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ON THE SOLUTION SETS OF SOME NONCONVEX HYPERBOLIC DIFFERENTIAL INCLUSIONS

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper \( \mathbb{R}^q \) denotes a real \( q \)-dimensional Euclidean space with norm \( | \cdot | \), \( M \) a separable metric space, \( Q \) the square \( I \times I \) with \( I = [0,1] \). Let \( F \) be a multifunction from \( Q \times \mathbb{R}^q \times M \) to the nonempty compact subsets of \( \mathbb{R}^q \). Let \( \lambda(x,y) = \alpha(x) + \beta(y) - \alpha(0) \), where \( \alpha \) and \( \beta \) are continuous functions from \( I \) to \( \mathbb{R}^q \) satisfying \( \alpha(0) = \beta(0) \).

Under suitable assumptions on \( F \), we consider the Darboux problem for hyperbolic differential inclusions of the form

\[
(D_{\lambda,\mu}) \quad \begin{cases} u_{xy}(x,y) \in F(x,y,u(x,y),\mu), \\ u(x,0) = \lambda(x,0), \quad u(0,y) = \lambda(0,y). \end{cases}
\]

Denote by \( \mathcal{F}(\lambda,\mu) \) the solution set of \( (D_{\lambda,\mu}) \). We prove that if \( F \) satisfies (among other assumptions) a Lipschitz condition with respect to \( u \), then \( \mathcal{F}(\lambda,\mu) \) is a retract of a convex subset of a Banach space. Furthermore, the retraction can be constructed as to depend continuously upon \( (\lambda,\mu) \). From this it follows that \( \mathcal{F}(\lambda,\mu) \) is contractible in itself, and that the multifunction \( (\lambda,\mu) \rightarrow \mathcal{F}(\lambda,\mu) \) admits a continuous selection. Finally it is shown that any two continuous selections of this multifunction can be joined by a homotopy with values in \( \mathcal{F}(\lambda,\mu) \).

Contributions to the study of the topological structure of the solution sets to hyperbolic differential equations or inclusions of the form \( (D_{\lambda,\mu}) \) can be found in Górniiewicz and Pruszko [6], Teodoru [12], Staicu [11]. In particular, in [6] it is shown that the solution set of \( (D_{\lambda,\mu}) \) with \( F \) single valued is an \( R_\delta \)-set. Similar problems for other types of differential equations or inclusions have been studied by
many authors, including Himmelberg and Van Vleck [9], Cellina [3], Deimling [4], Papageorgiou [10].

2. NOTATION AND PRELIMINARIES

Let $Z$ be a metric space with distance $d_Z$. For $a \in Z$ and $B$ a nonempty subset of $Z$, we put $d_Z(a, B) = \inf_{b \in B} d_Z(a, b)$. We denote by $\mathcal{C}(Z)$ the space of all nonempty closed bounded subsets of $Z$, endowed with the Hausdorff metric

$$H_Z(A, B) = \max \left\{ \sup_{a \in A} d_Z(a, B), \sup_{b \in B} d_Z(b, A) \right\}, \quad A, B \in \mathcal{C}(Z).$$

Let $Y$ be a measurable metric space with $\sigma$-algebra $\mathcal{A}$ and let $Z$ be a separable metric space. A multifunction $F: Y \to \mathcal{C}(Z)$ is called measurable (see Himmelberg [8]) if \{ $y \in Y \mid F(y) \cap D \neq \emptyset$ \} $\in \mathcal{A}$ for every closed subset $D$ of $Z$. The Borel $\sigma$-algebra of $Z$ is denoted by $\mathcal{B}(Z)$. In the sequel $Q$, as measurable space, is given the $\sigma$-algebra $\mathcal{L}$ of the Lebesgue measurable subsets of $Q$.

