations:
\[
\alpha(X, Y) \circ p \circ j = p_* \circ (\hat{X}(\hat{Y}) + f(\hat{X}(\hat{Y})) + f(\hat{X}, \hat{Y})(n \circ j)
- \hat{X}(\hat{Y}) - f(\hat{X}(\hat{Y})) - f(\hat{X}, \hat{Y})(z + f(z)))
= (g(X, Y) \circ p \circ j)p_* \circ (x - z + f(x) - \mu).
\]

Thus, \( p(j(\Sigma)) \) is a totally umbilical submanifold of \( P(E \oplus E^*) \) with normal curvature vector field given by \( \xi \circ p \circ j = p_* \circ (x - z + f(x) - \mu) \). We have \( \langle x - z + f(x) - \mu, v_{j(x)} \rangle = -\langle z + \mu, x - f(x) \rangle = -\frac{1}{2}(\mu(x) + \mu(x)) = 0 \). Therefore \( l = g(\xi, \xi) = (f(x) - \mu)(x - z) = 1 - \mu(x) - \mu(x) = -1 \), as desired. We shall call parallels of \( P(E \oplus E^*) \) the totally umbilical submanifolds defined in this section.

5. Construction of all the totally umbilical submanifolds of \( P(E \oplus E^*) \)

Until near the end, we shall assume in this section that \( V \) is a non totally geodesic, totally umbilical submanifold of \( P(E \oplus E^*) \) so that \( \xi \neq 0 \). First of all we shall prove that the inclusion \( J(TV) \subset \mathcal{N} \) is strict. From (1), we have now, for \( X, Y \in \mathcal{X}(V) \) and \( \eta, \mu \in \Gamma(\mathcal{N}) \), that
\[
\tilde{R}(X, Y, \eta, \mu) = g(JX, \mu)g(JY, \eta) - g(JX, \eta)g(JY, \mu),
\]
and this is zero if \( \mu = \xi \), that is
\[
g(JX, \xi)g(JY, \eta) - g(JX, \eta)g(JY, \xi) = 0.
\]
Assume that \( J(TV) = \mathcal{N} \). Then we can put \( \beta = JX \), \( \mu = JY \) and consider that they are arbitrary sections of \( \mathcal{N} \). Thus \( g(\beta, \xi)g(\mu, \eta) - g(\beta, \eta)g(\mu, \xi) = 0 \), that is
Now, we prove that $J\xi \in \Gamma(\mathcal{N})$. In fact, since in (12) $\eta$ is arbitrary we have $g(JX,\xi)JY - g(JY,\xi)JX = 0$. By multiplication by $JX$ we get $g(JX,\xi)JX \wedge JY = 0$. Since $J$ is an isomorphism and $s \geq 2$, this implies $g(JX,\xi) = -g(X,J\xi) = 0$, that is $J\xi \in \Gamma(\mathcal{N})$. From this, we can prove that if $\xi$ is not an eigen-vector field of $J$ and $l = 0$ then $s \leq r - 1$. On these assumptions, let us consider the subbundle of $\mathcal{N}$ generated by $J(TV)$, $\xi$ and $J\xi$, and suppose that there is some vector in the intersection of $J(TV)$ with the subbundle generated by $\xi$ and $J\xi$, namely $JX = a\xi + bJ\xi$, with $X \in TV$. Then $X = aJ\xi + b\xi$, whence $X = 0$. Therefore rank $\mathcal{N} = 2r - s \geq s + 2$. Now, $g(\xi, J\xi) = g(\xi, \xi) = g(\xi, JX) = -g(J\xi, X) = 0$ for every $X \in TV$. The equal sign in $2r - s \geq s + 2$ would then imply that $g|_{\mathcal{N}}$ be degenerate, for $\xi$ would be orthogonal to the whole $\mathcal{N}$; so, $2r - s > s + 2$, that is $s < r - 1$.

The identity tensor field $I$ can be decomposed into two projectors on the eigenspaces of $J$ as $I = \frac{1}{2}(I + J) + \frac{1}{2}(I - J)$. Let $\pi_1 = \frac{1}{2}(I + J)|_{TV}$, $\pi_2 = \frac{1}{2}(I - J)|_{TV}$, $v \in T_mV$ and suppose that $\pi_1(v) = 0$. Then $Ju = -v \in M_m \cap T_mV$, whence $v = 0$. Thus, if $M_1 = \pi_1(TV)$ and $M_2 = \pi_2(TV)$, we have that $\pi_1 : TV \to M_1$ and $\pi_2 : TV \to M_2$ are isomorphisms. Let $h$ be the isomorphism $h : M_1 \to M_2$ given by $h = \pi_2 \circ \pi_1^{-1}$. We claim that

