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Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 1, 107–116

Persistent URL: `http://dml.cz/dmlcz/128505`

Czechoslovak Mathematical Journal, 45 (120) 1995, Praha

ON THE SOLUTION SETS OF SOME NONCONVEX HYPERBOLIC DIFFERENTIAL INCLUSIONS

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(Received December 31, 1992)

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper $\mathbb{R}^q$ denotes a real q-dimensional Euclidean space with norm $|\cdot|$, M a separable metric space, Q the square $I \times I$ with $I = [0,1]$. Let F be a multifunction from $Q \times \mathbb{R}^q \times M$ to the nonempty compact subsets of $\mathbb{R}^q$. Let $\lambda(x,y) = \alpha(x) + \beta(y) - \alpha(0)$, where α and β are continuous functions from I to $\mathbb{R}^q$ satisfying $\alpha(0) = \beta(0)$.

Under suitable assumptions on F, we consider the Darboux problem for hyperbolic differential inclusions of the form

$$(D_{\lambda,\mu}) \qquad \begin{cases} u_{xy}(x,y) \in F(x,y,u(x,y),\mu), \\ u(x,0) = \lambda(x,0), \quad u(0,y) = \lambda(0,y). \end{cases}$$

Denote by $\mathscr{T}(\lambda,\mu)$ the solution set of $(D_{\lambda,\mu})$. We prove that if F satisfies (among other assumptions) a Lipschitz condition with respect to u, then $\mathscr{T}(\lambda,\mu)$ is a retract of a convex subset of a Banach space. Furthermore, the retraction can be constructed as to depend continuously upon (λ,μ). From this it follows that $\mathscr{T}(\lambda,\mu)$ is contractible in itself, and that the multifunction $(\lambda,\mu) \to \mathscr{T}(\lambda,\mu)$ admits a continuous selection. Finally it is shown that any two continuous selections of this multifunction can be joined by a homotopy with values in $\mathscr{T}(\lambda,\mu)$.

Contributions to the study of the topological structure of the solution sets to hyperbolic differential equations or inclusions of the form $(D_{\lambda,\mu})$ can be found in Górniewicz and Pruszko [6], Teodoru [12], Staicu [11]. In particular, in [6] it is shown that the solution set of $(D_{\lambda,\mu})$ with F single valued is an R_δ-set. Similar problems for other types of differential equations or inclusions have been studied by many authors, including Himmelberg and Van Vleck [9], Cellina [3], Deimling [4], Papageorgiou [10].

2. Notation and preliminaries

Let Z be a metric space with distance d_Z. For $a \in Z$ and B a nonempty subset of Z, we put $d_Z(a,B) = \inf_{b\in B} d_Z(a,b)$. We denote by $\mathscr{C}(Z)$ the space of all nonempty closed bounded subsets of Z, endowed with the Hausdorff metric

$$H_Z(A,B) = \max\Big\{\sup_{a\in A} d_Z(a,B),\ \sup_{b\in B} d_Z(b,A)\Big\},\ A,B\in\mathscr{C}(Z).$$

Let Y be a measurable metric space with σ-algebra $\mathscr{A}$ and let Z be a separable metric space. A multifunction $F\colon Y \to \mathscr{C}(Z)$ is called *measurable* (see Himmelberg [8]) if $\{y\in Y \mid F(y)\cap D \neq \emptyset\} \in \mathscr{A}$ for every closed subset D of Z. The Borel σ-algebra of Z is denoted by $\mathscr{B}(Z)$. In the sequel Q, as measurable space, is given the σ-algebra $\mathscr{L}$ of the Lebesgue measurable subsets of Q.

We denote by C the Banach space of all continuous functions $u\colon Q \to \mathbb{R}^q$, equipped with the norm $\|u\|_C = \sup_{(x,y)\in Q} |u(x,y)|$. Given a continuous strictly positive function $a\colon Q\to\mathbb{R}$, we denote by L^1 the Banach space of all (equivalence classes of) Lebesgue measurable functions $\sigma\colon Q\to\mathbb{R}^q$, endowed with the norm

$$\|\sigma\|_{L^1} = \iint_Q a(x,y)|\sigma(x,y)|\,\mathrm{d}x\,\mathrm{d}y. \tag{2.1}$$

Furthermore, by V we mean the linear subspace of $C(Q,\mathbb{R}^q)$ consisting of all $\lambda\in C$ such that there exist continuous functions $\alpha\colon I\to\mathbb{R}^q$ and $\beta\colon I\to\mathbb{R}^q$, with $\alpha(0)=\beta(0)$, satisfying $\lambda(x,y)=\alpha(x)+\beta(y)-\alpha(0)$ for every $(x,y)\in Q$. Observe that V, equipped with the norm of C, is a separable Banach space.