We denote by $C$ the Banach space of all continuous functions $u: Q \to \mathbb{R}^q$, equipped with the norm $\|u\|_C = \sup_{(x,y)\in Q} |u(x,y)|$. Given a continuous strictly positive function $a: Q \to \mathbb{R}$, we denote by $L^1$ the Banach space of all (equivalence classes of) Lebesgue measurable functions $\sigma: Q \to \mathbb{R}^q$, endowed with the norm

$$\|\sigma\|_{L^1} = \iint_Q a(x,y) |\sigma(x,y)| \, dx \, dy.$$  

Furthermore, by $V$ we mean the linear subspace of $C(Q, \mathbb{R}^q)$ consisting of all $\lambda \in C$ such that there exist continuous functions $\alpha: I \to \mathbb{R}^q$ and $\beta: I \to \mathbb{R}^q$, with $\alpha(0) = \beta(0)$, satisfying $\lambda(x,y) = \alpha(x) + \beta(y) - \alpha(0)$ for every $(x,y) \in Q$. Observe that $V$, equipped with the norm of $C$, is a separable Banach space.

In the sequel, when a product $Z = Z_1 \times \ldots \times Z_n$ of metric spaces $Z_i$, $i = 1, \ldots, n$, is considered, it is assumed that $Z$ is given the metric $\max_{1 \leq i \leq n} d_{Z_i}(x_i, y_i)$, where $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in Z$.

Following Hiai and Umegaki [7], a set $K \subset L^1$ is called decomposable if for every $u, v \in K$ and $A \in \mathcal{L}$ we have $u\chi_A + v\chi_{\mathbb{R}^q \setminus A} \in K$, where $\chi_A$ stands for the characteristic function of $A$. We set $\mathcal{D}(L^1) = \{ X \in \mathcal{C}(L^1) \mid X \text{ is decomposable} \}$.

Let $T$ be a Hausdorff topological space. A subspace $X$ of $T$ is called a retract of $T$ if there is a continuous map $\varphi: T \to X$ such that $\varphi(x) = x$ for every $x \in X$.

In order to treat problem $(D_{\lambda, \mu})$ we introduce the following
Assumption (A). The multifunction $F: Q \times \mathbb{R}^q \times M \to \mathcal{C}(\mathbb{R}^q)$ satisfies:

(a1) $F$ is $\mathcal{L} \otimes \mathcal{B}(\mathbb{R}^q \times M)$-measurable,

(a2) for each $(x, y, u) \in Q \times \mathbb{R}^q$ the multifunction $\mu \to F(x, y, u, \mu)$ is Hausdorff continuous on $M$,

(a3) there exist positive integrable functions $h: Q \to \mathbb{R}$ and $k: Q \to \mathbb{R}$ such that

$$H_{\mathbb{R}^q}(F(x, y, u_1, \mu), F(x, y, u_2, \mu)) \leq k(x, y)|u_1 - u_2| \quad \text{for every } (x, y, u_i, \mu) \in Q \times \mathbb{R}^q \times M, \quad i = 1, 2.$$

For $(x, y) \in Q$ and $\varepsilon > 0$, we put:

$$Q(x, y) = [0, x] \times [0, y], \quad R(x, y) = [x, 1] \times [y, 1],$$

$$P(x, y; \varepsilon) = [x - \varepsilon, x + \varepsilon] \times [y - \varepsilon, y + \varepsilon].$$

For $(\lambda, \sigma) \in V \times L^1$, consider the following Darboux problem

$$(C_{\lambda, \sigma}) \begin{cases} u_{xy}(x, y) = \sigma(x, y), \\
 u(x, 0) = \lambda(x, 0), \quad u(0, y) = \lambda(0, y). \end{cases}$$

**Definition 1.** Let $(\lambda, \sigma) \in V \times L^1$. The function $u \in C$ given by

$$u(x, y) = \lambda(x, y) + \iint_{Q(x, y)} \sigma(\xi, \eta) \, d\xi \, d\eta \quad \text{for } (x, y) \in Q,$$

is said to be solution of $(C_{\lambda, \sigma})$.

Clearly $(C_{\lambda, \sigma})$ has a unique solution which, in the sequel, will be denoted by $u^{\lambda, \sigma}$.

**Definition 2.** Let $(A)$ be satisfied. Let $(\lambda, \mu) \in V \times M$. A function $u \in C$ is said to be solution of $(D_{\lambda, \mu})$ if there exists a function $\sigma \in L^1$ such that:

$$\sigma(x, y) \in F(x, y, u(x, y), \mu) \quad \text{for } (x, y) \in Q \text{ a.e.,}$$

$$u(x, y) = \lambda(x, y) + \iint_{Q(x, y)} \sigma(\xi, \eta) \, d\xi \, d\eta \quad \text{for every } (x, y) \in Q.$$

We denote by $\mathcal{S}(\lambda, \mu)$ the solution set of $(D_{\lambda, \mu})$, i.e. the set of all solutions of $(D_{\lambda, \mu})$.

