Czechoslovak Mathematical Journal

Kamil John

On a result of J. Johnson

Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 2, 235–240

Persistent URL: http://dml.cz/dmlcz/128513

Czechoslovak Mathematical Journal, 45 (120) 1995, Praha

ON A RESULT OF J. JOHNSON

Kamil John, Praha

(Received March 3, 1993)

J. Johnson proved in [4] that if Y is a Banach space having the bounded approximation property then the anulator $K(X,Y)$ in $L(X,Y)^*$ is the kernel of a projection P in $L(X,Y)^*$. Here X is an arbitrary Banach space and $K(X,Y)=K$, $L(X,Y)=L$, denote respectively the space of all compact or bounded operators $f\colon X\to Y$. Moreover, the range space of the projection P is isomorphic to K^*. In [3] the same statement was shown to be true for the spaces $X=P$ and $Y=P^*$ where P is any separable Pisier space. Notice that here Johnson's result cannot be applied since P^* (and P) do not even have the approximation property. The proof in [3] was based on the fact that every $f\colon P\to P^*$ is factorable through a Hilbert space. In this note we observe (see Proposition 2 and Remarks 1 and 2) that Johnson's result holds for any couples of Banach spaces X, Y such that any $f\colon X\to Y$ is factorable through a Banach space Z, Z^* having the bounded approximation property and Z^* being separable. In fact much weaker assumptions are shown to be sufficient for J. Johnson's result (Proposition 1 and Remark 5).

Following N. Kalton [6] we denote on by w' the topology $L(X,Y)=L$ (projectively) generated by all $x^{**}\otimes y^*$ where $x^{**}\in X^{**}$ and $y^*\in Y^*$. Thus we write $f_n\overset{w'}{\to}f$ to denote that for any x^{**}, and y^* we have $x^{**}\big(f_n^*(y^*)\big)\to x^{**}\big(f^*(y^*)\big)$. We will make crucial use of the following result of Kalton:

(K) If $\{f_n\}\subset K$ is a sequence of compact operators such that $f_n\overset{w'}{\to}f$ and if $f\colon x\to y$ is compact then $f_n\to f$ in the weak topology of $L(X,Y)$.

We say that the operator $f\colon X\to Y$ is factorable through a Banach space Z if $f=f_1f_2$ where $f_2\colon X\to Z$ and $f_1\colon Z\to Y$ are operators. All operators in the paper are bounded linear operators.

Proposition 1. *Let X, Y be Banach spaces such that for every $f\in L(X,Y)=L$ there is a sequence $\{f_n\}\subset K(X,Y)=K$ such that $f_n\overset{w'}{\longrightarrow}f$. Then there exists a continuous bilinear form $J\colon K^*\times L\to R$ (scalars) and a number $c>0$ such that*

a) *if* $f \in K$ *and* $\Phi \in K^*$ *then* $J(\Phi, f) = \Phi(f)$;

b) $|J(\Phi, f)| \leqslant c\|\Phi\| \cdot \|f\|$ *for all* $f \in L$ *and* $\Phi \in K^*$;

c) $J(\Phi, f) = \lim \Phi(f_n)$ *where* $\{f_n\}$ *is any sequence of compact operators* $f_n \in K$ *tending* w' *to* f.

Proof. First we observe that if $f_n \xrightarrow{w'} f$, $f \in L$ and $f_n \in K$ then $\lim \Phi(f_n)$ exists for all $\Phi \in K^*$. Indeed, $\{\Phi(f_n)\}$ is bounded by the uniform boundedness principle and thus $\limsup_n \Phi(f_n) = \lim_k \Phi(f_{n_k})$ and $\liminf_n \Phi(f_n) = \lim_k \Phi(f_{m_k})$ for suitable subsequences $\{n_k\}$ and $\{m_k\}$ of natural numbers. Thus $\limsup \Phi(f_n) - \liminf \Phi(f_n) = \lim_k \Phi(f_{n_k} - f_{m_k}) = 0$, because $f_{n_k} - f_{m_k} \to 0$ weakly by (K). Similarly we show that if $f_n \xrightarrow{w'} f$ and $g_n \xrightarrow{w'} f$ with $\{f_n\} \subset K$ and $\{g_n\} \subset K$ then $\lim \Phi(f_n) = \lim \Phi(g_n)$ for any $\Phi \in K^*$. Thus we may define $J(\Phi, f)$ by c). J is evidently bilinear and if $f \in K$ then $J(\Phi, f) = \lim \Phi(f_n) = \Phi(f)$ because $f_n = f \xrightarrow{w'} f$. To show b) let us assume