In the sequel, when a product $Z = Z_1\times\ldots\times Z_n$ of metric spaces Z_i, $i=1,\ldots,n$, is considered, it is assumed that Z is given the metric $\max_{1\leqslant i\leqslant n} d_{Z_i}(x_i,y_i)$, where $(x_1,\ldots,x_n)$, $(y_1,\ldots,y_n)\in Z$.

Following Hiai and Umegaki [7], a set $K\subset L^1$ is called *decomposable* if for every $u,v\in K$ and $A\in\mathscr{L}$ we have $u\chi_A + v\chi_{Q\setminus A}\in K$, where χ_A stands for the characteristic function of A. We set $\mathscr{D}(L^1) = \{X\in\mathscr{C}(L^1) \mid X \text{ is decomposable}\}$.

Let T be a Hausdorff topological space. A subspace X of T is called a *retract* of T if there is a continuous map $\varphi\colon T\to X$ such that $\varphi(x)=x$ for every $x\in X$.

In order to treat problem $(D_{\lambda,\mu})$ we introduce the following

Assumption (A). The multifunction $F\colon Q \times \mathbb{R}^q \times M \to \mathscr{C}(\mathbb{R}^q)$ satisfies:

(a_1) F is $\mathscr{L} \otimes \mathscr{B}(\mathbb{R}^q \times M)$-measurable,

(a_2) for each $(x, y, u) \in Q \times \mathbb{R}^q$ the multifunction $\mu \to F(x, y, u, \mu)$ is Hausdorff continuous on M,

(a_3) there exist positive integrable functions $h\colon Q \to \mathbb{R}$ and $k\colon Q \to \mathbb{R}$ such that

$$H_{\mathbb{R}^q}(F(x, y, u, \mu), \{0\}) \leqslant h(x, y) \quad \text{for every } (x, y, u, \mu) \in Q \times \mathbb{R}^q \times M,$$
$$H_{\mathbb{R}^q}(F(x, y, u_1, \mu), F(x, y, u_2, \mu)) \leqslant k(x, y)|u_1 - u_2| \quad \text{for every } (x, y, u_i, \mu) \in Q \times \mathbb{R}^q \times M,\ i = 1, 2.$$

For $(x, y) \in Q$ and $\varepsilon > 0$, we put:

$$Q(x, y) = [0, x] \times [0, y], \quad R(x, y) = [x, 1] \times [y, 1],$$
$$P(x, y; \varepsilon) = [x - \varepsilon, x + \varepsilon] \times [y - \varepsilon, y + \varepsilon].$$

For $(\lambda, \sigma) \in V \times L^1$, consider the following Darboux problem

$$(C_{\lambda,\sigma}) \qquad \begin{cases} u_{xy}(x, y) = \sigma(x, y), \\ u(x, 0) = \lambda(x, 0), \quad u(0, y) = \lambda(0, y). \end{cases}$$

Definition 1. Let $(\lambda, \sigma) \in V \times L^1$. The function $u \in C$ given by

$$u(x, y) = \lambda(x, y) + \iint_{Q(x,y)} \sigma(\xi, \eta)\, d\xi\, d\eta \quad \text{for } (x, y) \in Q,$$

is said to be *solution* of $(C_{\lambda,\sigma})$.

Clearly $(C_{\lambda,\sigma})$ has a unique solution which, in the sequel, will be denoted by $u^{\lambda,\sigma}$.

Definition 2. Let (A) be satisfied. Let $(\lambda, \mu) \in V \times M$. A function $u \in C$ is said to be *solution* of $(D_{\lambda,\mu})$ if there exists a function $\sigma \in L^1$ such that:

$$\sigma(x, y) \in F(x, y, u(x, y), \mu) \quad \text{for } (x, y) \in Q \text{ a.e.},$$
$$u(x, y) = \lambda(x, y) + \iint_{Q(x,y)} \sigma(\xi, \eta)\, d\xi\, d\eta \quad \text{for every } (x, y) \in Q.$$

We denote by $\mathscr{T}(\lambda, \mu)$ the *solution set* of $(D_{\lambda,\mu})$, i.e. the set of all solutions of $(D_{\lambda,\mu})$.

