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ON THE TENSOR PRODUCT OF A BOOLEAN ALGEBRA AND AN ORTHOALGEBRA

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1. ORTHOALGEBRAS

Orthoalgebras are algebraic systems that generalize Boolean algebras, orthomodular lattices, and orthomodular posets. They were originally introduced in [13]. The following simplified definition is due to Golfin [6].

Definition 1.1. An *orthoalgebra* (OA) is a system $(L, 0, 1, \oplus)$ consisting of a set L containing two special elements $0, 1 \in L$ and a partially defined binary operation $\oplus$ on L that satisfies the following conditions for all $p, q, r \in L$:

(i) [*Commutative Law*] If $p \oplus q$ is defined, then so is $q \oplus p$ and $p \oplus q = q \oplus p$.

(ii) [*Associative Law*] If $p \oplus r$ and $p \oplus (q \oplus r)$ are defined, then so are $p \oplus q$ and $(p \oplus q) \oplus r$ and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(iii) [*Orthocomplementation Law*] For each $p \in L$ there is a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = 1$.

(iv) [*Consistency Law*] If $p \oplus p$ is defined, then $p = 0$.

Example 1.2. Let L be an orthomodular poset (OMP). If $p, q \in L$, define $p \oplus q$ iff $p \perp q$, in which case $p \oplus q := p \vee q$. Then $(L, 0, 1, \oplus)$ is an OA.

It can be shown [4] that an OA $(L, 0, 1, \oplus)$ arises as in Example 1.2 from an OMP iff it satisfies the following condition: If $p, q, r \in L$ and $p \oplus q$, $p \oplus r$, and $q \oplus r$ are defined, then $p \oplus (q \oplus r)$ is defined. This is the sense in which orthoalgebras generalize OMP's.

For simplicity, we usually refer to L, rather than to $(L, 0, 1, \oplus)$, as being an OA.

Definition 1.3. Let L be an OA and let $p, q \in L$. We say that p and q are *orthogonal* and write $p \perp q$ iff $p \oplus q$ is defined. If q is the unique element in L for which $p \perp q$ and $p \oplus q = 1$, we say that q is the *orthocomplement* of p and write

$q = p'$. The relation $p \leqslant q$ means that there is an element $r \in L$ such that $p \perp r$ and $p \oplus r = q$.

One can easily prove [4] that if L is an OA, then $(L, 0, 1, \leqslant, ')$ forms an orthocomplemented poset.

Definition 1.4. Let L be an OA and let $P \subseteq L$. We say that P is a *suborthoalgebra* of L iff $0, 1 \in P$, $p \in P \Longrightarrow p' \in P$, and $p, q \in P$ with $p \perp q \Longrightarrow p \oplus q \in P$.

Evidently, a suborthoalgebra P of an OA L is an OA in its own right under the restriction of $\oplus$ to P. As such, if P is a Boolean algebra, we refer to P as a Boolean suborthoalgebra of L.

Definition 1.5. A subset D of an OA L is said to be *orthogonal* if its elements are pairwise orthogonal and there is a Boolean suborthoalgebra P of L with $D \subseteq P$.

2. Tensor products of orthoalgebras

In this section we outline the basic facts about tensor products of OA's (see [3]).

Definition 2.1. If P, Q are OA's, then a *morphism* from P to Q is a mapping $\gamma\colon P \to Q$ such that $\gamma(1) = 1$ and, whenever $a, b \in P$ with $a \perp b$, it follows that $\gamma(a) \perp \gamma(b)$ and $\gamma(a \oplus b) = \gamma(a) \oplus \gamma(b)$. If, in addition, $a, b \in P$ with $\gamma(a) \perp \gamma(b) \Longrightarrow a \perp b$, then $\gamma\colon P \to Q$ is called a *monomorphism*. An *isomorphism* is a surjective monomorphism.

If $\gamma\colon P \to Q$ is a morphism, then $\gamma(0) = 0$ and, for every $p \in P$, $\gamma(p') = \gamma(p)'$. Also, if $a, b \in P$ with $a \leqslant b$, then $\gamma(a) \leqslant \gamma(b)$. Furthermore, if $\gamma\colon P \to Q$ is an isomorphism, then it is a bijection and $\gamma^{-1}\colon Q \to P$ is a morphism.