**Proposition 1.** Let $k: Q \to \mathbb{R}$ be a positive integrable function. Then there exists a continuous strictly positive function $a: Q \to \mathbb{R}$ which, for each $(x, y) \in Q$, satisfies

$$(2.2) \quad \iint_{R(x, y)} k(\xi, \eta)a(\xi, \eta) \, d\xi \, d\eta = \frac{1}{2} (a(x, y) - 1).$$
Proof. For \( n \in \mathbb{N} \) set \( x_i = i/n, \ i = 0,1,\ldots, n \). Fix \( n \in \mathbb{N} \) so that 
\[
2k(\xi, \eta) d\xi d\eta < 1, \quad i = 1,2,\ldots, n.
\]
By using the Banach-Caccioppoli fixed point theorem, it is easy to show that there is
a continuous strictly positive function \( a_n : [x_{n-1}, x_n] \times I \rightarrow \mathbb{R} \) satisfying (2.2) (with \( a_n \)
in the place of \( a \)) for every \((x, y) \in [x_{n-1}, x_n] \times I \). Then, recursively, one can construct
continuous strictly positive functions \( a_i : [x_{i-1}, x_i] \times I \rightarrow \mathbb{R}, \ i = 1,2,\ldots, n - 1, \)
satisfying
\[
\int_{[x,x_i] \times [y,1]} k(\xi, \eta) a_i(\xi, \eta) d\xi d\eta = \frac{1}{2} (a_i(x,y) - a_{i+1}(x,y)),
\]
for every \((x, y) \in [x_{i-1}, x_i] \times I \). Define \( a : Q \rightarrow \mathbb{R} \) by \( a(x, y) = \sum a_i(x, y) \chi_{U_i}(x, y) \),
where \( U_1 = [x_0, x_1] \times I \) and \( U_i = (x_{i-1}, x_i) \times I, \ i = 2,\ldots, n \). It is routine to verify
that the function \( a \) is continuous, strictly positive, and that \( a \) satisfies (2.2) for every
\((x, y) \in Q \). This completes the proof. \( \square \)

Proposition 2. The map \( T : V \times L^1 \rightarrow C \) given by \( T(\lambda, \sigma) = u^{\lambda, \sigma} \), where \( u^{\lambda, \sigma} \)
is the solution of \((C_{\lambda, \sigma})\), is linear and one-to-one.

Proof. Clearly \( T \) is linear. To show that \( T \) is one-to-one, suppose that
\( T(\lambda_1, \sigma_1) = T(\lambda_2, \sigma_2) \) for some \((\lambda_i, \sigma_i) \in V \times L^1, \ i = 1,2 \). This implies \( \lambda_1 = \lambda_2 \)
and thus, setting \( \sigma = \sigma_1 - \sigma_2 \), we have
\[
\int_{Q(x,y)} \sigma(\xi, \eta) d\xi d\eta = 0 \quad \text{for every} \quad (x, y) \in Q.
\]
Let \( L \) be the set of all Lebesgue points of \( \sigma \) belonging to the interior of \( Q \), and
observe that \( Q \setminus L \) has Lebesgue measure zero. Let \((\xi, \eta) \in L \) be arbitrary. For
\( \varepsilon > 0 \) sufficiently small, we have
\[
\sigma(\xi, \eta) = \frac{1}{4\varepsilon^2} \int_{P(\xi,\eta;\varepsilon)} (\sigma(\xi, \eta) - \sigma(x,y)) dx dy + \frac{1}{4\varepsilon^2} \int_{P(\xi,\eta;\varepsilon)} \sigma(x,y) dx dy.
\]
The first integral vanishes as \( \varepsilon \to 0 \) by virtue of a result from [5, p. 217]. The second
one is zero, as consequence of (2.3) and of the equality
\[
\int_{P(\xi,\eta;\varepsilon)} \sigma(x,y) dx dy = \int_{Q(\xi+\varepsilon,\eta+\varepsilon)} \sigma(x,y) dx dy + \int_{Q(\xi-\varepsilon,\eta-\varepsilon)} \sigma(x,y) dx dy - \int_{Q(\xi+\varepsilon,\eta-\varepsilon)} \sigma(x,y) dx dy - \int_{Q(\xi-\varepsilon,\eta+\varepsilon)} \sigma(x,y) dx dy.
\]
Letting \( \varepsilon \to 0 \), (2.4) gives \( \sigma(\xi, \eta) = 0 \), thus \( \sigma_1 = \sigma_2 \). Hence \((\lambda_1, \sigma_1) = (\lambda_2, \sigma_2)\), which
implies that \( T \) is one-to-one. This completes the proof. \( \square \)
3. MAIN RESULTS

