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ON REGULARITY OF INDUCTIVE LIMITS

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Throughout the paper $E_1 \subset E_2 \subset \dots$ is a sequence of Hausdorff locally convex spaces with continuous identity maps: $E_n \to E_{n+1}$, $n \in \mathbb{N}$, and $E = \operatorname{ind} E_n$ their inductive limit.

We use the following notation: The convex, resp. linear, hull of a set $S \subset E$ is denoted by $\operatorname{co} S$, resp. E_S; the symbol $\operatorname{cl}_E S$ stands for the closure of S in the space E. For any $n \in \mathbb{N}$, we write $\tau_n = \operatorname{top} E_n$, $\tau = \operatorname{top} E$, $\sigma_n = \sigma(E_n, E_n')$ is the weak topology on E_n, and $\tau(S)$ is the topology on S generated by τ.

In [2] Makarov introduced the following terminology: An inductive limit $\operatorname{ind} E_n$ is called

α-regular if any set bounded in $\operatorname{ind} E_n$ is contained in some E_n

β-regular if any set, which is bounded in $\operatorname{ind} E_n$ and contained in E_n is bounded in some E_m,

regular if it is both α- and β-regular.

We need two more notions, $\operatorname{ind} E_n$ is called:

uniformly β-regular if for any $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that any set bounded in $\operatorname{ind} E_n$ and contained in E_n is bounded in E_m,

uniformly regular if it simultaneously α-regular and uniformly β-regular.

The Dieudonné-Schwartz Theorem, [1; §4, Prop. 4] or [3; Ch. 2, §12, Th. 2] states that $E = \operatorname{ind} E_n$ is regular provided that:

(H-1) each space E_n is closed in E,

(H-2) each $\tau_n = \tau(E_n)$.

Theorem 1. (a) H-1 $\Longrightarrow$ *E is α-regular,*

(b) H-2 $\Longrightarrow$ *E is uniformly β-regular.*

P r o o f. Put $F_n = (\operatorname{cl}_E E_n, \tau)$, $n \in \mathbb{N}$. Then the $\operatorname{ind} F_n$ is strict and equal to $\operatorname{ind} E_n$. Hence, by Dieudonné-Schwartz Theorem, each set bounded in E is contained in some $\operatorname{cl}_E E_n = E_n$.

The second claim is evident.

We use four more hypotheses:

(H-3) for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\mathrm{cl}_E E_n \subset E_m$,

(H-4) there exists a sequence $\{G_n\}$, where each G_n is a 0-heighborhood in E_n, such that for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ for which $\mathrm{cl}_E \operatorname{co} \bigcup \{G_k\,;\ k \leqslant n\} \subset E_m$,

(H-5) for every $n \in \mathbb{N}$, there exists $m \geqslant n$ such that $\tau(E_n) \supset \sigma_m(E_n)$,

(H-6) for every set $B \subset E_n$, bounded in E, there exists $m \geqslant n$ such that $\tau(E_B) \supset \sigma_m(E_B)$. □

Theorem 2. H-1 $\Longrightarrow$ H-3 $\Longrightarrow$ H-4 $\Longrightarrow$ *E is α-regular. If all spaces E_n are normable, the last implication can be reversed.*

Proof. The first two implications are evident. To prove the third one, assume H-4 and E not α-regular. Then there exists an absolutely convex set $B \subset E$ which is bounded in E and not contained in any space E_n. By taking a subsequence of $\{E_n\}$, we may assume that in the hypothesis H-4 we can put $m = n + 1$.

Take a sequence $\{b_n\} \subset B$ such that $b_n \notin E_n$, $n \in \mathbb{N}$. Since $b_1 \neq 0$, there exists an absolutely convex, closed in E, 0-neighborhood $U_1 \subset E$ such that $b_1 \notin U_1$. Put $V_1 = U_1 \cap G_1$ and $W_1 = \mathrm{cl}_E V_1$. Then $W_1 \subset U_1$ and $b_1 \notin W_1$. Further, by H-4, we have $W_1 \subset \mathrm{cl}_E G_1 \subset E_2$ which implies $\frac{1}{2}b_2 \notin W_1$. Hence there exists an absolutely convex, closed in E, 0-neighborhood $U_2 \subset E$ such that b_1, $\frac{1}{2}b_2 \notin W_1 + U_2 + U_2$. Put $V_2 = U_2 \cap G_2$ and $W_2 = \mathrm{cl}_E \operatorname{co}(V_1 \cup V_2)$. Then $\operatorname{co}(V_1 \cup V_2) \subset W_1 + U_2$, $W_2 \subset \mathrm{cl}_E(W_1 + U_2) \subset W_1 + U_2 + U_2$, and b_1, $\frac{1}{2}b_2 \notin W_2$. Since $\operatorname{co}(V_1 \cup V_2) \subset \operatorname{co}(G_1 \cup G_2)$, H-4 implies $W_2 \subset \mathrm{cl}_E \operatorname{co}(G_1 \cup G_2) \subset E_3$ and $\frac{1}{3}b_3 \notin W_2$. Hence there exists an absolutely convex, closed in E, 0-neighborhood $U_3 \subset E$ such that b_1, $\frac{1}{2}b_2$, $\frac{1}{3}b_3 \notin W_2 + U_3 + U_3$, etc.