Definition 2.2. Let P, Q, L be OA's. A mapping $\beta\colon P \times Q \to L$ is called a *bimorphism* iff it satisfies the following conditions:

(i) $a, b \in P$ with $a \perp b$, $q \in Q \Longrightarrow \beta(a, q) \perp \beta(b, q)$ and $\beta(a \oplus b, q) = \beta(a, q) \oplus \beta(b, q)$.

(ii) $p \in P$ and $c, d \in Q$ with $c \perp d \Longrightarrow \beta(p, c) \perp \beta(p, d)$ and $\beta(p, c \oplus d) = \beta(p, c) \oplus \beta(p, d)$.

(iii) $\beta(1, 1) = 1$.

If $\beta\colon P \times Q \to L$ is a bimorphism, then $\beta(\cdot, 1)\colon P \to L$ and $\beta(1, \cdot)\colon Q \to L$ are morphisms. Also, if $a, b \in P$ and $c, d \in Q$, then

$$a \leqslant b,\ c \leqslant d \Longrightarrow \beta(a, c) \leqslant \beta(b, d) \text{ and } \beta(a, 0) = \beta(0, c) = 0.$$

Definition 2.3. If P, Q are OA's, then a *tensor product* of P and Q is a pair (T, τ) consisting of an orthoalgebra T and a bimorphism $\tau\colon P \times Q \to T$ such that the following conditions are satisfied:

(i) If L is an OA and $\beta\colon P \times Q \to L$ is a bimorphism, there exists a morphism $\gamma\colon T \to L$ such that $\beta = \gamma \circ \tau$.

(ii) Every element of T is a finite orthogonal sum of elements of the form $\tau(p, q)$ with $p \in P$, $q \in Q$.

A tensor product of P and Q, if it exists, is unique up to isomorphism in the following sense: If (T, τ) and (T^*, τ^*) are tensor products of P and Q, then there exists a unique isomorphism $\sigma\colon T \to T^*$ such that $\tau^* = \sigma \circ \tau$. Thus, if P, Q admit a tensor product, we may speak of *the* tensor product of P and Q and denote it by $(P \otimes Q, \otimes)$, or simply by $P \otimes Q$.

Theorem 2.4 [3]. *Let P, Q be OA's. Then the tensor product $P \otimes Q$ exists iff there is at least one OA L for which there is a bimorphism $\beta\colon P \times Q \to L$.*

Although there are examples of OA's P and Q having no tensor product, the tensor product usually exists except for rather bizarre OA's [3].

3. The sum of a Boolean algebra and an orthoalgebra

In this section, we assume that B is a Boolean algebra and L is an OA. Our purpose is to construct the *sum* S of B and L. (Prior to that, let us call a finite subset D of L orthogonal if its elements are pairwisely orthogonal and there is a Boolean subalgebra P of L with $D \subseteq P$. It can be easily proved [4] that there is an element $\bigoplus D \in L$, called the orthogonal sum of D, such that $\bigoplus D$ is the least upper bound of D in any Boolean subalgebra of L that contains D.)

Definition 3.1. A subset E of B is called a *finite partition* (FP) if $0 \notin E$, E is a finite orthogonal set, and $\bigoplus E = 1$.

If $E \subseteq B$ is an FP and $b \in B$, then $b = \bigoplus\{b \wedge e \mid e \in E\}$ follows from the fact that $\bigoplus E = 1$ and the distributive law. In particular, if $b \neq 0$, there exists $e \in E$ with $b \wedge e \neq 0$. Also, if $E, F \subseteq B$ are FP's, then

$$G := \{e \wedge f \mid e \in E,\ f \in F,\ e \wedge f = 0\}$$

is an FP. Furthermore, each element $g \in G$ can be written uniquely in the form $g = e \wedge f$ with $e \in E$, $f \in F$.

Definition 3.2. Let $\Sigma := \{\varphi\colon E \to L \mid E \subseteq B$ is an FP$\}$. If $\varphi, \psi \in \Sigma$ with $E = \mathrm{dom}(\varphi)$, $F = \mathrm{dom}(\psi)$, we define:

(i) $\varphi \leqslant \psi$ iff $e \in E$, $f \in F$, $e \wedge f \neq 0 \Longrightarrow \varphi(e) \leqslant \psi(f)$.

