Czechoslovak Mathematical Journal

Radomír Halaš

Annihilators and ideals in ordered sets

Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 1, 127–134

Persistent URL: http://dml.cz/dmlcz/128502

Czechoslovak Mathematical Journal, 45 (120) 1995, Praha

ANNIHILATORS AND IDEALS IN ORDERED SETS

RADOMÍR HALAŠ, Olomouc

(Received February 2, 1993)

M. Mandelker [5] introduced and studied the concept of an annihilator in a lattice. If L is a lattice, $a, b \in L$, then the annihilator of a relative to b is the set

$$\langle a, b\rangle = \{x \in L;\ a \wedge x \leqslant b\},$$

the dual annihilator is

$$\langle a, b\rangle_d = \{x \in L;\ a \vee x \geqslant b\}.$$

He proved that L is distributive iff each annihilator of L is an ideal. Further, he found a necessary and sufficient condition for a distributive lattice to satisfy the identity $\langle a, b\rangle \vee \langle b, a\rangle_d = L$, where the symbol $\vee$ denotes the join in the lattice of all ideals of L.

Moreover, Davey and Nieminen [1] studied connections between modularity of L and the so called prime annihilator conditions.

The aim of this paper is to study analogous connections in the case of ordered sets. Let us recall some basic notions. Let $(S, \leqslant)$ be an ordered set, $X \subseteq S$; then we denote $U(X) = \{y \in S;\ y \geqslant x$ for all $x \in X\}$, $L(X) = \{y \in S;\ y \leqslant x$ for all $x \in X\}$. A subset $I \subseteq S$ is called an ideal (filter) of S if $LU(x, y) \subseteq I$ ($UL(x, y) \subseteq I$) whenever $x, y \in I$. An ideal (filter) $I \neq \emptyset$, $I \neq S$, is called prime if $L(x, y) \subseteq I$ ($U(x, y) \subseteq I$) implies $x \in I$ or $y \in I$. If an ideal (filter) is an up (down) directed set then it is called a u-ideal (l-filter). The set of all ideals (filters) of S forms a lattice $\mathrm{Id}(S)$ ($\mathrm{Fil}(S)$) with respect to set inclusion, see [2].

Recall from [4] that an ordered set S is called
distributive if $\forall a, b, c \in S\colon L(U(a, b), c) = LU(L(a, c), L(b, c))$,
modular if $\forall a, b, c \in S\colon a \leqslant c \Rightarrow L(U(a, b), c) = LU(a, L(b, c))$.
For $a, b \in S$ the annihilator of a relative to b is the set

$$\langle a, b\rangle = \{x \in S;\ UL(a, x) \supseteq U(b)\},$$

the dual annihilator is

$$\langle a,b\rangle_d = \{x \in S;\ LU(a,x) \supseteq L(b)\}.$$

The annihilator $\langle a,b\rangle$ of S is called prime if

(i) $\langle a,b\rangle \cap \langle b,a\rangle_d = \emptyset$,

(ii) $\langle a,b\rangle \cup \langle b,a\rangle_d = S$.

Proposition 1. *Let F, G be l-filters of an ordered set S. Then $F \vee G = \bigcup\{UL(a,b);\ a \in F, b \in G\}$, where the symbol $\vee$ denotes the join in* $\mathrm{Fil}(S)$.

Proof. See [2], Theorem 3. □

Lemma 1. *Let S be a distributive set and F an l-filter of S satisfying the condition*

$$(*)\qquad \forall x \in F\ \forall a,b \in S\colon U(a,b) \subseteq F \ \Rightarrow\ UL\big(x,U(a,b)\big) \subseteq F.$$

If the set of all filters of S containing F forms a chain, then F is a prime filter.

Proof. Let $a,b \in S$, $U(a,b) \subseteq F$. Let us denote $G = F \vee U(a)$, $H = F \vee U(b)$. Since filters containing F form a chain, we have e.g. $H \subseteq G$. Since both F and $U(a)$ are l-filters, by Proposition 1 we have $b \in UL(x,a)$ for some $x \in F$. Hence we obtain $U(b) \subseteq UL(x,a)$ and, by the condition $(*)$,

$$UL\big(x,U(a,b)\big) \subseteq F.$$

Using distributivity, we can derive

$$L\big(x,U(a,b)\big) = LU\big(L(a,x),L(b,x)\big) = L\big(UL(a,x) \cap UL(b,x)\big) \subseteq L(b).$$

Finally, we have $F \supseteq UL\big(x,U(a,b)\big) \supseteq U(b)$, hence $b \in F$ and F is prime. □

Remark. In a finite set S every l-filter F satisfies the condition $(*)$.

