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EXISTENCE OF SOLUTIONS OF A FUNCTIONAL-INTEGRAL EQUATION IN INFINITE DIMENSIONAL BANACH SPACES¹

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Let Ω be a bounded closed subset of \mathbb{R}^n . If f, k, g are functions defined, respectively, in $\Omega \times E$, $\Omega \times \Omega$, $\Omega \times E$ (E a Banach space) with values into, respectively, E , $L(F, E)$, F (F a Banach space, $L(F, E)$ the space of all linear continuous operators from F into E), we consider the functional-integral equation

$$(1) \quad x(t) = f\left(t, r \int_{\Omega} k(t, s)g(s, x(s)) ds\right), \quad t \text{ a.e. in } \Omega$$

and look for solutions of (1) lying in $L^1(\Omega, E)$, the usual Bochner function space on (Ω, \mathcal{L}, m) , the usual Lebesgue measure space. The equation (1) is quite general, because for $f(t, x) = x$ we get the Hammerstein integral equation, whereas if $g(t, x) = x$ we get an equation recently considered in [2] and in [4]. (In particular, we improve the result in [4] because, in the case of $E = F = \mathbb{R}$ and $\Omega = [0, 1] \subset \mathbb{R}$, we are able to dispense with one of the hypotheses used in that paper.) For several applications of the Hammerstein integral equation to partial differential equations we refer to [3] and [8].

The technique we use in the main theorem is the usual one: we construct an operator A mapping continuously a suitable bounded, closed and convex subset Q of $L^1(\Omega, E)$ into itself, and prove that $A(Q)$ is relatively compact. Hence the Schauder fixed point theorem can be applied. The choice of the set Q is such that it allows us to avoid the use of certain monotonicity assumptions contained in [2] (see also results and examples in [3] and in [8]) that are not always extendible to the case of functions with values in infinite dimensional Banach spaces; the hypotheses we consider are quite general and "natural" in the sense that they are necessary and sufficient for certain operators to take $L^1(\Omega, E)$ continuously into itself (see [7]).

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One of the main tools we use is the following generalization of the Ascoli-Arzelà theorem to the case of vector-valued continuous functions.

Theorem 1 (see [1], for instance). *Let T be a compact metric space and let $(f_n) \subset C^0(T, E)$ be a sequence of equicontinuous and equibounded functions. If for each $t \in T$, $\{f_n(t)\}$ is relatively compact in E , then (f_n) is relatively compact in $C^0(T, E)$. Moreover, the set $\{f_n(t) : t \in T, n \in \mathbb{N}\}$ is relatively compact in E , too.*

We even need the following extension of a theorem of Scorza-Drăgoni to be found in [9].

Theorem 2 ([9]). *Let T be a compact metric space with a Radon measure defined on it, E a separable metric space, F a Banach space. If $f : T \times E \rightarrow F$ is a function verifying the Carathéodory hypotheses, i.e. f is measurable with respect to $t \in T$ for all $x \in E$ and continuous with respect to $x \in E$ for almost all $t \in T$, given $\varepsilon > 0$ there is a measurable closed subset T_ε of T with $m(T \setminus T_\varepsilon) < \varepsilon$ and $f|_{T_\varepsilon \times E}$ continuous.*

We shall make use of the following result concerning compact subsets of separable Banach spaces (see [5]).

Theorem 3 ([5]). *Let M be a bounded subset of a separable Banach space E . M is relatively compact if and only if for any w^* -null sequence $(x_n^*) \subset E^*$ one has $\limsup_n \sup_{x \in M} |x_n^*(x)| = 0$.*

We are now ready to give the proof of our result for which we need an easy lemma:

Lemma 4. *Let us assume*

(k₁) $h_1, h_2 \in L^1(\Omega, \mathbb{R})$, $h_1(t) \geq 0, h_2(t) \geq 0$ a.e. on Ω ;

(k₂) $\psi : \Omega \times \Omega \rightarrow \mathbb{R}_+$ verifies the Carathéodory hypotheses and the linear operator

$$(\Psi z)(t) = \int_{\Omega} \psi(t, s) z(s) \, ds$$

maps $L^1(\Omega, \mathbb{R})$ into $L^1(\Omega, \mathbb{R})$ (in this case Ψ is continuous ([10]) and $\|\Psi\|$ denotes its norm);

(k₃) $b_1, b_2, r \geq 0$ are such that $rb_1 b_2 \|\Psi\| < 1$.