(ii) $\varphi \equiv \psi$ iff $\varphi \leqslant \psi$ and $\psi \leqslant \varphi$.

(iii) $\varphi'\colon E \to L$ by $\varphi'(e) := \varphi(e)'$, for all $e \in E$.

(iv) $\varphi \perp \psi$ iff $\varphi \leqslant \psi'$.

Lemma 3.3. *$\leqslant$ is a reflexive, transitive relation on Σ and $\equiv$ is an equivalence relation on Σ.*

Proof. It is clear that $\leqslant$ is reflexive. To prove that it is transitive, suppose that $\varphi, \xi, \psi \in \Sigma$ with $\varphi \leqslant \xi$ and $\xi \leqslant \psi$. Let $E = \mathrm{dom}(\varphi)$, $G = \mathrm{dom}(\xi)$, $F = \mathrm{dom}(\psi)$, and let $e \in E$, $f \in F$ with $e \wedge f \neq 0$. Then there exists $g \in G$ with $e \wedge f \wedge g \neq 0$. Thus, $e \wedge g \neq 0$, so that $\varphi(e) \leqslant \xi(g)$, and $g \wedge f \neq 0$, so that $\xi(g) \leqslant \psi(f)$. Consequently, $\varphi(e) \leqslant \psi(f)$, proving that $\varphi \leqslant \psi$. Since $\leqslant$ is reflexive and transitive, it follows that $\equiv$ is an equivalence relation. □

For $\varphi, \psi \in \Sigma$, it is clear that $\varphi \leqslant \psi \Longrightarrow \psi' \leqslant \varphi'$ and that $\varphi'' = \varphi$. Consequently, if $\varphi^*, \psi^* \in \Sigma$ with $\varphi \equiv \varphi^*$ and $\psi \equiv \psi^*$, then

$$\varphi \perp \psi \Longleftrightarrow \varphi^* \perp \psi^* \text{ and } \varphi \equiv \psi' \Longleftrightarrow \varphi^* \equiv (\psi^*)'.$$

Definition 3.4. Let $\varphi, \psi \in \Sigma$ with $\varphi \perp \psi$. Let $E = \mathrm{dom}(\varphi)$, $F = \mathrm{dom}(\psi)$, and $G := \{e \wedge f \mid e \in E,\ f \in F,\ e \wedge f \neq 0\}$. Define $(\varphi \oplus \psi)\colon G \to L$ for $e \in E$, $f \in F$, with $e \wedge f \neq 0$ by

$$(\varphi \oplus \psi)(e \wedge f) = \varphi(e) \oplus \psi(f).$$

Theorem 3.5. *Let $\varphi, \varphi^*, \psi, \psi^* \in \Sigma$ with $\varphi^* \leqslant \varphi$, $\psi^* \leqslant \psi$, and $\varphi \perp \psi$. Then $\varphi^* \perp \psi^*$ and $\varphi^* \oplus \psi^* \leqslant \varphi \oplus \psi$.*

Proof. Let $e^* \in \mathrm{dom}(\varphi^*)$, $f^* \in \mathrm{dom}(\psi^*)$, $e \in \mathrm{dom}(\varphi)$, and $f \in \mathrm{dom}(\psi)$ and assume that $e^* \wedge f^* \wedge e \wedge f \neq 0$. We have to prove that $\varphi^*(e^*) \oplus \psi^*(f^*) \leqslant \varphi(e) \oplus \psi(f)$. But this follows immediately from $\varphi^*(e^*) \leqslant \varphi(e)$, $\psi^*(f^*) \leqslant \psi(f)$ and $\varphi(e) \perp \psi(f)$. □

Corollary 3.6. *Let $\varphi, \varphi^*, \psi, \psi^* \in \Sigma$ with $\varphi^* \equiv \varphi$, $\psi^* \equiv \psi$, and $\varphi \perp \psi$. Then $\varphi^* \oplus \psi^* \equiv \varphi \oplus \psi$.*

Lemma 3.7. *Let $\varphi, \psi, \xi \in \Sigma$ with $\varphi \perp \xi$ and $\varphi \perp (\psi \oplus \xi)$. Then $\varphi \perp \psi$, $(\varphi \oplus \psi) \perp \xi$, and $\varphi \oplus (\psi \oplus \xi) = (\varphi \oplus \psi) \oplus \xi$.*

The proof is easy.