Lemma 2. *Let S be a distributive set satisfying the condition*

$(**)$ *for every $a,b \in S$ there exists $x \in F$ such that the sets $UL(a,x)$, $UL(b,x)$ are comparable,*

where F is a filter of S. Then the set of all filters of S containing F forms a chain.

Proof. Let $a,b \in S$, $x \in F$, $G,H \in \mathrm{Fil}(S)$, $G,H \supseteq F$. If $G \parallel H$, then let $a \in G \setminus H$, $b \in H \setminus G$. Further, let $UL(a,x) \supseteq UL(b,x)$. Since $a,x \in G$ we have $UL(a,x) \subseteq G$, thus $G \supseteq UL(a,x) \supseteq UL(b,x) \supseteq U(b)$, therefore $b \in G$, a contradiction with $b \notin G$. □

Now, by Lemmas 1 and 2 we infer the following

Theorem 1. *Let S be a distributive set and F a l-filter of S satisfying the condition $(*)$. Let us consider the conditions*

(1) $\forall a, b \in S\ \exists x \in F$: $UL(a,x)$ *and* $UL(b,x)$ *are comparable;*

(2) *filters of S containing F form a chain;*

(3) *F is a prime filter;*

(4) *F contains a prime filter.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Lemma 3. *Let S be a distributive set, $a, b \in S$. Then $(***)$ implies $(****)$, where*

$(***)$ *both $\langle a,b\rangle$ and $\langle b,a\rangle$ are up directed sets and $\langle a,b\rangle \vee \langle b,a\rangle = S$,*

$(****)$ *if F is a filter of S containing a prime filter P, then for $a, b \in S$ there exists $x \in F$ such that the sets $UL(a,x)$, $UL(b,x)$ are comparable.*

Proof. Let $P \subseteq F$, $z \in P$. Then $z \in S = \langle a,b\rangle \vee \langle b,a\rangle$ and since both $\langle a,b\rangle$, $\langle b,a\rangle$ are u-ideals, we obtain by Proposition 1 that $z \in LU(x,y)$ fore some $x \in \langle a,b\rangle$, $y \in \langle b,a\rangle$. This implies $z \leqslant k$ for every $k \in U(x,y)$. Since P is a filter and $z \in P$, we have $U(x,y) \subseteq P$ and as P is prime, $x \in P$ or $y \in P$. Suppose $x \in F$. Since $x \in \langle a,b\rangle$ we observe that $L(a,x) \subseteq L(b)$, $L(a,x) \subseteq L(b,x)$ and finally,

$$UL(a,x) \supseteq UL(b,x).$$

□

Definition. An ordered set S is called s-distributive if it satisfies the condition

$$L\big(U(a,b), U(c,d)\big) = LU\big(L(a,U(c,d)), L(b,U(c,d))\big)$$

for all $a, b, c, d \in S$.

Theorem 2. *Let S be an s-distributive set, $I \in \mathrm{Id}(S)$, $D \in \mathrm{Fil}(S)$, $D \cap I = \emptyset$. Let I_D be a maximal ideal of S satisfying the conditions $I_D \supseteq I$, $I_D \cap D = \emptyset$. If an ideal I_D satisfies also the conditions*

(i) *I_D is a u-ideal,*

(ii) $\forall a, b \notin I_D\ \forall x, x' \in I_D$:

$$L(a,b) \subseteq I_D \ \Rightarrow\ U\big(L(a,b), L(a,x'), L(b,x), L(x,x')\big) \cap I_D \neq \emptyset,$$

then I_D is a prime ideal.