Proposition 1. *Let $k\colon Q \to \mathbb{R}$ be a positive integrable function. Then there exists a continuous strictly positive function $a\colon Q \to \mathbb{R}$ which, for each $(x, y) \in Q$, satisfies*

$$\iint_{R(x,y)} k(\xi, \eta) a(\xi, \eta)\, d\xi\, d\eta = \frac{1}{2}(a(x, y) - 1). \tag{2.2}$$

Proof. For $n \in \mathbb{N}$ set $x_i = i/n$, $i = 0, 1, \ldots, n$. Fix $n \in \mathbb{N}$ so that

$$\iint_{[x_{i-1},x_i]\times I} 2k(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta < 1, \qquad i = 1, 2, \ldots, n.$$

By using the Banach-Caccioppoli fixed point theorem, it is easy to show that there is a continuous strictly positive function $a_n\colon [x_{n-1}, x_n]\times I \to \mathbb{R}$ satisfying (2.2) (with a_n in the place of a) for every $(x, y) \in [x_{n-1}, x_n]\times I$. Then, recursively, one can construct continuous strictly positive functions $a_i\colon [x_{i-1}, x_i] \times I \to \mathbb{R}$, $i = 1, 2, \ldots, n-1$, satisfying

$$\iint_{[x,x_i]\times[y,1]} k(\xi,\eta)a_i(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = \frac{1}{2}\left(a_i(x,y) - a_{i+1}(x_i,y)\right),$$

for every $(x, y) \in [x_{i-1}, x_i] \times I$. Define $a\colon Q \to \mathbb{R}$ by $a(x,y) = \sum\limits_{i=1}^{n} a_i(x,y)\ \chi_{U_i}(x,y)$, where $U_1 = [x_0, x_1] \times I$ and $U_i = (x_{i-1}, x_i] \times I$, $i = 2, \ldots, n$. It is routine to verify that the function a is continuous, strictly positive, and that a satisfies (2.2) for every $(x, y) \in Q$. This completes the proof. □

Proposition 2. *The map $T\colon V \times L^1 \to C$ given by $T(\lambda,\sigma) = u^{\lambda,\sigma}$, where $u^{\lambda,\sigma}$ is the solution of $(C_{\lambda,\sigma})$, is linear and one-to-one.*

Proof. Clearly T is linear. To show that T is one-to-one, suppose that $T(\lambda_1, \sigma_1) = T(\lambda_2, \sigma_2)$ for some $(\lambda_i, \sigma_i) \in V \times L^1$, $i = 1, 2$. This implies $\lambda_1 = \lambda_2$ and thus, setting $\sigma = \sigma_1 - \sigma_2$, we have

$$\iint_{Q(x,y)} \sigma(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = 0 \quad \text{for every } (x,y) \in Q. \tag{2.3}$$

Let L be the set of all Lebesgue points of σ belonging to the interior of Q, and observe that $Q \setminus L$ has Lebesgue measure zero. Let $(\xi, \eta) \in L$ be arbitrary. For $\varepsilon > 0$ sufficiently small, we have

$$\sigma(\xi,\eta) = \frac{1}{4\varepsilon^2}\iint_{P(\xi,\eta;\varepsilon)} (\sigma(\xi,\eta) - \sigma(x,y))\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{4\varepsilon^2}\iint_{P(\xi,\eta;\varepsilon)} \sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.4}$$

The first integral vanishes as $\varepsilon \to 0$ by virtue of a result from [5, p. 217]. The second one is zero, as consequence of (2.3) and of the equality

$$\begin{aligned}\iint_{P(\xi,\eta;\varepsilon)} \sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \iint_{Q(\xi+\varepsilon,\eta+\varepsilon)} \sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{Q(\xi-\varepsilon,\eta-\varepsilon)} \sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \iint_{Q(\xi-\varepsilon,\eta+\varepsilon)} \sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{Q(\xi+\varepsilon,\eta-\varepsilon)} \sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Letting $\varepsilon \to 0$, (2.4) gives $\sigma(\xi,\eta) = 0$, thus $\sigma_1 = \sigma_2$. Hence $(\lambda_1, \sigma_1) = (\lambda_2, \sigma_2)$, which implies that T is one-to-one. This completes the proof. □