Definition 3.8. Define $\zeta \in \Sigma$ by $\mathrm{dom}(\zeta) = \{1\}$ and $\zeta(1) = 0$.

If $\varphi \in \Sigma$, it is clear that $\varphi \leqslant \zeta \iff \varphi \equiv \zeta \iff \varphi(e) = 0$ for all $e \in \mathrm{dom}(\varphi)$. Consequently, $\zeta' \leqslant \varphi \iff \zeta' \equiv \varphi \iff \varphi(e) = 1$ for all $e \in \mathrm{dom}(\varphi)$. Also, $\varphi \leqslant \varphi' \iff \varphi \equiv \zeta$.

The proof of the following lemma is straightforward.

Lemma 3.9. *Let* $\varphi, \psi \in \Sigma$. *Then:*
(i) *If* $\varphi \perp \psi$, *then* $\varphi \oplus \psi \equiv \zeta' \iff \psi \equiv \varphi'$.
(ii) $\varphi \leqslant \psi \iff \exists \xi \in \Sigma,\ \varphi \perp \xi,\ \varphi \oplus \xi \equiv \psi$.

Definition 3.10. For $\varphi \in \Sigma$, define $[\varphi] := \{\psi \in \Sigma \mid \varphi \equiv \psi\}$ and define $S := \{[\varphi] \mid \varphi \in \Sigma\}$. For $\varphi, \psi \in \Sigma$, define:
(i) $[\varphi] \leqslant [\psi]$ iff $\varphi \leqslant \psi$,
(ii) $[\varphi] \perp [\psi]$ iff $\varphi \perp \psi$,
(iii) $[\varphi]' := [\psi]'$,
(iv) $0 := [\zeta]$,
(v) $1 := [\zeta']$,
(vi) If $\varphi \perp \psi$, $[\varphi] \oplus [\psi] := [\varphi \oplus \psi]$.

Our work thus far shows that all notions introduced in Definition 3.10 are well defined.

Theorem 3.11. $(S, 0, 1, \oplus)$ *is an orthoalgebra.*

Proof. The commutative and consistency laws are obvious, the associative law follows from Lemma 3.7, and the orthocomplementation law follows from Part (i) of Lemma 3.9. □

We refer to the orthoalgebra S in Theorem 3.11 as the *sum* of the Boolean algebra B and the OA L.

4. The isomorphism of $B \oplus L$ and the sum S

In this section, we continue with the notation of Section 3, and prove that the tensor product $B \oplus L$ exists and is isomorphic to the sum S of B and L.

Definition 4.1. Let $b \in B$, $p \in L$. Define $b \cdot p \in \Sigma$ as follows:
(i) If $b = 0$, then $b \cdot p := \zeta$.
(ii) If $b = 1$, then $\mathrm{dom}(b \cdot p) = \{1\}$ and $(b \cdot p)(1) := p$.
(iii) If $b \neq 0, 1$, then $\mathrm{dom}(b \cdot p) = \{b, b'\} \cdot (b \cdot p)(b) := p$, and $(b \cdot p)(b') = 0$.

The proof of the following lemma is a straightforward verification based on Section 3 and Definition 4.1.

Lemma 4.2. *Let $a, b \in B$, $p, q \in L$. Then:*

(i) $1 \cdot 1 \equiv \zeta'$.

(ii) $a \cdot p \equiv \zeta \Longleftrightarrow a = 0$ *or* $b = 0$.

(iii) $a \cdot p \perp b \cdot q \Longleftrightarrow a \perp b$ or $p \perp q$.

(iv) $a \perp b \Longrightarrow a \cdot p \oplus b \cdot p \equiv (a \oplus b) \cdot p$

(v) $p \perp q \Longrightarrow b \cdot (p \oplus q) \equiv b \cdot p \oplus b \cdot q$