Proof. According to (i), I_D is a u-ideal. Let I_D be not prime, i.e. $a, b \notin I_D$ but $L(a,b) \subseteq I_D$ for some $a, b \in S$. Since the ideals $L(a)$, $L(b)$ are u-ideals, by Proposition 1 we obtain in $\mathrm{Id}(S)$:

$$\begin{aligned} I_D \vee L(a) &= \cup\{LU(a,x)\,;\ x \in I_D\}, \\ I_D \vee L(b) &= \cup\{LU(b,y)\,;\ y \in I_D\}. \end{aligned}$$

Obviously, $\big(I_D \vee L(a)\big) \cap D \neq \emptyset$ and $\big(I_D \vee L(b)\big) \cap D \neq \emptyset$. Thus there exists $x, x' \in I_D$, $p, q \in D$ such that

$$p \in LU(a,x), \quad q \in LU(b,x').$$

Then we have $L(p) \subseteq LU(a,x)$, $L(q) \subseteq LU(b,x')$, therefore

$$UL(p,q) \supseteq U\big(LU(a,x) \cap LU(b,x')\big) = UL\big(U(a,x), U(b,x')\big).$$

Since S is an s-distributive set, we can derive

$$\begin{aligned} UL\big(U(a,x), U(b,x')\big) &= ULU\big(L(a, U(b,x')), L(x, U(b,x'))\big) \\ &= U\big(LU\big(L(a,b), L(a,x')\big), LU\big(L(b,x), L(x,x')\big)\big) \\ &= U\big(L(a,b), L(a,x'), L(b,x), L(x,x')\big). \end{aligned}$$

By the condition (ii) we have $U\big(L(a,b), L(a,x'), L(b,x), L(x,x')\big) \cap I_D \neq \emptyset$. Since $UL\big(U(b,x'), U(a,x)\big) \subseteq UL(p,q) \subseteq D$, we obtain $D \cap I_D \neq \emptyset$, a contradiction. □

Remark. If S is a finite set, then the condition (i) implies the condition (ii): $X = L(a,b) \cup L(a,x') \cup L(b,x) \cup L(x,x')$ is a finite subset of I_D, hence $I_D \cap U(X) \neq \emptyset$.

Example 1. Let us consider an ordered set S whose diagram is visualized in Fig. 1.

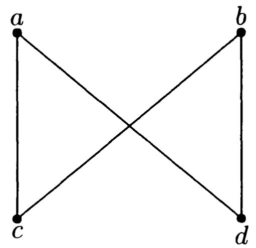

Fig. 1

It is an s-distributive set, $D = \{a, b\}$ is a filter of S, $I_D = I = \{c, d\}$. We can verify that neither I_D is a u-ideal nor I_D is prime.

Definition. Let S be an ordered set, $I \in \mathrm{Id}(S)$, $D \in \mathrm{Fil}(S)$, $I \cap D = \emptyset$. If I_D is the maximal ideal with $I_D \supseteq I$, $I_D \cap D = \emptyset$ satisfying the conditions (i) and (ii) of Theorem 2, then I is called a D-strong ideal.

Theorem 3. *Let S be a distributive set, $a, b \in S$, $I = \langle a, b\rangle \vee \langle b, a\rangle$. If there exists an element $z \in S \setminus I$ such that the filter $U(z)$ is I-strong, then the condition* (∗∗∗∗) *of Lemma 3 implies $I = S$.*

P r o o f. Let I be a proper ideal of S. According to theorem dual to Theorem 2, there exists a prime filter F containing z such that $F \cap I = \emptyset$. By the condition (∗∗∗∗) of Lemma 3 we have that there exists $x \in F$ such that $UL(a, x) \supseteq UL(b, x) \supseteq U(b)$, hence $x \in \langle a, b\rangle$, $x \in I$, thus $F \cap I \neq \emptyset$, a contradiction. □

Now, we shall show connection between prime annihilators and prime ideals in ordered sets.

Definition. An ordered set S is called 3-distributive if

$$U\big(b, L(a, x, y)\big) = UL\big(U(a, b), U(b, x), U(b, y)\big)$$

holds for all $a, b, x, y \in S$.