Let assumption (A) be satisfied. Let \((\lambda, \mu, \sigma) \in V \times M \times L^1\). Let \(u^{\lambda, \sigma}: Q \to \mathbb{R}^q\) be the solution of \((C_{\lambda, \sigma})\). We put

\[
\begin{align*}
\mathcal{Y}(\lambda, \mu, \sigma) &= \{ \phi \in L^1 | \phi(x, y) \in F(x, y, u^{\lambda, \sigma}(x, y), \mu), (x, y) \in Q \text{ a.e.} \}, \\
\mathcal{I}(\lambda, \mu) &= \{ \phi \in L^1 | \phi \in \mathcal{Y}(\lambda, \mu, \phi) \}.
\end{align*}
\]

Observe that \(\mathcal{Y}(\lambda, \mu, \sigma)\) is a decomposable closed bounded subset of \(L^1\), thus (3.1) defines a multifunction \(\mathcal{Y}: V \times M \times L^1 \to \mathcal{D}(L^1)\).

Furthermore, set

\[W = \{ u \in C | u = u^{\lambda, \sigma} \text{ for some } (\lambda, \sigma) \in V \times L^1 \}.
\]

By Proposition 2, for each \(u \in W\) there is one and only one \((\lambda, \sigma) \in V \times L^1\) such that \(u = u^{\lambda, \sigma}\). In view of that, we write \(u^{\lambda, \sigma}\) to denote an arbitrary member of \(W\). Let \(k\) be the positive integrable function occurring in assumption (A). By Proposition 1, there is a continuous strictly positive function \(a: Q \to \mathbb{R}\) satisfying (2.2) for every \((x, y) \in Q\). With this choice of \(a\), for arbitrary \(u^{\lambda, \sigma} \in W\) we set

\[
(3.3) \quad \|u^{\lambda, \sigma}\|_W = \|u^{\lambda, \sigma}\|_C + \|\sigma\|_{L^1},
\]

where \(\|\sigma\|_{L^1}\) is given by (2.1). By using Proposition 2, it is easy to check that (3.3) defines a norm on \(W\) and that, under this norm, \(W\) is a Banach space.

For \(\lambda \in V\), set

\[W(\lambda) = \{ u \in W | u(x, 0) = \lambda(x, 0) \text{ for } x \in I, \ u(0, y) = \lambda(0, y) \text{ for } y \in I \}.
\]

We observe that \(W(\lambda)\) is a nonempty convex closed subset of \(W\) satisfying

\[\mathcal{I}(\lambda, \mu) \subset W(\lambda) \text{ for every } \mu \in M.\]

**Theorem 1.** Let assumption (A) be satisfied. Let \(G = \{ (\lambda, \mu, u) \in V \times M \times W | (\lambda, \mu) \in V \times M, u \in W(\lambda) \}\). Then there exists a continuous function \(\Phi: G \to W\) satisfying, for each \((\lambda, \mu) \in V \times M\), the following properties:

\[
\begin{align*}
(3.4) \quad & \Phi(\lambda, \mu, u) \in \mathcal{I}(\lambda, \mu) \text{ for every } u \in W(\lambda), \\
(3.5) \quad & \Phi(\lambda, \mu, u) = u \text{ for every } u \in \mathcal{I}(\lambda, \mu).
\end{align*}
\]