Lemma 4.3. *Let D be a finite, nonempty, orthogonal set of nonzero elements of B and let $\eta\colon D \to L$. Let $E \subseteq B$ be an FP with $D \subseteq E$, and define $\varphi\colon E \to L$ by $\varphi(d) := \eta(d)$ for $d \in D$ and $\varphi(e) := 0$ for $e \in E \setminus D$. Then $\{[d \cdot \varphi(d)] \mid d \in D\}$ is an orthogonal subset of S and*

$$[\varphi] = \bigoplus_{d \in D} [d \cdot \varphi(d)].$$

Proof. The proof is by induction on $\sharp D$, the cardinal number of D. The result is obvious for $\sharp D = 1$. Assume that it holds for $\sharp D = n$, and suppose $\sharp D = n + 1$. Choose and fix $d_0 \in D$. By the induction hypothesis, the theorem holds for $D \setminus \{d_0\}$ and the restriction of η to $D \setminus \{d_0\}$. Therefore, with $F := (D \setminus \{d_0\}) \cup \{f_0\}$, $f_0 := \left(\bigoplus (D \setminus \{d_0\})\right)' = d_0 \oplus (\bigoplus D)'$, and $\psi\colon F \to L$ defined by $\psi(d) := \eta(d)$ for $d \in D \setminus \{d_0\}$ and $\psi(f_0) := 0$, we have that $\{[d \cdot \psi(d)] \mid d \in D,\ d \neq d_0\}$ is an orthogonal subset of S and

$$[\psi] = \bigoplus_{d \in D,\, d = d_0} [d \cdot \psi(d)].$$

Evidently, $d_0 \cdot \varphi(d_0) \perp [\psi]$. $[\psi] \oplus [d_0 \cdot \varphi(d_0)] = [\varphi]$, and the induction argument is complete. □

Corollary 4.4. *If $\varphi \in \Sigma$, and $E = \operatorname{dom}(\varphi)$, then $\{[e \cdot \varphi(r)] \mid e \in E\}$ is an orthogonal subset of S and*

$$[\varphi] = \bigoplus_{e \in E} [e \cdot \varphi(e)].$$

Lemma 4.5. *The tensor product $B \otimes L$ exists and there is a surjective morphism $\gamma\colon B \otimes L \to S$ such that, for $b \in B$, $p \in L$, $\gamma(b \otimes p) = [b \cdot p]$. Furthermore, for $a, b \in B$, $p, q \in L$,*

$$(a \otimes p) \perp (b \otimes q) \Longleftrightarrow a \perp b \text{ or } p \perp q.$$

Proof. By Parts (i), (iv), and (v) of Lemma 4.2, the mapping $(b,p) \mapsto [b \cdot p]$ is a bimorphism from $P \times L$ to S; hence, $B \otimes L$ exists by Theorem 2.4. Therefore, by Part (i) of Definition 2.3, there is a morphism $\gamma\colon B \times L \leftarrow S$ such that $\gamma(b \otimes p) = [b \cdot p]$ for every $b \in B$, $p \in L$. If $\varphi \in \Sigma$ with $E = \operatorname{dom}(\varphi)$, then

$$\gamma\Big(\bigoplus_{e\in E} e \otimes \varphi(e)\Big) = \bigoplus_{e\in E} \gamma(e \otimes \varphi(e)) = \bigoplus_{e\in E} [e \cdot \varphi(e)] = [\varphi]$$

by Corollary 4.4, and it follows that $\gamma\colon B \otimes L \to S$ is surjective. Finally, $a \otimes p \perp b \otimes q \Longrightarrow \gamma(a \otimes p) = [a \cdot p] \perp \gamma(b \otimes q) = [b \cdot q] \Longrightarrow a \cdot p \perp b \cdot q \Longrightarrow a \perp b$ or $p \perp q$ by Part (iii) of Lemma 4.2. $\square$

Corollary 4.6. *If $0 \neq b \in B$, P is a finite subset of L, and $\{b \otimes p \mid p \in P\}$ is an orthogonal subset of $B \otimes L$, then P is an orthogonal subset of L and $\bigoplus_{p\in P} b \otimes p = b \otimes \bigoplus P$.*

Lemma 4.7. *Suppose that $t \in B \otimes L$ has the form $t = \bigoplus_{a\in A} a \otimes \sigma(a)$, where A is a finite subset of B and $\sigma\colon A \to L$. Let $E \subseteq B$ be an FP such that, $a \in A \Longrightarrow a = \bigoplus_{e\in E,\, e\leqslant a} e$. Then:*

(i) *$e \in E \Rightarrow \{\sigma(a) \mid a \in A,\ e \leqslant a\}$ is an orthogonal set.*

(ii) *If $\varphi\colon E \to L$ is defined by $\varphi(e) := \bigoplus_{a\in A,\, e\leqslant a} \sigma(a)$, then $t = \bigoplus_{e\in E} e \otimes \varphi(e)$.*