Then there is a nonnegative $\varphi_0 \in L^1(\Omega, \mathbb{R})$ such that

$$\varphi_0(t) = h_1(t) + rb_1 \int_{\Omega} \psi(t, s) (h_2(s) + b_2 \varphi_0(s)) \, ds, \quad t \text{ a.e. in } \Omega.$$

PROOF. Let us put $p = \frac{\|h_1\| + \|a\|}{1 - rb_1 b_2 \|\Psi\|}$ where $a(t) = rb_1 \int_{\Omega} \psi(t, s) h_2(s) ds \in L^1(\Omega, \mathbb{R})$. It is easy to see that $Mx \in B_p = \{x: x \in L^1(\Omega, \mathbb{R}), \|x\| \leq p\}$ whenever $x \in B_p$, where

$$Mx(t) = h_1(t) + rb_1 \int_{\Omega} \psi(t, s)(h_2(s) + b_2 x(s)) ds, \quad t \in \Omega.$$

It is also clear that $Mx(t) \geq 0$ a.e. on Ω when $x(t) \geq 0$ a.e. on Ω and so $M(B_p^+) \subset B_p^+$ with $B_p^+ = B_p \cap \{x: x \in L^1(\Omega, \mathbb{R}), x(t) \geq 0 \text{ a.e. on } \Omega\}$; furthermore, B_p^+ is a complete metric space. It is also easy to prove that M is a contraction when restricted to B_p^+ . Then the Banach-Caccioppoli fixed point theorem applies to give the result. We are done. \square

Theorem 5. *Let E be a separable Banach space, F an arbitrary Banach space and Ω a bounded, closed subset of \mathbb{R}^n . Let us assume*

(h₁) $f: \Omega \times E \rightarrow E$ verifies the Carathéodory hypotheses and, moreover, there exist $h_1 \in L^1(\Omega, \mathbb{R})$ and $b_1 \geq 0$ such that

$$\|f(t, x)\|_E \leq h_1(t) + b_1 \|x\|_E \quad \text{for a.a. } t \in \Omega \text{ and all } x \in E;$$

(h₂) $k: \Omega \times \Omega \rightarrow C(F, E)$ (the Banach space of linear compact operators from F into E with the usual operator norm) verifies the Carathéodory hypotheses and the linear operator K defined by

$$(Kz)(t) = \int_{\Omega} \|k(t, s)\|_{C(F, E)} z(s) ds, \quad t \text{ a.e. in } \Omega$$

maps $L^1(\Omega, \mathbb{R})$ into itself (this last fact implies that K is continuous, see [10]; let $\|K\|$ denote its norm);

(h₃) $g: \Omega \times E \rightarrow F$ verifies the Carathéodory hypotheses and, moreover, there exist $h_2 \in L^1(\Omega, \mathbb{R})$ and $b_2 \geq 0$ such that

$$\|g(t, x)\|_F \leq h_2(t) + b_2 \|x\|_E \quad \text{for a.a. } t \in \Omega \text{ and all } x \in E;$$

(h₄) $rb_1 b_2 \|K\| < 1$.

Then the equation (1) has a solution x in $L^1(\Omega, E)$.