3. Main Results

Let assumption (A) be satisfied. Let $(\lambda,\mu,\sigma) \in V \times M \times L^1$. Let $u^{\lambda,\sigma}\colon Q \to \mathbb{R}^q$ be the solution of $(C_{\lambda,\sigma})$. We put

$$
\begin{aligned}
&(3.1)\quad \mathscr{V}(\lambda,\mu,\sigma) = \{\varrho \in L^1 \mid \varrho(x,y) \in F(x,y,u^{\lambda,\sigma}(x,y),\mu),\ (x,y) \in Q \text{ a.e.}\},\\
&(3.2)\qquad\qquad\qquad \mathscr{F}(\lambda,\mu) = \{\varrho \in L^1 \mid \varrho \in \mathscr{V}(\lambda,\mu,\varrho)\}.
\end{aligned}
$$

Observe that $\mathscr{V}(\lambda,\mu,\sigma)$ is a decomposable closed bounded subset of L^1, thus (3.1) defines a multifunction $\mathscr{V}\colon V \times M \times L^1 \to \mathscr{D}(L^1)$.

Furthermore, set

$$W = \{u \in C \mid u = u^{\lambda,\sigma} \quad \text{for some } (\lambda,\sigma) \in V \times L^1\}.$$

By Proposition 2, for each $u \in W$ there is one and only one $(\lambda,\sigma) \in V \times L^1$ such that $u = u^{\lambda,\sigma}$. In view of that, we write $u^{\lambda,\sigma}$ to denote an arbitrary member of W. Let k be the positive integrable function occurring in assumption (A). By Proposition 1, there is a continuous strictly positive function $a\colon Q \to \mathbb{R}$ satisfying (2.2) for every $(x,y) \in Q$. With this choice of a, for arbitrary $u^{\lambda,\sigma} \in W$ we set

$$\|u^{\lambda,\sigma}\|_W = \|u^{\lambda,\sigma}\|_C + \|\sigma\|_{L^1}, \tag{3.3}$$

where $\|\sigma\|_{L^1}$ is given by (2.1). By using Proposition 2, it is easy to check that (3.3) defines a norm on W and that, under this norm, W is a Banach space.

For $\lambda \in V$, set

$$W(\lambda) = \{u \in W \mid u(x,0) = \lambda(x,0) \quad \text{for } x \in I,\ u(0,y) = \lambda(0,y) \quad \text{for } y \in I\}.$$

We observe that $W(\lambda)$ is a nonempty convex closed subset of W satisfying

$$\mathscr{T}(\lambda,\mu) \subset W(\lambda) \qquad \text{for every } \mu \in M.$$

Theorem 1. *Let assumption* (A) *be satisfied. Let* $G = \{(\lambda,\mu,u) \in V \times M \times W \mid (\lambda,\mu) \in V \times M,\ u \in W(\lambda)\}$. *Then there exists a continuous function* $\Phi\colon G \to W$ *satisfying, for each* $(\lambda,\mu) \in V \times M$, *the following properties:*

$$\Phi(\lambda,\mu,u) \in \mathscr{T}(\lambda,\mu) \qquad \text{for every } u \in W(\lambda), \tag{3.4}$$

$$\Phi(\lambda,\mu,u) = u \qquad \text{for every } u \in \mathscr{T}(\lambda,\mu). \tag{3.5}$$

Proof. Let $\mathscr{V}\colon V \times M \times L^1 \to L^1$ be defined by (3.1).

(i) $\mathscr{V}$ is Hausdorff continuous. To this end we prove first that $\mathscr{V}$ is Hausdorff lower semicontinuous. Suppose the contrary. Then there exist an $\varepsilon > 0$, a sequence $\{(\lambda_n, \mu_n, \sigma_n)\}$ converging to $(\lambda_0, \mu_0, \sigma_0)$ in $V \times M \times L^1$, and a sequence $\{\varrho_n\} \subset L^1$, with $\varrho_n \in \mathscr{V}(\lambda_0, \mu_0, \sigma_0)$ for each $n \in \mathbb{N}$, such that

$$d_{L^1}(\varrho_n, \mathscr{V}(\lambda_n, \mu_n, \sigma_n)) \geqslant \varepsilon \qquad \text{for every } n \in \mathbb{N}. \tag{3.6}$$