Proof. For each fixed $e \in E$, we have $a \in A$ with $e \leqslant a \Rightarrow e \otimes \sigma(a) \leqslant a \otimes \sigma(a)$, and it follows that $\{e \otimes \sigma(a) \mid e \leqslant a \in A\}$ is an orthogonal subset of $B \otimes L$. Hence, by Corollary 4.6, $e \in E \Rightarrow \{\sigma(a) \mid e \leqslant a\}$ is an orthogonal subset of L and $\bigoplus_{a\in A,\, e\leqslant a} e \otimes \sigma(a) = e \otimes \varphi(e)$. Therefore,

$$\begin{aligned} t = \bigoplus_{a\in A} a \otimes \sigma(a) &= \bigoplus_{a\in A} \Big(\bigoplus_{e\in E,\, e\leqslant a} e\Big) \otimes \sigma(a) \\ = \bigoplus_{a\in A} \Big(\bigoplus_{e\in E,\, e\leqslant a} e \otimes \sigma(a)\Big) &= \bigoplus_{e\in E} \Big(\bigoplus_{a\in A,\, e\leqslant a} e \otimes \sigma(a)\Big) \\ &= \bigoplus_{e\in E} e \otimes \varphi(e). \end{aligned}$$

$\square$

Lemma 4.8. *Every element $t \in B \otimes L$ can be written in the form $t = \bigoplus_{e\in E} e \otimes \varphi(e)$, where $E \subseteq B$ is an FP and $\varphi\colon E \to L$.*

P r o o f. We can write t in the form $t = \bigoplus_{i \in I} a_i \otimes p_i$, where I is a finite, nonempty indexing set, $a_i \in B$, and $p_i \in L$ for all $i \in I$. Let $A := \{a_i \mid i \in I\}$ and, for each $a \in A$, let $I_a := \{i \in I \mid a_i = a\}$. By Corollary 4.6, $a \in A \Longrightarrow \{p_i \mid i \in I_a\}$ is an orthogonal subset of L and $\bigoplus_{i \in I_a} a \otimes p_i = a \otimes \sigma(a)$, where $\sigma\colon A \to L$ is defined by $\sigma(a) := \bigoplus_{i \in I_a} p_i$. Therefore, $t = \bigoplus_{a \in A} (\bigoplus_{i \in I_a} a \otimes p_i) = \bigoplus_{a \in A} a \otimes \sigma(a)$. Let E be the set of all nonzero elements of B having the form $e = \bigwedge_{a \in A} \varepsilon(a)$, where, for each $a \in A$, $\varepsilon(a)$ is either a of a'. Then E is a FP and $a \in A \Longrightarrow a = \bigoplus_{e \in E,\, e \leqslant a} e$. An application of Lemma 4.7 now completes the proof. □

Corollary 4.9. *If $t \in B \otimes L$, there exists $\varphi \in \Sigma$ such that $t = \bigoplus_{e \in \mathrm{dom}(\varphi)} e \otimes \varphi(e)$ and $\gamma(t) = [\varphi]$.*

P r o o f. Lemmas 4.8, 4.5, and 4.3. □

Lemma 4.10. *If $E \subseteq B$ is an FP, $\varphi\colon E \to L$, and $t = \bigoplus_{e \in E} e \otimes \varphi(e)$, then $t' = \bigoplus_{e \in E} e \otimes \varphi(e)'$.*

P r o o f. $1 = 1 \otimes 1 = (\bigoplus_{e \in E} e) \otimes 1 = \bigoplus_{e \in E} e \otimes 1 = \bigoplus_{e \in E} e \otimes \big(\varphi(e) \oplus \varphi(e)'\big) = (\bigoplus_{e \in E} e \otimes \varphi(e)) \oplus (\bigoplus_{e \in E} e \otimes \varphi(e)')$. □