PROOF. Putting $\psi(t, s) = \|k(t, s)\|_{C(E, F)}$ in Lemma 4, we get that there is a nonnegative $\varphi_0 \in L^1(\Omega, \mathbb{R})$ such that

$$\varphi_0(t) = h_1(t) + rb_1 \int_{\Omega} \|k(t, s)\|_{C(E, F)} (h_2(s) + b_2 \varphi_0(s)) ds, \quad t \text{ a.e. in } \Omega.$$

First of all, assume $\varphi_0 = \theta_{L^1(\Omega, \mathbb{R})}$. In such a case we easily get that $\theta_{L^1(\Omega, E)}$ is a solution of (1). Indeed, we have

$$\begin{aligned} & \left\| f\left(t, r \int_{\Omega} k(t, s)g(s, \theta_{L^1(\Omega, E)}(s)) \, ds\right) \right\| \\ & \leq h_1(t) + b_1 \left\| r \int_{\Omega} k(t, s)g(s, \theta_{L^1(\Omega, E)}(s)) \, ds \right\| \\ & \leq h_1(t) + b_1 r \int \|k(t, s)\|_{C(E, F)} \|g(s, \theta_{L^1(\Omega, E)}(s))\| \, ds \\ & \leq h_1(t) + b_1 r \int \|k(t, s)\|_{C(E, F)} (h_2(s) + b_2 \|\theta_{L^1(\Omega, E)}(s)\|) \, ds \\ & \leq h_1(t) + b_1 r \int \|k(t, s)\|_{C(E, F)} (h_2(s) + b_2 \theta_{L^1(\Omega, \mathbb{R})}(s)) \, ds \\ & = \theta_{L^1(\Omega, \mathbb{R})}(t), \qquad t \text{ a.e. in } \Omega, \end{aligned}$$

which means that

$$f\left(t, r \int_{\Omega} k(t, s)g(s, \theta_{L^1(\Omega, E)}(s)) \, ds\right) = \theta_{L^1(\Omega, E)}(t), \quad t \text{ a.e. in } \Omega.$$

So let us assume $\varphi_0 \neq \theta_{L^1(\Omega, \mathbb{R})}$ and consider the following subset of $L^1(\Omega, E)$:

$$Q = \{x : x \in L^1(\Omega, E), \|x(t)\|_E \leq \varphi_0(t) \quad \text{a.e. on } \Omega\}.$$

Q is clearly bounded, closed and convex in $L^1(\Omega, E)$; furthermore, Q is uniformly integrable, i.e. $\lim_{m(S) \rightarrow 0} \int_S \|x(s)\| \, ds = 0$ uniformly on Q . We consider the operator

$$(Ax)(t) = f\left(t, r \int_{\Omega} k(t, s)g(s, x(s)) \, ds\right).$$

We shall prove that

- (i) $A(L^1(\Omega, E)) \subset L^1(\Omega, E)$,
- (ii) $A(Q) \subset Q$,
- (iii) $A|_Q$ is continuous,
- (iv) $A(Q)$ is relatively compact.

Hence an easy application of the Schauder fixed point theorem will give the existence of a solution of (1). That (i) is true is an easy consequence of our assumptions

$(h_1), (h_2), (h_3)$. Let us show (ii). If $x \in Q$ we have, for a.a. $t \in \Omega$,

$$\begin{aligned} \|(Ax)(t)\|_E &= \left\| f\left(t, r \int_{\Omega} k(t, s)g(s, x(s)) \, ds\right) \right\|_E \\ &\leq h_1(t) + b_1 r \left\| \int_{\Omega} k(t, s)g(s, x(s)) \, ds \right\|_E \\ &\leq h_1(t) + b_1 r \int_{\Omega} \|k(t, s)\|_{C(F, E)} \|g(s, x(s))\|_F \, ds \\ &\leq h_1(t) + b_1 r \int_{\Omega} \|k(t, s)\|_{C(F, E)} (h_2(s) + b_2 \|x(s)\|_E) \, ds \\ &\leq h_1(t) + b_1 r \int_{\Omega} \|k(t, s)\|_{C(F, E)} (h_2(s) + b_2 \varphi_0(s)) \, ds = \varphi_0(t) \end{aligned}$$