**Proof.** Let \(\mathcal{Y}: V \times M \times L^1 \to L^1\) be defined by (3.1).
(i) \( \mathcal{Y} \) is Hausdorff continuous. To this end we prove first that \( \mathcal{Y} \) is Hausdorff lower semicontinuous. Suppose the contrary. Then there exist an \( \varepsilon > 0 \), a sequence \( \{ (\lambda_n, \mu_n, \sigma_n) \} \) converging to \( (\lambda_0, \mu_0, \sigma_0) \) in \( V \times M \times L^1 \), and a sequence \( \{ q_n \} \subseteq L^1 \), with \( q_n \in \mathcal{Y} (\lambda_0, \mu_0, \sigma_0) \) for each \( n \in \mathbb{N} \), such that

\[
d_{L^1} (q_n, \mathcal{Y} (\lambda_n, \mu_n, \sigma_n)) \geq \varepsilon \quad \text{for every } n \in \mathbb{N}.
\]

For \( n \in \mathbb{N} \) define \( M_n : Q \to \mathcal{C}(\mathbb{R}^q) \) by

\[
M_n (x, y) = F(x, y, u^\lambda_n, \sigma_n (x, y), \mu_n)
\]

\[
\cap \mathcal{B}_{\mathbb{R}^q} (q_n (x, y), d_{\mathbb{R}^q} (q_n (x, y), F(x, y, u^\lambda_n, \sigma_n (x, y), \mu_n))),
\]

where, for \( a \in \mathbb{R}^q \) and \( r \geq 0 \), \( \mathcal{B}_{\mathbb{R}^q} (a, r) = \{ x \in \mathbb{R}^q \mid |x - a| \leq r \} \). As \( M_n \) is measurable, there exists a measurable selection \( \tilde{q}_n \in \mathcal{Y} (\lambda_n, \mu_n, \sigma_n) \) such that

\[
|q_n (x, y) - \tilde{q}_n (x, y)| = d_{\mathbb{R}^q} (q_n (x, y), F(x, y, u^\lambda_n, \sigma_n (x, y), \mu_n)) \quad \text{for } (x, y) \in Q \text{ a.e.}
\]

From this, observing that \( q_n (x, y) \in F(x, y, u^{\lambda_0, \sigma_0} (x, y), \mu_0) \), one has:

\[
\int_{Q} a(x, y) |q_n (x, y) - \tilde{q}_n (x, y)| \, dx \, dy
\]

\[
= \int_{Q} a(x, y) d_{\mathbb{R}^q} (q_n (x, y), F(x, y, u^\lambda_n, \sigma_n (x, y), \mu_n)) \, dx \, dy
\]

\[
\leq \int_{Q} a(x, y) H_{\mathbb{R}^q} (F(x, y, u^{\lambda_0, \sigma_0} (x, y), \mu_0), F(x, y, u^\lambda_n, \sigma_n (x, y), \mu_n)) \, dx \, dy
\]

\[
\leq \int_{Q} a(x, y) H_{\mathbb{R}^q} (F(x, y, u^\lambda_n, \sigma_n (x, y), \mu_n), F(x, y, u^{\lambda_0, \sigma_0} (x, y), \mu_n)) \, dx \, dy
\]

\[
+ \int_{Q} a(x, y) H_{\mathbb{R}^q} (F(x, y, u^{\lambda_0, \sigma_0} (x, y), \mu_n), F(x, y, u^{\lambda_0, \sigma_0} (x, y), \mu_0)) \, dx \, dy.
\]

Denoting by \( w_n (x, y) \) the function under the sign of the last integral, and using assumption (A) (a_3), it follows that

\[
\|q_n - \tilde{q}_n\|_{L^1} \leq \int_{Q} a(x, y) k(x, y) |u^\lambda_n, \sigma_n (x, y) - u^{\lambda_0, \sigma_0} (x, y)| \, dx \, dy
\]

\[
+ \int_{Q} w_n (x, y) \, dx \, dy.
\]

Let \( n \to +\infty \). The first integral vanishes, for \( \{ u^\lambda_n, \sigma_n \} \) converges to \( u^{\lambda_0, \sigma_0} \) in \( C \). Likewise does the second integral, because of the Lebesgue dominated convergence
theorem. Therefore, there is \( n_0 \in \mathbb{N} \) such that \( \|\varrho_n - \tilde{\varrho}_n\|_{L^1} < \frac{1}{2}\varepsilon \) for \( n \geq n_0 \). A fortiori
\[
d_{L^1}(\varrho_n, \mathcal{V}(\lambda_n, \mu_n, \sigma_n)) < \frac{\varepsilon}{2} \quad \text{for } n \geq n_0,
\]
which contradicts (3.6). Consequently \( \mathcal{V} \) is Hausdorff lower semicontinuous. The proof that \( \mathcal{V} \) is Hausdorff upper semicontinuous is similar, and thus it is omitted. Hence \( \mathcal{V} \) is Hausdorff continuous.