Theorem 4.11. *$\gamma\colon B \otimes L \to S$ is an isomorphism.*

P r o o f. Since γ is surjective, it suffices to prove that it is a monomorphism. Thus, let $s, t \in B \otimes L$ with $\gamma(s) \perp \gamma(t)$. By Corollary 4.9, there exist $\sigma, \tau \in \Sigma$ with $\mathrm{dom}(\sigma) = G$, $\mathrm{dom}(\tau) = H$ such that $s = \bigoplus_{g \in G} g \otimes \sigma(g)$, $t = \bigoplus_{h \in H} h \otimes \tau(h)$, $\gamma(s) = [\sigma]$, $\gamma(t) = [\tau]$ and $\sigma \perp \tau$. Let $E := \{g \wedge h \mid g \in G,\ h \in H,\ g \wedge h \neq 0\}$. Noting that E is an FP, $g \in G \Longrightarrow g = \bigoplus_{e \in E,\, e \leqslant g} e$ and $h \in H \Longrightarrow h = \bigoplus_{e \in E,\, e \leqslant h} e$. Applying Lemma 4.7 with t replaced by s and A replaced by G , we find that $s = \bigoplus_{e \in E} e \otimes \varphi(e)$, where $\varphi\colon E \to L$ is defined for $e \in E$ by $\varphi(e) := \bigoplus_{g \in G,\, e \leqslant g} \sigma(g)$. Likewise, $t = \bigoplus_{e \in E} e \otimes \psi(e)$, where $\psi\colon E \to L$ is defined for $e \in E$ by $\psi(e) := \bigoplus_{h \in H,\, e \leqslant h} \tau(h)$. By Corollary 4.9, $[\sigma] = \gamma(s) = [\varphi]$ and $[\tau] = \gamma(t) = [\psi]$, and it follows from $\sigma \perp \tau$ that $\varphi \perp \psi$. Therefore, $e \in E \Longrightarrow \varphi(e) \perp \psi(e) \Longrightarrow e \otimes \varphi(e) \leqslant e \otimes \psi(e)' \Longrightarrow s \leqslant t'$ by Lemma 4.10. Therefore, $\gamma(s) \perp \gamma(t) \Longrightarrow s \perp t$. □

5. Concluding remarks

In [10] the sum S of a Boolean algebra B and an OML L is shown to have the following properties:

(i) There exist isomorphism $f\colon B \to S_B$ and $g\colon L \to S_L$, where S_B, S_L are sub-OML's of S, such that $f(b) \wedge g(p) = 0$ iff $b = 0$ or $p = 0$.

(ii) There is no proper sub-OML of S that contains $f(B) \cup g(L)$.

(iii) If μ is a probability measure on B and ν is a probability measure on L, then there exists a probability measure $\mu\nu$ on S such that $\mu\nu(f(b)) = \mu(b)$ and $\mu\nu(g(p)) = \nu(p)$ for all $b \in B$, $p \in L$.

It is not difficult to show that, even if L is only an orthoalgebra, the sum S has analogous properties. Indeed, if we identify S with $B \otimes L$ by the isomorphism of Theorem 4.11, we can define $S_B := \{b\otimes 1 \mid b \in B\}$, $S_L := \{1\otimes p \mid p \in L\}$, $f(b) := b\otimes 1$ for $b \in B$, and $g(p) := 1 \otimes p$ for $p \in L$. Then S_B and S_L are suborthoalgebras of S and $f\colon B \to S_B$, $g\colon L \to S_L$ are isomorphisms. Even though S need not be a lattice, it turns out that the infimum $f(b) \wedge g(p)$ exists in S for all $b \in B$, $p \in L$, and we have $f(b) \wedge g(p) = (b \otimes 1) \wedge (1 \otimes p) = b \otimes p$. In particular, $f(b) \wedge g(p) = 0$ iff $b = 0$ or $p = 0$. Thus, the analogue of Condition (i) holds. The analogue of Condition (ii) would state that there is no proper suborthoalgebra of $B \otimes L$ that contains $f(B) \cup g(L)$ and is closed under existing finite infima. The analogue of Condition (iii) is a direct consequence of Theorem 2.7.

In [1] and [7] (see also [11]) it is shown that the sum S of a Boolean algebra B and an OML L is isomorphic to the bounded Boolean power $L[B]^*$ of L by B. By exactly the same argument, this result holds even if L is only an orthoalgebra. Therefore, we may conclude that the sum S, the tensor product $B \otimes L$, and the bounded Boolean power $L[B]^*$ are mutually isomorphic. The tensor product seems to be the only one of these three constructions that is available for the more general case in which B is replaced by an OML, and OMP, or an orthoalgebra (see [5] and [12]).

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