by virtue of Lemma 4. Let us prove (iii). Let $(x_n), (x_0) \subset Q$ with $x_n \rightarrow x_0$. This means that $x_n(s) \rightarrow x_0(s)$ almost everywhere on Ω (by passing to a subsequence if necessary). Fix $\bar{t} \in \Omega$; we have $k(\bar{t}, s)g(s, x_n(s)) \rightarrow k(\bar{t}, s)g(s, x_0(s))$ for a.a. $s \in \Omega$ because of (h_3) and (h_2) . Thanks to (h_3) we also have that, for a.a. $s \in \Omega$,

$$\begin{aligned} &\|k(\bar{t}, s)[g(s, x_n(s)) - g(s, x_0(s))]\|_E \\ &\leq \|k(\bar{t}, s)\|_{C(E, F)} [2h_2(s) + b_2(\|x_n(s)\| + \|x_0(s)\|)] \\ &\leq \|k(\bar{t}, s)\|_{C(E, F)}^2 (h_2(s) + b_2 \varphi_0(s)); \end{aligned}$$

this easily yields

$$\int_{\Omega} k(\bar{t}, s)g(s, x_n(s)) \, ds \rightarrow \int_{\Omega} k(\bar{t}, s)g(s, x_0(s)) \, ds.$$

Hence

$$f\left(\bar{t}, r \int_{\Omega} k(\bar{t}, s)g(s, x_n(s)) \, ds\right) \rightarrow f\left(\bar{t}, r \int_{\Omega} k(\bar{t}, s)g(s, x_0(s)) \, ds\right),$$

i.e. $Ax_n(\bar{t}) \rightarrow Ax_0(\bar{t})$, thanks to (h_1) . But $\|Ax_n(t) - Ax_0(t)\|_E \leq 2\varphi_0(t)$ and so $\|Ax_n - Ax_0\|_{L^1(\Omega, E)} \rightarrow 0$. It remains to show the most difficult (iv). It is clear that we can assume Q countable; so we do it. First of all, we observe that thanks to Theorem 2, given $\sigma > 0$ there is $\Omega_{\sigma} \subset \Omega$, closed, with $m(\Omega \setminus \Omega_{\sigma}) < \sigma$, such that $f|_{\Omega_{\sigma} \times E}, k|_{\Omega_{\sigma} \times \Omega}$ are continuous. First we shall prove that

$$(j) \quad B = \{Hx|_{\Omega_{\sigma}} : x \in Q\} \subset C^0(\Omega_{\sigma}), \text{ where } (Hx)(t) = \int_{\Omega} k(t, s)g(s, x(s)) \, ds, \\ t \in \Omega,$$

(jj) B is relatively compact in $C^0(\Omega_{\sigma})$.

Let $t', t'' \in \Omega$. We have

$$\begin{aligned} \|(Hx)(t') - (Hx)(t'')\|_E &\leq \int_{\Omega} \|k(t', s) - k(t'', s)\|_{C(F,E)} \|g(s, x(s))\| ds \\ &\leq \int_{\Omega} \|k(t', s) - k(t'', s)\|_{C(F,E)} (h_2(s) + b_2\varphi_0(s)) ds. \end{aligned}$$

Since $k|_{\Omega_{\sigma} \times \Omega}$ is uniformly continuous, we get that B is an equicontinuous subset of $C^0(\Omega_{\sigma})$; it is also clear that B is an equibounded subset of $C^0(\Omega_{\sigma})$. It remains to show that for all $t \in \Omega_{\sigma}$, $B(t) = \{Ax|_{\Omega_{\sigma}}(t) : x \in Q\}$ is relatively compact, so that we can apply Theorem 1 to B . Now we use Theorem 3. Let (x_n^*) be a w^* -null sequence in E^* ; for $\bar{t} \in \Omega_{\sigma}$ we have

$$(2) \quad x_n^* \int_{\Omega} k(\bar{t}, s)g(s, x(s)) ds = \int_{\Omega} x_n^* k(\bar{t}, s)g(s, x(s)) ds, \quad n \in \mathbb{N}.$$