Theorem 4. *Let S be a distributive and 3-distributive set. Then every prime annihilator of S is a prime ideal.*

P r o o f. In a distributive set every prime annihilator is an ideal (see [3]). Let $\langle a, b\rangle$ be a prime annihilator. Let $L(x, y) \subseteq \langle a, b\rangle$ but $x, y \notin \langle a, b\rangle$, i.e. $x, y \in \langle b, a\rangle_d$. Then we obtain

$$U(x, b) \subseteq U(a),$$
$$U(y, b) \subseteq U(a),$$
$$L(z, a) \subseteq L(b) \text{ for every } z \in L(x, y). \tag{1}$$

We shall prove that $L(a, x, y) \subseteq L(b)$. Let $z^* \in L(a, x, y) \subseteq L(x, y)$. By (1) we have $L(z^*, a) \subseteq L(b)$. But $z^* \leqslant a$, hence $L(z^*) \subseteq L(b)$, thus $L(a, x, y) \subseteq L(b)$. Further, $U\big(b, L(a, x, y)\big) = U(b)$. By 3-distributivity we can derive

$$UL\big(U(b, x), U(b, a), U(b, y)\big) = U\big(b, L(a, x, y)\big) = U(b).$$

The inclusions (1) imply

$$UL\big(U(a), U(b, a), U(a)\big) = U(a) \supseteq U(b),$$

hence $a \leqslant b$. Then $\langle a, b\rangle = \langle b, a\rangle_d = S$, a contradiction with $\langle a, b\rangle$ being prime. □

Theorem 5. *Let S be an ordered set, $a, b \in S$, $a > b$. Then $\langle a, b\rangle \cup \langle b, a\rangle_d = S$ if and only if $a \succ b$ and the following condition* (P) *is satisfied:*

$$\text{(P)} \qquad \forall x \in S\colon \big(\exists z \in U\big(b, L(a,x)\big),\ z \parallel a\big) \ \Rightarrow\ x \in \langle a, b\rangle\,.$$

Proof. Let $\langle a, b\rangle \cup \langle b, a\rangle_d = S$, $z \in U\big(b, L(a,x)\big)$, $z \parallel a$. Then $z \notin \langle b, a\rangle_d$ since $U(b,z) = U(z) \nsubseteq U(a)$; hence $z \in \langle a, b\rangle$, i.e. $L(a,z) \subseteq L(b)$. But the opposite inclusion is trivially valid, thus $L(a,z) = L(b)$, i.e. there exists an infimum $a \wedge z = b$. Since $L(a,x) \subseteq L(a,z)$, we have $L(a,x) \subseteq L(b)$, thus $x \in \langle a, b\rangle$. If there exists $q \in S$ such that $a > q > b$, then $q \notin \langle a, b\rangle$ and $q \notin \langle b, a\rangle_d$, a contradiction.

Conversely, let the condition (P) be valid and $a \succ b$. Let $x \in S$, $x \notin \langle a, b\rangle$. We shall prove that $U\big(b, L(a,x)\big) = U(a)$. Obviously, $U\big(b, L(a,x)\big) \supseteq U(a)$. If $z \in U\big(b, L(a,x)\big)$, then $z \neq b$ since $x \notin \langle a, b\rangle$. Now, we have the following possibilities:

(1) $b < z < a$ — it can not occur since $a \succ b$,

(2) $z \parallel a$ — it can not occur since using the condition (P) we get $x \in \langle a, b\rangle$, a contradiction with the choice of x,

(3) $z \geqslant a$.

So we have proved that $U\big(b, L(a,x)\big) \subseteq U(a)$, thus

$$U\big(L(a,x), b\big) = U(a).$$

Further, this implies

$$U(x,b) = U\big(x, L(a,x), b\big) = U\big(b, L(a,x)\big) \cap U(x) = U(a,x) \subseteq U(a),$$

i.e. $LU(b,x) \supseteq L(a)$, i.e. $x \in \langle b, a\rangle_d$. □

Proposition 2. *Let S be a modular set, $a, b \in S$, $a > b$. Then*

$$\langle a, b\rangle \cap \langle b, a\rangle_d = \emptyset.$$

Proof. Let $x \in \langle a, b\rangle \cap \langle b, a\rangle_d$. Then $L(a,x) \subseteq L(b)$, $U(b,x) \subseteq U(a)$. Now, by modularity of S we obtain

$$L\big(a, U(b,x)\big) = L(a) = LU\big(b, L(a,x)\big) = LU\big(L(b)\big) = L(b),$$

hence $a = b$, a contradiction. □

By Proposition 2 and Theorem 5 we have the following consequence:

Corollary. *Let S be a modular set, $a, b \in S$, $a > b$. Then $\langle a, b\rangle$ is a prime annihilator iff $a \succ b$ and S satisfies the condition* (P).