(i) there is $c > 0$ such that for any $f \in L$ there is $\{f_n\} \subset K$ with $f_n \xrightarrow{w'} f$ and $\|f_n\| \leqslant c\|f\|$.

If (i) is satisfied and $\Phi \in K^*$ then

$$|J(\Phi, f)| = |\lim \Phi(f_n)| \leqslant \|\Phi\| \sup \|f_n\| \leqslant c\|\Phi\| \cdot \|f\|.$$

To complete the proof it is sufficient to show (i). □

Lemma. *Let* X, Y *be such that for every* $f \in L(X,Y)$ *there is a sequence* $\{f_n\} \subset K(X,Y)$ *such that* $f_n \xrightarrow{w'} f$. *Then the condition (i) is satisfied. In deed, the norm* $|||\cdot|||$

$$|||f||| = \inf\{\sup_n \|f_n\|;\ f_n \subset K,\ f_n \xrightarrow{w'} f\} \quad \text{for } f \in L(X,Y)$$

is an equivalent norm on $L(X,Y)$.

Proof. The uniform boundedness theorem yields that if $f_n \xrightarrow{w'} f$ then $\{f_n\}$ is bounded in the norm so that $|||f|||$ is finite. We observe that $\|\cdot\| \leqslant |||\cdot|||$ on L. In fact for any $\varepsilon > 0$ let $\|x\| \leqslant 1$ and $\|y^*\| \leqslant 1$ be such that

$$\|f\| - \varepsilon \leqslant \left|y^*\big(f(x)\big)\right| = \lim \left|y^*\big(f_n(x)\big)\right| \leqslant \sup \|f_n\|.$$

Passing to the infimum gives the claim. Evidently $|||\cdot|||$ is a norm on L. Now we observe that $(L, |||\cdot|||)$ is complete. To prove this it is sufficient to show that if $f_p \in L$, $\sum_{p=1}^{\infty} |||f_p||| < \infty$ then $\sum_{p=1}^{\infty} f_p \in L$ exists in L and $|||\sum f_p||| \leqslant \sum |||f_p|||$ (cf. Theorem 6.2.3

[7]). To see this let $f_{np} \in K$ be such that $f_{np} \xrightarrow[n]{w'} f_p$, $\sup_n \|f_{np}\| \leqslant |||f_p||| + \frac{\varepsilon}{2^p}$. If $\|x^{**}\| \leqslant 1$, $\|y^*\| \leqslant 1$ then we have

$$\left|x^{**}\left(f_{np}^*(y^*)\right)\right| \leqslant |||f_p||| + \frac{\varepsilon}{2^p} \quad \text{for all } n.$$

Thus $\sum_p x^{**}\left(f_{np}^*(y^*)\right)$ converges uniformly in n and

$$\lim_n \sum_{p=1}^{\infty} x^{**}\left(f_{np}^*(y^*)\right) = \sum_{p=1}^{\infty} \lim x^{**}\left(f_{np}^*(y^*)\right) = \sum_{p=1}^{\infty} x^{**}\left(f_p^*(y^*)\right). \tag{1}$$

Observe now that $\sum_p f_p \in L$ exists because $\|f_p\| \leqslant |||f_p|||$ and similarly also $\sum_{p=1}^{\infty} f_{np} \in K$ exists because K is $\|\ \|$-complete. Thus (1) implies that

$$\sum_p f_{np} \xrightarrow[n]{w'} \sum f_p.$$

Then $|||\sum f_p||| \leqslant \sup_n \|\sum_p f_{np}\| \leqslant \sup_n \sum_p \|f_{np}\| \leqslant \varepsilon + \sum_p |||f_p|||$, showing that $|||\sum_p f_p||| \leqslant \sum_p |||f_p|||$. Finally, the open mapping theorem gives that $|||\cdot||| \leqslant \frac{c}{2}\|\cdot\|$ which implies (i). □