For almost all $s \in \Omega$, the set $\{g(s, x(s)) : x \in Q\}$ is bounded by virtue of (h_3) and because of the very definition of Q ; hence $\{k(\bar{t}, s)g(s, x(s)) : x \in Q\}$ is compact in E for a.a. $s \in \Omega$ and so $x_n^* k(\bar{t}, s)g(s, x(s)) \rightarrow 0$ uniformly on $x \in Q$; furthermore, $|x_n^* k(\bar{t}, s)g(s, x(s))| \leq \sup_n \|x_n^*\| \|k(\bar{t}, s)\|_{C(F,E)} (h_2(s) + b_2\varphi_0(s))$, which implies

$$\sup_{x \in Q} \int_{\Omega} x_n^* k(\bar{t}, s)g(s, x(s)) ds \rightarrow 0.$$

Thanks to (2) we are done: $B(t)$ is relatively compact for all $t \in \Omega$. Hence B is relatively compact in $C^0(\Omega_{\sigma})$. Once we have (j) and (jj) for any $\sigma > 0$, we can conclude our proof as follows. Given a sequence $(x_h) \subset Q$, it is easy to get a sequence (Ω_n) of closed subsets of Ω with $m(\Omega \setminus \Omega_n) \rightarrow 0$ and a subsequence (y_h) of (x_h) such that (Hy_h) is a Cauchy sequence in any $C^0(\Omega_n)$. Again thanks to Theorem 1 we have that

$$C_n = \{Hy_h(t) : t \in \Omega_n, h \in \mathbb{N}\}$$

is a relatively compact subset of E and so $f|_{\Omega_n \times \bar{C}_n}$ is uniformly continuous. It is then very easy to see (use again Theorem 1) that

$$\{f(\cdot, Hy_h(\cdot)) : \Omega_n \rightarrow E, h \in \mathbb{N}\}$$

is a Cauchy sequence in $C^0(\Omega_n)$ for all $n \in \mathbb{N}$, by passing to a suitable subsequence if necessary. Hence, if $\varepsilon > 0$, let $\sigma > 0$ be such that

$$\sup_{x \in Q} \int_S \|Ax(s)\| ds < \frac{\varepsilon}{4} \quad \text{for all } S \subset \Omega, m(S) < \sigma.$$

Choose $\bar{n} \in \mathbb{N}$ with $m(\Omega \setminus \Omega_{\bar{n}}) < \sigma$. We have

$$\begin{aligned} \int_{\Omega} \|Ay_{h'}(t) - Ay_{h''}(t)\|_E dt &= \int_{\Omega_{\bar{n}}} \|Ay_{h'}(t) - Ay_{h''}(t)\|_E dt \\ &\quad + \int_{\Omega \setminus \Omega_{\bar{n}}} \|Ay_{h'}(t) - Ay_{h''}(t)\|_E dt \\ &\leq \int_{\Omega_{\bar{n}}} \|f(t, Hy_{h'}(t)) - f(t, Hy_{h''}(t))\|_E dt + \frac{\varepsilon}{2} \\ &\leq m(\Omega_{\bar{n}}) \|f(\cdot, Hy_{h'}(\cdot)) - f(\cdot, Hy_{h''}(\cdot))\|_{C^0(\Omega_{\bar{n}})} + \frac{\varepsilon}{2} \end{aligned}$$

$h', h'' \in \mathbb{N}$.

Since $(f(\cdot, Hy_h(\cdot)))_{h \in \mathbb{N}}$ is a Cauchy sequence in $C^0(\Omega_{\bar{n}})$ we are done. \square

Remark. If one of the two spaces E and F is finite dimensional, then any continuous and linear operator from F into E is compact, but this even happens for suitable infinite dimensional Banach spaces; we refer to [6] for a list of such pairs of Banach spaces.

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