When all 0-neighborhoods $V_n \subset E_n$, $n \in \mathbb{N}$, are constructed, the set $V = \operatorname{co} \bigcup \{V_n\,;\ n \in \mathbb{N}\}$ is a 0-neighborhood in E for which $\frac{1}{k}b_k \notin V$, $k \in \mathbb{N}$. Thus V does not absorb B, a contradiction.

Assume all spaces E_n are normable and E is α-regular. For each $n \in \mathbb{N}$, let G_n be an open ball in E_n. Since all maps $\mathrm{id} : E_n \to E_{n+1}$ are continuous, we may choose each G_n so that $G_1 \subset G_2 \subset \ldots$. In this case $\operatorname{co} \bigcup \{G_k\,;\ k \leqslant n\} = G_n$. Now, G_n is bounded in E_n, hence also bounded, together with its $\mathrm{cl}_E G_n$, in the space E. By α-regularity of E, there exist $m \in \mathbb{N}$ for which $\mathrm{cl}_E G_n \subset E_m$, i.e. H-4 holds. □

Theorem 3. H-2 $\Longrightarrow$ H-5 $\Leftrightarrow$ *E is uniformly β-regular.*

Proof. The first implication is evident.

Assume H-5 and fix $n \in \mathbb{N}$. There exists $m \in \mathbb{N}$ such that every τ-bounded set in E_n is weakly bounded in E_m, hence also bounded in E_m.

Assume H-5 does not hold. Then there exists $n \in \mathbb{N}$ such that, for any $m \geqslant n$, the topology $\tau(E_n)$ is not stronger than $\sigma_m(E_n)$. This implies that, for each $m \geqslant n$, the set families $\mathscr{D}_m = \{D \subset E_n;\, D \text{ is } \sigma_m\text{-bounded}\}$ and $\mathscr{D} = \{D \subset E_n;\, D \text{ is } \tau\text{-bounded}\}$ are not equal. Since $\mathscr{D}_n \subset \mathscr{D}_{n+1} \subset \ldots \subset \mathscr{D}$, we have $\mathscr{D} \setminus \mathscr{D}_m \neq \emptyset$, for any $m \geqslant n$, i.e. E is not uniformly β-regular. □

Theorem 4. H-6 $\Longrightarrow$ *E is β-regular.*

Proof. Let $B \subset E_n$ be bounded in E. Then B is also $\tau(E_B)$-bounded. By H-6, there exists $m \in \mathbb{N}$, $m \geqslant n$, such that $\tau(E_B) \supset \sigma_m(E_B)$. Thus B is $\sigma_m(E_B)$-bounded, hence $\sigma_m(E_m)$-bounded and also τ_m-bounded. □

Example. We construct a regular inductive limit of Hilbert spaces which does not satisfy H-6. So the implication is Theorem 4 cannot be reversed in case that all spaces E_n are normable.

Let $E_n = \{x\colon [0,\infty) \to R,\ \|x\|_n^2 = \int_0^\infty x^2(t)\exp(-2nt)\,\mathrm{d}t < +\infty\}$, $n \in \mathbb{N}$. Then all spaces E_n are Hilbert and, by [4, Th. 4], their inductive limits is regular, hence also β-regular. For each $k, m \in \mathbb{N}$, put $x_{k,m}(t) = \psi_{[0,k]}(t)\exp(mt)$, where $\psi_{[0,k]}$ is the charakteristic function of $[0,k]$. It is easy to establish that:

(a) $x_{k,m} \in E_1$, $k, m \in \mathbb{N}$,

(b) $\lim\limits_{k\to\infty} \|x_{k,m} - \exp(mt)\|_{m+1} = 0$, $m \in \mathbb{N}$,

(c) $\lim\limits_{k\to\infty} \|x_{k,m}\|_m = +\infty$, $m \in \mathbb{N}$.

Denote by B the unit ball in E_1 and for any $m \in \mathbb{N}$ put $B_m = \{x_{k,m};\, k \in \mathbb{N}\}$. Then, by (a), each $B_m \subset E_1 = E_B$. By (b), B_m is bounded in E_{m+1}, hence it is also bounded in $\operatorname{ind} E_n$. On the other hand, by (c), B_m is not bounded in E_m. This implies that the topology $\tau(E_B)$ is not stronger than $\sigma_m(E_B)$ for any $m \in \mathbb{N}$.

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