(ii) For every \((\lambda, \mu, \sigma_1), (\lambda, \mu, \sigma_2) \in V \times M \times L^1\),
\[
(3.7) \quad H_{L^1}(\mathcal{V}(\lambda, \mu, \sigma_1), \mathcal{V}(\lambda, \mu, \sigma_2)) \leq \frac{1}{2} \|\sigma_1 - \sigma_2\|_{L^1}.
\]
Indeed, let \((\lambda, \mu, \sigma_1) \in V \times M \times L^1, i = 1, 2\). Denoting by \( u^{\lambda, \sigma_1} \) and \( u^{\lambda, \sigma_2} \) the solutions of \((C_{\lambda, \sigma_1})\) and \((C_{\lambda, \sigma_2})\), respectively, one has
\[
(3.8) \quad |u^{\lambda, \sigma_1}(x, y) - u^{\lambda, \sigma_2}(x, y)| \leq \int_{Q(x, y)} |\sigma_1(\xi, \eta) - \sigma_2(\xi, \eta)| \, d\xi \, d\eta, \text{ for } (x, y) \in Q.
\]
Let \( \varrho_1 \in \mathcal{V}(\lambda, \mu, \sigma_1) \) be arbitrary. Take \( \varrho_2 \in \mathcal{V}(\lambda, \mu, \sigma_2) \) so that
\[
|\varrho_1(x, y) - \varrho_2(x, y)| = d_{R^2}(\varrho_1(x, y), F(x, y, u^{\lambda, \sigma_1}(x, y), \mu)), \text{ for } (x, y) \in Q \text{ a.e.}
\]
From this, observing that \( \varrho_1(x, y) \in F(x, y, u^{\lambda, \sigma_1}(x, y), \mu) \), and using (A) \((a_3)\), (3.8) and Proposition 1, one has:
\[
\|\varrho_1 - \varrho_2\|_{L^1} \leq \int_{Q} a(x, y) H_{R^2}(F(x, y, u^{\lambda, \sigma_1}(x, y), \mu), F(x, y, u^{\lambda, \sigma_2}(x, y), \mu)) \, dx \, dy
\leq \int_{Q} a(x, y) k(x, y) |u^{\lambda, \sigma_1}(x, y) - u^{\lambda, \sigma_2}(x, y)| \, dx \, dy
\leq \int_{Q} a(x, y) k(x, y) \left( \int_{Q(x, y)} |\sigma_1(\xi, \eta) - \sigma_2(\xi, \eta)| \, d\xi \, d\eta \right) \, dx \, dy
= \int_{Q} |\sigma_1(\xi, \eta) - \sigma_2(\xi, \eta)| \left( \int_{R(\xi, \eta)} k(x, y) a(x, y) \, dx \, dy \right) \, d\xi \, d\eta
\leq \frac{1}{2} \int_{Q} |\sigma_1(\xi, \eta) - \sigma_2(\xi, \eta)| a(\xi, \eta) \, d\xi \, d\eta
\]
\[
= \frac{1}{2} \|\sigma_1 - \sigma_2\|_{L^1}.
\]
A fortiori, \( d_{L^1}(\varrho_1, \mathcal{V}(\lambda, \mu, \sigma_2)) \leq \frac{1}{2} \|\sigma_1 - \sigma_2\|_{L^1} \) and thus, since \( \varrho_1 \in \mathcal{V}(\lambda, \mu, \sigma_1) \) is arbitrary,
\[
\sup_{\varrho_1 \in \mathcal{V}(\lambda, \mu, \sigma_1)} d_{L^1}(\varrho_1, \mathcal{V}(\lambda, \mu, \sigma_2)) \leq \frac{1}{2} \|\sigma_1 - \sigma_2\|_{L^1}.
\]
From this, and the analogous inequality obtained by interchanging the roles of $\varrho_1$ and $\varrho_2$, (3.7) follows.