For $n \in \mathbb{N}$ define $M_n \colon Q \to \mathscr{C}(\mathbb{R}^q)$ by

$$\begin{aligned} M_n(x,y) = {} & F(x,y,u^{\lambda_n,\sigma_n}(x,y),\mu_n) \\ & \cap B_{\mathbb{R}^q}\big(\varrho_n(x,y), d_{\mathbb{R}^q}\big(\varrho_n(x,y), F(x,y,u^{\lambda_n,\sigma_n}(x,y),\mu_n)\big)\big), \end{aligned}$$

where, for $a \in \mathbb{R}^q$ and $r \geqslant 0$, $B_{\mathbb{R}^q}(a,r) = \{x \in \mathbb{R}^q \mid |x-a| \leqslant r\}$. As M_n is measurable, there exists a measurable selection $\tilde{\varrho}_n \in \mathscr{V}(\lambda_n, \mu_n, \sigma_n)$ such that

$$|\varrho_n(x,y) - \tilde{\varrho}_n(x,y)| = d_{\mathbb{R}^q}(\varrho_n(x,y),\ F(x,y,u^{\lambda_n,\sigma_n}(x,y),\mu_n)) \quad \text{for } (x,y) \in Q \text{ a.e.}$$

From this, observing that $\varrho_n(x,y) \in F(x,y,u^{\lambda_0,\sigma_0}(x,y),\mu_0)$, one has:

$$\begin{aligned} & \iint_Q a(x,y)|\varrho_n(x,y) - \tilde{\varrho}_n(x,y)|\,\mathrm{d}x\,\mathrm{d}y \\ & \quad = \iint_Q a(x,y) d_{\mathbb{R}^q}(\varrho_n(x,y),\ F(x,y,u^{\lambda_n,\sigma_n}(x,y),\mu_n))\,\mathrm{d}x\,\mathrm{d}y \\ & \quad \leqslant \iint_Q a(x,y) H_{\mathbb{R}^q}(F(x,y,u^{\lambda_0,\sigma_0}(x,y),\mu_0),\ F(x,y,u^{\lambda_n,\sigma_n}(x,y),\mu_n))\,\mathrm{d}x\,\mathrm{d}y \\ & \quad \leqslant \iint_Q a(x,y) H_{\mathbb{R}^q}(F(x,y,u^{\lambda_n,\sigma_n}(x,y),\mu_n),\ F(x,y,u^{\lambda_0,\sigma_0}(x,y),\mu_n))\,\mathrm{d}x\,\mathrm{d}y \\ & \qquad + \iint_Q a(x,y) H_{\mathbb{R}^q}(F(x,y,u^{\lambda_0,\sigma_0}(x,y),\mu_n),\ F(x,y,u^{\lambda_0,\sigma_0}(x,y),\mu_0))\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Denoting by $w_n(x,y)$ the function under the sign of the last integral, and using assumption (A) (a_3), it follows that

$$\begin{aligned} \|\varrho_n - \tilde{\varrho}_n\|_{L^1} \leqslant {} & \iint_Q a(x,y)k(x,y)|u^{\lambda_n,\sigma_n}(x,y) - u^{\lambda_0,\sigma_0}(x,y)|\,\mathrm{d}x\,\mathrm{d}y \\ & + \iint_Q w_n(x,y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Let $n \to +\infty$. The first integral vanishes, for $\{u^{\lambda_n,\sigma_n}\}$ converges to u^{λ_0,σ_0} in C. Likewise does the second integral, because of the Lebesgue dominated convergence theorem. Therefore, there is $n_0 \in \mathbb{N}$ such that $\|\varrho_n - \tilde{\varrho}_n\|_{L^1} < \frac{1}{2}\varepsilon$ for $n \geqslant n_0$. A fortiori

$$d_{L^1}(\varrho_n, \mathscr{V}(\lambda_n, \mu_n, \sigma_n)) < \frac{\varepsilon}{2} \quad \text{for } n \geqslant n_0,$$

which contradicts (3.6). Consequently $\mathscr{V}$ is Hausdorff lower semicontinuous. The proof that $\mathscr{V}$ is Hausdorff upper semicontinuous is similar, and thus it is omitted. Hence $\mathscr{V}$ is Hausdorff continuous.