$$T_mV = \{v + hv : v \in (M_1)_m\}.$$

In fact, if $v \in (M_1)_m$, then $v = \pi_1 w$, for some $w \in T_mV$. Thus $w = \pi_1 w + \pi_2 w = \pi_1 w + \pi_2 \circ \pi_1^{-1} \circ \pi_1 w = \pi_1 w + h(\pi_1 w) = v + hv$. Moreover, $h$ is self-adjoint, that is $g(hX,Y) = g(X,hY)$ for every $X,Y \in \Gamma(M_1)$, and the bilinear symmetric tensor field given by $X,Y \in \Gamma(M_1) \mapsto g(hX,Y)$ is non-degenerate at each point. To show this, we note that since $Y + hY \in \mathcal{X}(V)$ we have $J(Y + hY) \in \Gamma(\mathcal{N})$, that is $g(X + hX, J(Y + hY)) = g(X + hX, Y - hY) = g(hX,Y) - g(X,hY) = 0$. Also, $g(X + hX, Y + hY) = 2g(hX,Y)$ by the above result, and the non-degeneracy of this bilinear symmetric tensor field follows from that of $i^*g$.

Let $l + 1 \neq 0$, so that $V$ is not flat. Then, given a point $m = p(y + \beta) \in V$, with $y + \beta \in S$, we want to show that there is some parallel of $P(E \oplus E^*)$, $j(\Sigma)$, defined as in Section 4, that passes by $m$ having $T_mV$ as tangent space at $m$ and $\xi_m$ as normal curvature vector at $m$. With the notations of Section 4, we want to determine $x$, $x_0$, $\eta \otimes \xi = \xi \otimes \eta$ for every $\eta \in \Gamma(\mathcal{N})$, and this would imply $s = \dim V = \text{rank } \mathcal{N} = 1$, which is contrary to the assumption $s \geq 2$. 

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\( f(x), \alpha_0, a \) such that if \( u + \gamma \) is the \( p \)-horizontal lift of \( \xi_m \) to \( y + \beta \), then

\[
\begin{align*}
x + x_0 + f(x) + \alpha_0 & = y + \beta, \\
\frac{a - 1}{a} x + x_0 + \frac{a - 1}{a} f(x) + \alpha_0 & = u + \gamma, \\
\frac{1 - a}{a} & = l.
\end{align*}
\]

The solution of this system is the following

\[
a = \frac{1}{1 + l}, \quad x = \frac{y - u}{1 + l}, \quad x_0 = \frac{ly + u}{1 + l}, \quad f(x) = \frac{\beta - \gamma}{1 + l}, \quad \alpha_0 = \frac{l\beta + \gamma}{1 + l}.
\]

We have

\[
\beta(y) = 1, \quad \gamma(u) = l, \quad \gamma(y) = 0, \quad \beta(u) = 0,
\]

formulae that express that \( y + \beta \in S \), \( g(\xi, \xi) = l \), \( u + \gamma \in T_{y+\beta}S \) and \( u + \gamma \) is \( p \)-horizontal. Let the superscript \( H \) denote the \( p \)-horizontal lift of \( T_mV \) to \( T_{y+\beta}S \). This lift preserves \( J \) and the inner product. Thus, as before we can see that \( (T_mV)^H = \{ v + f(v) : v \in M_1^H \equiv \pi_1((T_mV)^H) \} \), where \( f = \pi_2 \circ \pi_1^{-1} \) with the obvious meaning. We put \( F = M_1^H + Rx \).

The formula for \( f(x) \), that until now was just a form, gives \( f(x)(x) = (\beta - \gamma)(y - u)/(1 + l)^2 = 1/(1 + l) = a \). Also, \( f(x)(v) = 0 \) if \( v \in M_1^H \). In fact, we have then \( v + f(v) \in (T_mV)^H \). But \( \xi, J\xi \in \Gamma(N) \), whence \( (v + f(v), u + \gamma) = f(v)(u) + \gamma(v) = 0 \) and \( (v + f(v), u - \gamma) = f(v)(u) - \gamma(v) = 0 \). Therefore \( f(v)(u) = \gamma(v) = 0 \), and \( f(x)(v) = (\beta(v) - \gamma(v))/(1 + l) = \beta(v)/(1 + l) \) and this is zero. In fact, \( v + f(v) \) is \( p \)-horizontal and tangent to \( S \), so that \( \langle v + f(v), y + \beta \rangle = \langle v + f(v), y - \beta \rangle = 0 \), whence \( \beta(v) = f(v)(y) = 0 \). As a consequence, \( f(x) \) allows us to extend \( f \) to \( F \) by putting \( f(x, x) = f(x)(x) = a \), \( f(x, v) = f(x)(v) = 0 \), and we have \( x \in \Sigma \), with \( \Sigma \) defined as in the preceding section. To complete our construction, we need to show that \( \alpha_0 \in F^\perp \), \( x_0 \in f(F)^\perp \), and this is easily done using the same techniques used for proving that \( f(x)(v) = 0 \).