Since the multifunction $\mathcal{Y}: V \times M \times L^1 \to \mathcal{D}(L^1)$ satisfies (i) and (ii), by a result of Bressan, Cellina and Fryszkowski [2] there is continuous map $\varphi: V \times M \times L^1 \to L^1$ satisfying, for each $(\lambda, \mu) \in V \times M$, the following properties:

\[(3.9) \quad \varphi(\lambda, \mu, \sigma) \in \mathcal{F}(\lambda, \mu) \quad \text{for every} \quad \sigma \in L^1,\]

\[(3.10) \quad \varphi(\lambda, \mu, \sigma) = \sigma \quad \text{for every} \quad \sigma \in \mathcal{F}(\lambda, \mu).\]

Let $(\lambda, \mu, u) \in G$ be arbitrary. Since $u \in W(\lambda)$, for some $\sigma \in L^1$ we have $u = u^\lambda,\sigma$, where $u^\lambda,\sigma$ is the solution of $(C,\sigma)$. Hence $(\lambda, \mu, u) = (\lambda, \mu, u^\lambda,\sigma)$. Let $\Phi(\lambda, \mu, u^\lambda,\sigma): Q \to \mathbb{R}^q$ be given by

\[(3.11) \quad \Phi(\lambda, \mu, u^\lambda,\sigma)(x, y) = \lambda(x, y) + \int \int_{Q(x, y)} \varphi(\lambda, \mu, \sigma)(\xi, \eta) \, d\xi \, d\eta.\]

As $\Phi(\lambda, \mu, u^\lambda,\sigma) = u^\lambda,\varphi(\lambda, \mu, \sigma)$, this equality defines a map $\Phi: G \to W$.

If will be shown that $\Phi$ is continuous and that satisfies (3.4) and (3.5).

The multifunction $\Phi$ is continuous. To see this, let $\varepsilon > 0$. For arbitrary $(\lambda_0, \mu_0, u^{\lambda_0,\sigma_0})$, $(\lambda, \mu, u^\lambda,\sigma)$ in $G$, we have

\[(3.12) \quad \|\Phi(\lambda, \mu, u^\lambda,\sigma) - \Phi(\lambda_0, \mu_0, u^{\lambda_0,\sigma_0})\|_W \leq \|\lambda - \lambda_0\|_C + \left(1 + \frac{1}{m}\right)\|\varphi(\lambda, \mu, \sigma) - \varphi(\lambda_0, \mu_0, \sigma_0)\|_{L^1},\]

where $m$ denotes the absolute minimum of the continuous strictly positive function $\alpha: Q \to \mathbb{R}$. As $\varphi$ is continuous, there is $0 < \delta < \varepsilon$ so that $\|\varphi(\lambda, \mu, \sigma) - \varphi(\lambda_0, \mu_0, \sigma_0)\|_{L^1} < \varepsilon$, provided that $\|\lambda - \lambda_0\|_C < \delta$, $d_M(\mu, \mu_0) < \delta$ and $\|\sigma - \sigma_0\|_{L^1} < \delta$. Now, let $(\lambda, \mu, u^\lambda,\sigma) \in G$ satisfy $\|\lambda - \lambda_0\|_C < \delta$, $d_M(\mu, \mu_0) < \delta$ and $\|u^\lambda,\sigma - u^{\lambda_0,\sigma_0}\|_W < \delta$. Since $\|\sigma - \sigma_0\|_{L^1} \leq \|u^\lambda,\sigma - u^{\lambda_0,\sigma_0}\|_W < \delta$, we have $\|\varphi(\lambda, \mu, \sigma) - \varphi(\lambda_0, \mu_0, \sigma_0)\|_{L^1} < \varepsilon$. Hence, from (3.12),

\[\|\Phi(\lambda, \mu, u^\lambda,\sigma) - \Phi(\lambda_0, \mu_0, u^{\lambda_0,\sigma_0})\|_W < \delta + \left(1 + \frac{1}{m}\right)\varepsilon < \left(2 + \frac{1}{m}\right)\varepsilon,\]

and thus $\Phi$ is continuous.