(ii) For every (λ, μ, σ_1), $(\lambda, \mu, \sigma_2) \in V \times M \times L^1$,

$$H_{L^1}(\mathscr{V}(\lambda, \mu, \sigma_1), \mathscr{V}(\lambda, \mu, \sigma_2)) \leqslant \frac{1}{2}\|\sigma_1 - \sigma_2\|_{L^1}. \tag{3.7}$$

Indeed, let $(\lambda, \mu, \sigma_i) \in V \times M \times L^1$, $i = 1, 2$. Denoting by u^{λ,σ_1} and u^{λ,σ_2} the solutions of (C_{λ,σ_1}) and (C_{λ,σ_2}), respectively, one has

$$|u^{\lambda,\sigma_1}(x,y) - u^{\lambda,\sigma_2}(x,y)| \leqslant \iint_{Q(x,y)} |\sigma_1(\xi,\eta) - \sigma_2(\xi,\eta)|\, \mathrm{d}\xi\, \mathrm{d}\eta, \text{ for } (x,y) \in Q. \tag{3.8}$$

Let $\varrho_1 \in \mathscr{V}(\lambda, \mu, \sigma_1)$ be arbitrary. Take $\varrho_2 \in \mathscr{V}(\lambda, \mu, \sigma_2)$ so that

$$|\varrho_1(x,y) - \varrho_2(x,y)| = d_{\mathbb{R}^q}(\varrho_1(x,y),\ F(x,y,u^{\lambda,\sigma_2}(x,y),\mu)), \text{ for } (x,y) \in Q \text{ a.e.}$$

From this, observing that $\varrho_1(x,y) \in F(x,y,u^{\lambda,\sigma_1}(x,y),\mu)$, and using (A) (a_3), (3.8) and Proposition 1, one has:

$$\begin{aligned}
\|\varrho_1 - \varrho_2\|_{L^1} &\leqslant \iint_Q a(x,y) H_{\mathbb{R}^q}(F(x,y,u^{\lambda,\sigma_1}(x,y),\mu),\ F(x,y,u^{\lambda,\sigma_2}(x,y),\mu))\, \mathrm{d}x\, \mathrm{d}y \\
&\leqslant \iint_Q a(x,y)k(x,y)|u^{\lambda,\sigma_1}(x,y) - u^{\lambda,\sigma_2}(x,y)|\, \mathrm{d}x\, \mathrm{d}y \\
&\leqslant \iint_Q a(x,y)k(x,y)\Big(\iint_{Q(x,y)} |\sigma_1(\xi,\eta) - \sigma_2(\xi,\eta)|\, \mathrm{d}\xi\, \mathrm{d}\eta\Big)\, \mathrm{d}x\, \mathrm{d}y \\
&= \iint_Q |\sigma_1(\xi,\eta) - \sigma_2(\xi,\eta)|\Big(\iint_{R(\xi,\eta)} k(x,y)a(x,y)\, \mathrm{d}x\, \mathrm{d}y\Big)\, \mathrm{d}\xi\, \mathrm{d}\eta \\
&\leqslant \frac{1}{2}\iint_Q |\sigma_1(\xi,\eta) - \sigma_2(\xi,\eta)|a(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \\
&= \frac{1}{2}\|\sigma_1 - \sigma_2\|_{L^1}.
\end{aligned}$$

A fortiori, $d_{L^1}(\varrho_1, \mathscr{V}(\lambda, \mu, \sigma_2)) \leqslant \frac{1}{2}\|\sigma_1 - \sigma_2\|_{L^1}$ and thus, since $\varrho_1 \in \mathscr{V}(\lambda, \mu, \sigma_1)$ is arbitrary,

$$\sup_{\varrho_1 \in \mathscr{V}(\lambda,\mu,\sigma_1)} d_{L^1}(\varrho_1, \mathscr{V}(\lambda, \mu, \sigma_2)) \leqslant \frac{1}{2}\|\sigma_1 - \sigma_2\|_{L^1}.$$

From this, and the analogous inequality obtained by interchanging the roles of ϱ_1 and ϱ_2, (3.7) follows.