Let \( l + 1 = 0 \), so that \( V \) is flat. Now, we want to determine \( x, z, f(x), \mu \) such that if \( u + \gamma \) is the \( p \)-horizontal lift of \( \xi_m \) to \( y + \beta \), then

\[
\begin{align*}
x + f(x) & = y + \beta, \\
x - z + f(x) - \mu & = u + \gamma.
\end{align*}
\]

Clearly, this implies

\[
\begin{align*}
z & = y - u, \quad x = y, \quad \mu = \beta - \gamma, \quad f(x) = \beta.
\end{align*}
\]
$F$ and $f$ are defined as before. Then, $f(x)(x) = \beta(y) = 1$, $\mu(z) = (\beta - \gamma)(y - u) = 1 - 1 = 0$. As in the other case, one can easily verify that this construction gives the desired parallel of $P(E \oplus E^*)$.

**Theorem 3.** Let $V$ be a connected totally umbilical pseudo-Riemannian submanifold of $P(E \oplus E^*)$ with $s = \dim V > 1$ and assume that it is not totally geodesic. Then, $V$ is contained in a parallel of $P(E \oplus E^*)$ of the same dimension.

**Proof.** As proved above, if $m \in V$, there is a parallel of $P(E \oplus E^*)$, $p(\jmath(\Sigma))$, that passes by $m$ having $T_mV$ as tangent space at $m$ and $\zeta_m$ as normal curvature vector at $m$. Let $\gamma$ be a geodesic of $V$. Then, we have

$$\tilde{\nabla}_\gamma \dot{\gamma} = \nabla_\gamma \dot{\gamma} + g(\dot{\gamma}, \dot{\gamma})(\xi \circ \gamma) = g(\dot{\gamma}, \dot{\gamma})(\xi \circ \gamma),$$

$$\tilde{\nabla}_\gamma \dot{\xi} = D_\gamma \xi - A_\xi \dot{\gamma} = -l \dot{\gamma}.$$  

Thus, if we put $\chi = \xi \circ \gamma$, we have a curve $\chi$ in $TP(E \oplus E^*)$, with projection $\gamma$ on $P(E \oplus E^*)$, that satisfies the following differential equations

$$\tilde{\nabla}_\gamma \dot{\gamma} = g(\dot{\gamma}, \dot{\gamma})\chi,$$

$$\tilde{\nabla}_\gamma \chi = -l \dot{\gamma},$$

where $l$ is a constant. To convince oneself that this is a well posed system of ordinary differential equations, we can write it locally as

$$\ddot{x}^i + (\Gamma^i_{jk} \circ \gamma)\dot{x}^j \dot{x}^k - (g_{jk} \circ \gamma)\dot{x}^j \dot{x}^k \chi^i = 0,$$

$$\chi^i + (\Gamma^i_{jk} \circ \gamma)\dot{x}^j \dot{x}^k + l \dot{x}^i = 0.$$  

Since the geodesics of both $V$ and $p(\jmath(\Sigma))$ starting from $m$ satisfy the same system with the same initial conditions, we conclude that there is some open neighborhood of $m$ where both submanifolds coincide. By a standard argument, we have our claim. 

As for totally geodesic submanifolds of $P(E \oplus E^*)$, we can separate them in three classes [6]. First, totally geodesic submanifolds with a degenerate metric $i^*g$, which are of no interest here in the context of umbilical pseudo-Riemannian submanifolds.

The second consists of the paracomplex projective subspaces. Let $E = F \oplus G$ be a splitting of $E$ in two subspaces, and let $E^* = F^* \oplus G^*$ be the corresponding splitting for $E^*$. Then, the inclusion $i: F \oplus F^* \to E \oplus E^*$ passes to the quotient and gives the paracomplex projective subspace $P(F \oplus F^*) \hookrightarrow P(E \oplus E^*)$, which is a totally geodesic pseudo-Riemannian submanifold.

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Submanifolds $V$ of the third class are such that for each point $m \in V$, $T_mV = \{v + h_m v : v \in (T_mV)_1\}$, where $(T_mV)_1 = (I + J)(T_mV)$ and $h_m$ is a symmetric isomorphism from $(T_mV)_1$ to $(T_mV)_2 = (I - J)(T_mV)$. These are parallels of $P(E \oplus E^*)$ given by the preceding formulae for the non-flat case when $\xi = 0$. Then $l = 0$, $a = 1$, $x_0 + \alpha_0 = 0$, and $p \circ j : \Sigma \to P(E \oplus E^*)$ is a totally geodesic isometric immersion of the pseudo-Riemannian sphere $\Sigma = \{x \in F : f(x, x) = 1\}$. Let us call meridians these submanifolds $p(j(\Sigma))$.

**Theorem 4.** Let $V$ be a connected totally umbilical pseudo-Riemannian submanifold of the paracomplex projective space $P(E \oplus E^*)$ with $s = \dim V > 1$. Then:

1. If $V$ is not totally geodesic, it is contained in a parallel of $P(E \oplus E^*)$ of the same dimension $s$, and then $V$ has constant sectional curvature.
2. If $V$ is totally geodesic, then either it is contained in a paracomplex projective subspace $P(F \oplus F^*)$ of $P(E \oplus E^*)$ with $\dim F = \frac{1}{2}s + 1$ and then $V$ has constant para-holomorphic sectional curvature, or it is contained in a meridian of $P(E \oplus E^*)$ of the same dimension $s$ and then $V$ has constant sectional curvature.

**References**


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