Let $(\lambda, \mu) \in V \times M$. Let $u \in V(\lambda)$ be arbitrary, thus $u = u^\lambda,\sigma$ for some $\sigma \in L^1$. By (3.9), $\varphi(\lambda, \mu, \sigma) \in \mathcal{F}(\lambda, \mu)$ and hence $u^\lambda,\varphi(\lambda, \mu, \sigma) \in \mathcal{F}(\lambda, \mu)$. As $\Phi(\lambda, \mu, u^\lambda,\sigma) = u^\lambda,\varphi(\lambda, \mu, \sigma)$ and $u^\lambda,\sigma = u$, it follows that $\Phi(\lambda, \mu, u) \in \mathcal{F}(\lambda, \mu)$, proving (3.4).
Let \((\lambda, \mu) \in V \times M\). Let \(u \in \mathcal{T}(\lambda, \mu)\) be arbitrary, that is \(u = u^\lambda, \sigma\) for some \(\sigma \in \mathcal{Y}(\lambda, \mu, \sigma)\). Hence \(\sigma \in \mathcal{T}(\lambda, \mu)\) and so, by (3.10), \(\varphi(\lambda, \mu, \sigma) = \sigma\). From this and (3.11) it follows that

\[
\Phi(\lambda, \mu, u) = \Phi(\lambda, \mu, u^\lambda, \sigma) = u^{\lambda, \varphi(\lambda, \mu, \sigma)} = u^\lambda, \sigma = u,
\]

proving (3.5). This completes the proof of the theorem. \(\square\)

**Corollary 1.** Let assumption (A) be satisfied. Then, for each \((\lambda, \mu) \in V \times M\), \(\mathcal{T}(\lambda, \mu)\) is an absolute retract. Furthermore, \(\mathcal{T}(\lambda, \mu)\) is a contractible closed subspace of \(W\).

**Proof.** By Theorem 1, \(\mathcal{T}(\lambda, \mu)\) is a retract of \(W(\lambda)\). As \(W(\lambda)\) is a convex subset of \(W\), by a result of Borsuk [1, p. 85] \(\mathcal{T}(\lambda, \mu)\) is an absolute retract. Consequently \(\mathcal{T}(\lambda, \mu)\) is a contractible closed subspace of \(W\), completing the proof. \(\square\)

The following result is of a type proved by Cellina [3].

**Corollary 2.** Let assumption (A) be satisfied. Then there exists a continuous map \(\tau: V \times M \to W\) satisfying

\[
(3.13) \quad \tau(\lambda, \mu) \in \mathcal{T}(\lambda, \mu) \quad \text{for every } (\lambda, \mu) \in V \times M.
\]

**Proof.** For \(\lambda \in V\) set \(u(\lambda) = u^\lambda, 0\), where \(u^\lambda, 0\) denotes the solution of \((C_{\lambda, 0})\). Define \(\tau: V \times M \to W\) by \(\tau(\lambda, \mu) = \Phi(\lambda, \mu, u(\lambda))\), where \(\Phi\) is the map constructed in Theorem 1. The function \(\tau\) is well defined, since \(u(\lambda) \in W(\lambda)\). Furthermore, \(\tau\) is continuous, as \(\|u(\lambda) - u(\lambda_0)\|_W = \|\lambda - \lambda_0\|_C\) for \(\lambda, \lambda_0 \in V\), and satisfies (3.13), by virtue of (3.4). Hence the result. \(\square\)

**Corollary 3.** Let assumption (A) be satisfied. For \(i = 1, 2\), let \(\tau_i: V \times M \to W\) be a continuous map such that \(\tau_i(\lambda, \mu) \in \mathcal{T}(\lambda, \mu)\), for every \((\lambda, \mu) \in V \times M\). Then there exists a continuous map \(h: V \times M \times I \to W\) satisfying:

(i) \(h(\lambda, \mu, 0) = \tau_1(\lambda, \mu)\) and \(h(\lambda, \mu, 1) = \tau_2(\lambda, \mu)\), for every \((\lambda, \mu) \in V \times M\),

(ii) \(h(\lambda, \mu, s) \in \mathcal{T}(\lambda, \mu)\) for every \((\lambda, \mu, s) \in V \times M \times I\).

**Proof.** Define \(h: V \times M \times I \to W\) by

\[
(3.14) \quad h(\lambda, \mu, s) = \Phi(\lambda, \mu, (1-s)\tau_1(\lambda, \mu) + s\tau_2(\lambda, \mu)),
\]

where \(\Phi\) is the map constructed in Theorem 1. By using (3.14), (3.5) and (3.4), it is routine to see that \(h\) has the required properties. Hence the result. \(\square\)
References


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