Since the multifunction $\mathscr{V}\colon V \times M \times L^1 \to \mathscr{D}(L^1)$ satisfies (i) and (ii), by a result of Bressan, Cellina and Fryszkowski [2] there is continuous map $\varphi\colon V \times M \times L^1 \to L^1$ satisfying, for each $(\lambda, \mu) \in V \times M$, the following properties:

$$\varphi(\lambda, \mu, \sigma) \in \mathscr{F}(\lambda, \mu) \qquad \text{for every } \sigma \in L^1, \tag{3.9}$$

$$\varphi(\lambda, \mu, \sigma) = \sigma \qquad \text{for every } \sigma \in \mathscr{F}(\lambda, \mu). \tag{3.10}$$

Let $(\lambda, \mu, u) \in G$ be arbitrary. Since $u \in W(\lambda)$, for some $\sigma \in L^1$ we have $u = u^{\lambda,\sigma}$, where $u^{\lambda,\sigma}$ is the solution of $(C_{\lambda,\sigma})$. Hence $(\lambda, \mu, u) = (\lambda, \mu, u^{\lambda,\sigma})$. Let $\Phi(\lambda, \mu, u^{\lambda,\sigma})\colon Q \to \mathbb{R}^q$ be given by

$$\Phi(\lambda, \mu, u^{\lambda,\sigma})(x, y) = \lambda(x, y) + \iint_{Q(x,y)} \varphi(\lambda, \mu, \sigma)(\xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{3.11}$$

As $\Phi(\lambda, \mu, u^{\lambda,\sigma}) = u^{\lambda,\varphi(\lambda,\mu,\sigma)}$, this equality defines a map

$$\Phi\colon G \to W.$$

If will be shown that Φ is continuous and that satisfies (3.4) and (3.5).

The multifunction Φ is continuous. To see this, let $\varepsilon > 0$. For arbitrary $(\lambda_0, \mu_0, u^{\lambda_0,\sigma_0})$, $(\lambda, \mu, u^{\lambda,\sigma})$ in G, we have

$$\begin{aligned} &\|\Phi(\lambda, \mu, u^{\lambda,\sigma}) - \Phi(\lambda_0, \mu_0, u^{\lambda_0,\sigma_0})\|_W \\ &= \|u^{\lambda,\varphi(\lambda,\mu,\sigma)} - u^{\lambda_0,\varphi(\lambda_0,\mu_0,\sigma_0)}\|_C + \|\varphi(\lambda, \mu, \sigma) - \varphi(\lambda_0, \mu_0, \sigma_0)\|_{L^1} \\ &\leqslant \|\lambda - \lambda_0\|_C + \Big(1 + \frac{1}{m}\Big)\|\varphi(\lambda, \mu, \sigma) - \varphi(\lambda_0, \mu_0, \sigma_0)\|_{L^1}, \end{aligned} \tag{3.12}$$

where m denotes the absolute minimum of the continuous strictly positive function $a\colon Q \to \mathbb{R}$. As φ is continuous, there is $0 < \delta < \varepsilon$ so that $\|\varphi(\lambda, \mu, \sigma) - \varphi(\lambda_0, \mu_0, \sigma_0)\|_{L^1} < \varepsilon$, provided that $\|\lambda - \lambda_0\|_C < \delta$, $d_M(\mu, \mu_0) < \delta$ and $\|\sigma - \sigma_0\|_{L^1} < \delta$. Now, let $(\lambda, \mu, u^{\lambda,\sigma}) \in G$ satisfy $\|\lambda - \lambda_0\|_C < \delta$, $d_M(\mu, \mu_0) < \delta$ and $\|u^{\lambda,\sigma} - u^{\lambda_0,\sigma_0}\|_W < \delta$. Since $\|\sigma - \sigma_0\|_{L^1} \leqslant \|u^{\lambda,\sigma} - u^{\lambda_0,\sigma_0}\|_W < \delta$, we have $\|\varphi(\lambda, \mu, \sigma) - \varphi(\lambda_0, \mu_0, \sigma_0)\|_{L^1} < \varepsilon$. Hence, from (3.12),

$$\|\Phi(\lambda, \mu, u^{\lambda,\sigma}) - \Phi(\lambda_0, \mu_0, u^{\lambda_0,\sigma_0})\|_W < \delta + \Big(1 + \frac{1}{m}\Big)\varepsilon < \Big(2 + \frac{1}{m}\Big)\varepsilon,$$

and thus Φ is continuous.

Let $(\lambda, \mu) \in V \times M$. Let $u \in V(\lambda)$ be arbitrary, thus $u = u^{\lambda,\sigma}$ for some $\sigma \in L^1$. By (3.9), $\varphi(\lambda, \mu, \sigma) \in \mathscr{F}(\lambda, \mu)$ and hence $u^{\lambda,\varphi(\lambda,\mu,\sigma)} \in \mathscr{T}(\lambda, \mu)$. As $\Phi(\lambda, \mu, u^{\lambda,\sigma}) = u^{\lambda,\varphi(\lambda,\mu,\sigma)}$ and $u^{\lambda,\sigma} = u$, it follows that $\Phi(\lambda, \mu, u) \in \mathscr{T}(\lambda, \mu)$, proving (3.4).

Let $(\lambda, \mu) \in V \times M$. Let $u \in \mathscr{T}(\lambda, \mu)$ be arbitrary, that is $u = u^{\lambda,\sigma}$ for some $\sigma \in \mathscr{V}(\lambda, \mu, \sigma)$. Hence $\sigma \in \mathscr{F}(\lambda, \mu)$ and so, by (3.10), $\varphi(\lambda, \mu, \sigma) = \sigma$. From this and (3.11) it follows that

$$\Phi(\lambda, \mu, u) = \Phi(\lambda, \mu, u^{\lambda,\sigma}) = u^{\lambda,\varphi(\lambda,\mu,\sigma)} = u^{\lambda,\sigma} = u,$$

proving (3.5). This completes the proof of the theorem. □

Corollary 1. *Let assumption* (A) *be satisfied. Then, for each* $(\lambda, \mu) \in V \times M$, $\mathscr{T}(\lambda, \mu)$ *is an absolute retract. Furthermore,* $\mathscr{T}(\lambda, \mu)$ *is a contractible closed subspace of* W.

Proof. By Theorem 1, $\mathscr{T}(\lambda, \mu)$ is a retract of $W(\lambda)$. As $W(\lambda)$ is a convex subset of W, by a result of Borsuk [1, p. 85] $\mathscr{T}(\lambda, \mu)$ is an absolute retract. Consequently $\mathscr{T}(\lambda, \mu)$ is a contractible closed subspace of W, completing the proof. □

The following result is of a type proved by Cellina [3].

Corollary 2. *Let assumption* (A) *be satisfied. Then there exists a continuous map* $\tau\colon V \times M \to W$ *satisfying*

$$\tau(\lambda, \mu) \in \mathscr{T}(\lambda, \mu) \qquad \text{for every } (\lambda, \mu) \in V \times M. \tag{3.13}$$

Proof. For $\lambda \in V$ set $u(\lambda) = u^{\lambda,0}$, where $u^{\lambda,0}$ denotes the solution of $(C_{\lambda,0})$. Define $\tau\colon V \times M \to W$ by $\tau(\lambda, \mu) = \Phi(\lambda, \mu, u(\lambda))$, where Φ is the map constructed in Theorem 1. The function τ is well defined, since $u(\lambda) \in W(\lambda)$. Furthermore, τ is continuous, as $\|u(\lambda) - u(\lambda_0)\|_W = \|\lambda - \lambda_0\|_C$ for λ, $\lambda_0 \in V$, and satisfies (3.13), by virtue of (3.4). Hence the result. □

Corollary 3. *Let assumption* (A) *be satisfied. For* $i = 1, 2$, *let* $\tau_i\colon V \times M \to W$ *be a continuous map such that* $\tau_i(\lambda, \mu) \in \mathscr{T}(\lambda, \mu)$, *for every* $(\lambda, \mu) \in V \times M$. *Then there exists a continuous map* $h\colon V \times M \times I \to W$ *satisfying:*

(i) $h(\lambda, \mu, 0) = \tau_1(\lambda, \mu)$ *and* $h(\lambda, \mu, 1) = \tau_2(\lambda, \mu)$, *for every* $(\lambda, \mu) \in V \times M$,

(ii) $h(\lambda, \mu, s) \in \mathscr{T}(\lambda, \mu)$ *for every* $(\lambda, \mu, s) \in V \times M \times I$.

Proof. Define $h\colon V \times M \times I \to W$ by

$$h(\lambda, \mu, s) = \Phi(\lambda, \mu, (1 - s)\tau_1(\lambda, \mu) + s\tau_2(\lambda, \mu)), \tag{3.14}$$

where Φ is the map constructed in Theorem 1. By using (3.14), (3.5) and (3.4), it is routine to see that h has the required properties. Hence the result. □

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