Proposition 2. *Suppose that every $f \in L(X,Y)$ is factorable through a Banach space Z, $f = f_1 f_2$ (Z depending on f) such that Z^* is separable and has the approximation property. Then for every $f \in L$ there is a sequence $\{f_n\} \subset K$ with $f_n \xrightarrow{w'} f$, i.e. the assumptions of Proposition 1 are satisfied.*

Proof. Under the assumptions Z^* has the metric approximation property. Let $f = f_1 f_2$ be any factorization of $f \in L$ through the Banach space Z, let $p_n(z^*) \to z^*$ for every $z^* \in Z^*$. We may suppose that $p_n = P_n^*$ where $P_n \in K(Z)$, $\|P_n\| \leqslant 1$ are finite-dimensional operators [5]. Let us define $f_n = f_1 P_n f_2 \in K$. Then $f_n \xrightarrow{w'} f$. □

Remark 1. J gives rise to two isomorphic imbeddings:

$$J_K\colon K^* \to L^* \quad J_K\Phi(f) = J(\Phi, f)$$

and

$$J_L\colon L \to K^{**} \quad J_L f(\Phi) = J(\Phi, f), \quad J_L = J_K^*/L.$$

Evidently $J_L f = f$ if $f \in K$.

Moreover,

$$\|\Phi\| \leqslant \|J_K \Phi\| \leqslant c\|\Phi\| \quad \text{for all } \Phi \in K^*$$

and

$$\|f\| \leqslant \|J_L f\| \leqslant c\|f\| \quad \text{for all } f \in L.$$

Thus J_K and J_L are c isomorphisms and $\|J_K\| \leqslant c$, $\|J_L\| \leqslant c$.

Proof. Indeed, given $f \in L$ and $\varepsilon > 0$ we have for suitable $\|x\| = 1$, $\|y^*\| = 1$

$$\begin{aligned}\|f\| - \varepsilon \leqslant \left|y^*\big(f(x)\big)\right| &= \left|\lim x\big(f_n^*(y^*)\big)\right| \\ &= |J(x \otimes y^*, f)| = |J_L f(x \otimes y^*)| \\ &\leqslant \sup\big\{|J_L f(\Phi)|;\ \|\Phi\| \leqslant 1\big\} = \|J_L f\|.\end{aligned}$$

Similarly $\|\Phi\| = \sup\big\{|\Phi(f)|;\ f \in K;\ \|f\| \leqslant 1\big\}$. But $\Phi(f) = J(\Phi, f) = J_K \Phi(f)$. Thus

$$\|\Phi\| \leqslant \sup\big\{|J_K \Phi(f)|;\ f \in L;\ \|f\| \leqslant 1\big\} = \|J_K \Phi\|.$$

□

Remark 2. If Re : $L^* \to K^*$ is the restriction operator then $P = J_K \operatorname{Re}$ is a projection in L^* whose range is c-isomorphic to K^* and $\operatorname{Ker} P = K^\circ$.

This is J. Johnson's type of statement and it follows immediately from Remark 1.

Remark 3. Let every $f \in L(X, Y)$ be factorable as indicated in the assumption of Proposition 2. Let us put

$$p(f) = \inf \|f_1\| \cdot \|f_2\|$$

where the infimum is taken over all factorizations of f through any Z such that Z^* has the bounded approximation property and is separable. Then

a) p is an equivalent norm on $L(X, Y)$;

b) for every $\varepsilon > 0$ there are $f_n \in K$ such that

$$f_n \xrightarrow{w'} f \quad \text{and} \quad p(f_n) \leqslant (1 + \varepsilon) p(f).$$

Thus

b_1) $$|J(\Phi, f)| \leqslant p^*(\Phi) p(f) \quad \text{for } f \in L \text{ and } \Phi \in K^*.$$

Easy observations similar as in Remark 1 give that J_K and J_L are p-isometries and $p(P) = 1$. Thus K is an ideal in (L, p) in the terminology of [2]. The question when e.g. (L, p) is a u-ideal or an M-ideal will be treated in a subsequent paper.