Example 2. Let the diagram of S be given in Fig. 2.

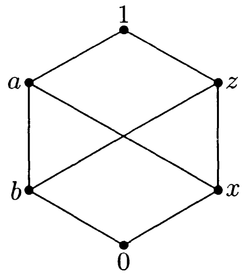

Fig. 2

It can be easily verified that $L(b)$ is prime ideal in the distributive set S. It is not a prime annihilator since $L(b) = \langle a, b\rangle = \langle z, b\rangle$, but these annihilators do not satisfy the condition (P) of Theorem 5: $z \in U\big(b, L(a, x)\big)$, $z \parallel a$, but $x \notin \langle a, b\rangle$.

Theorem 6. *Let S be a modular set, $m \in S$ such that there exists a unique minimal element m^* in $U(m) \setminus \{m\}$. Then $L(m) = \langle m^*, m\rangle$.*

Proof. Let m^* be the unique minimal element in $U(m) \setminus \{m\}$. Obviously, $L(m) \subseteq \langle m^*, m\rangle$. Let $x \in \langle m^*, m\rangle$, $x \nleq m$. Then $L(x, m^*) \subseteq L(m)$, $U(x, m) \subsetneqq U(m)$, hence for $z \in U(x, m)$ we get $z \geqslant m^*$, $U(z) \subseteq U(m^*)$, thus $m^* \in LU(m, x)$. This implies $m^* \in L\big(m^*, U(m, x)\big) = LU\big(m, L(x, m^*)\big)$ by modularity of S. Finally, we can derive $m^* \in LU\big(m, L(x, m^*)\big) \subseteq LU\big(m, L(m)\big) \subseteq L(m)$, thus $m^* \leqslant m$, a contradiction with the choice of m^*. □

Now, if m^* is the unique minimal element in $U(m) \setminus \{m\}$, then obviously $m^* \succ m$ and for the elements m^*, m the condition (P) is valid. Hence we obtain the next corollary:

Corollary. *Let S be a modular set, let $m \in S$ be such that there exists a unique minimal element in $U(m) \setminus \{m\}$. Then $L(m)$ is a prime annihilator.*

Definition. An ordered set S is called complemented if for each $a \in S$ there exists $a' \in S$ such that $UL(a, a') = LU(a, a') = S$. A distributive and complemented set is called boolean. An element $q \in S$ is called a coatom if either q is a maximal element of S if S has no greatest element or $1 \succ q$ whenever S has the greatest element 1.

Proposition 3. *Let S be a finite boolean set. Then every prime ideal of S is of the form $L(x)$, where x is a coatom of S.*

Proof. See [2]. □

By Proposition 3 and the preceding corollary we infer

Corollary. *Let S be a finite boolean set with the greatest element 1. Then every prime ideal of S is a prime annihilator.*

Proof. Prime ideals of S are of the form $L(m)$, where m is a maximal element of S. We put $m^* = 1$. Obviously, $m^* \succ m$ and by the preceding Corollary we have $L(m) = \langle 1, m \rangle$. Moreover, $\langle 1, m \rangle$ is a prime annihilator. □

Remark. If an ordered set is distributive only, the last corollary need not be true. It suffices to consider the set shown in Fig. 2.

References

[1] *Davey B., Nieminen J.*: Annihilators in modular lattices. Alg. Univ. *22* (1986), 154–158.
[2] *Halaš R.*: Ideals in ordered sets, preprint.
[3] *Halaš R.*: A characterization of distributive sets by generalized annihilators. To appear.
[4] *Chajda I., Rachůnek J.*: Forbidden configurations of distributive and modular ordered sets. Order *5* (1989), 407–423.
[5] *Mandelker M.*: Relative annihilators in lattices. Duke Math. J. *40* (1970), 377–386.

Author's address: Department of Algebra and Geometry, Palacký University Olomouc, Faculty of Sci., Tomkova 38, 779 00 Olomouc, Czech Republic.