P r o o f. We show e.g. a). As in the proof of Proposition 1 we have $\|\cdot\| \leqslant p(\cdot)$ on L. Evidently p is subadditive. In fact, let $f_i = B_i A_i$ be factorizations of f_i through suitable Z_i so that $\sum_i \|A_i\| \cdot \|B_i\| \leqslant \varepsilon + \sum_i p(f_i)$, $\|A_i\| = \|B_i\|$. Let us put

$$Z = (Z_i)_{\ell_2} \quad \text{and} \quad A = (A_i)\colon X \to Z,$$

$B\colon Z \to Y$, $B(\{z_i\}) = \sum B_i(z_i)$. Then $\|A\|^2 \leqslant \sum \|A_i\|^2$ and $\|B\|^2 = \|B^*\|^2 \leqslant \sum \|B_i\|^2$. Thus

$$p\Big(\sum f_i\Big) \leqslant \|A\| \cdot \|B\| \leqslant \sum \|A_i\| \cdot \|B_i\| \leqslant \varepsilon + \sum p(f_i).$$

To see that (L, p) is complete it suffices as in the proof of the Lemma to show the following: Let $f_i \in L$ be such that $\sum p(f_i) < \infty$. Then $\sum f_i \in L$ and $p(\sum f_i) \leqslant \sum p(f_i)$. But this is exactly the above proof of the subadditivity of p. □

Remark 4. The isomorphism $J_L\colon L \to K^{**}$ together with the local reflexivity of K gives: Under the assumptions of Proposition 1 the Banach space L is $(c+\varepsilon)$-finitely representable in K so that the representations are the identity on K.

Remark 5. It is not necessary to assume in Proposition 2 that Z^* is separable. In fact, the following is sufficient for the statement of Proposition 2 (and for Remark 3): Every $f \in L$ is factorable through a Banach space Z, $f = f_1 f_2$ (Z depending on f) such that Z^* has the bounded approximation property and $f_1^*(Y^*) \subset Z^*$ is separable.

Remark 6. Another modification of Proposition 2 is the following:

Suppose that every $f \in L(X, Y)$ is factorable through a Banach space Z, (Z depending on f) such that there is a sequence $\{P_n\}$ in the unit ball of $K(Z)$ such that $P_n \to Id_Z$ in the weak operator topology and such that Z has the property $(**)$ defined in [1, p. 678]. Then the assumptions of Proposition 1 are satisfied.

In order to have $(**)$ it is sufficient that Z has the unique extension property in the sense of [1].

Acknowledgement. The author is indebted to Gerhard Racher and Dirk Werner for valuable discussions on the subject.

References

[1] *G.Godefroy, P. Saphar:*: Duality in spaces of operators and smooth norms on Banach spaces. Illinois Journal Math. *32 No. 4* (1988), 672–695.

[2] *G. Godefroy, N.J. Kalton, P.D. Saphar*: Unconditional ideals in Banach spaces. Studia Math. *104* (1993), 13–59.

[3] *K. John*: On the space $K(P, P^*)$ of compact operators on Pisier space P. Note di Matematica *12* (1992), 69–75.

[4] *J. Johnson*: Remarks on Banach spaces of compact operators. Funct. Anal. *32* (1979), 304–311.

[5] *W.B. Johnson, H.P. Rosenthal, M. Zippin*: On bases, finite dimensional decompositions and weaker structures in Banach spaces. Israel J. M. *9* (1971), 488–506.

[6] *N.J. Kalton*: Spaces of compact operators. Math. Ann. *208* (1974), 267–278.

[7] *A. Pietsch*: Operator Ideals. Deutscher Verlag der Wissenschaften, Berlin, 1978.

Author's address: Mathematical Institute of the Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic.