Kamil John
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ON A RESULT OF J. JOHNSON

KAMIL JOHN, Praha

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J. Johnson proved in [4] that if Y is a Banach space having the bounded approximation property then the anulator $K(X,Y)$ in $L(X,Y)^*$ is the kernel of a projection $P$ in $L(X,Y)^*$. Here $X$ is an arbitrary Banach space and $K(X,Y) = K$, $L(X,Y) = L$, denote respectively the space of all compact or bounded operators $f: X \to Y$. Moreover, the range space of the projection $P$ is isomorphic to $K^*$. In [3] the same statement was shown to be true for the spaces $X = P$ and $Y = P^*$ where $P$ is any separable Pisier space. Notice that here Johnson’s result cannot be applied since $P^*$ (and $P$) do not even have the approximation property. The proof in [3] was based on the fact that every $f: P \to P^*$ is factorable through a Hilbert space. In this note we observe (see Proposition 2 and Remarks 1 and 2) that Johnson’s result holds for any couples of Banach spaces $X, Y$ such that any $f: X \to Y$ is factorable through a Banach space $Z, Z^*$ having the bounded approximation property and $Z^*$ being separable. In fact much weaker assumptions are shown to be sufficient for J. Johnson’s result (Proposition 1 and Remark 5).

Following N. Kalton [6] we denote by $w'$ the topology $L(X,Y) = L$ (projectively) generated by all $x^{**} \otimes y^*$ where $x^{**} \in X^{**}$ and $y^* \in Y^*$. Thus we write $f_n \stackrel{w'}{\to} f$ to denote that for any $x^{**}$, and $y^*$ we have $x^{**}(f_n^*(y^*)) \to x^{**}(f^*(y^*))$. We will make crucial use of the following result of Kalton:

(K) If $\{f_n\} \subset K$ is a sequence of compact operators such that $f_n \stackrel{w'}{\to} f$ and if $f: x \to y$ is compact then $f_n \to f$ in the weak topology of $L(X,Y)$.

We say that the operator $f: X \to Y$ is factorable through a Banach space $Z$ if $f = f_1 f_2$ where $f_2: X \to Z$ and $f_1: Z \to Y$ are operators. All operators in the paper are bounded linear operators.

**Proposition 1.** Let $X, Y$ be Banach spaces such that for every $f \in L(X,Y) = L$ there is a sequence $\{f_n\} \subset K(X,Y) = K$ such that $f_n \stackrel{w'}{\to} f$. Then there exists a continuous bilinear form $J: K^* \times L \to R$ (scalars) and a number $c > 0$ such that
a) if $f \in K$ and $\Phi \in K^*$ then $J(\Phi, f) = \Phi(f)$;
b) $|J(\Phi, f)| \leq c\|\Phi\| \cdot \|f\|$ for all $f \in L$ and $\Phi \in K^*$;
c) $J(\Phi, f) = \lim \Phi(f_n)$ where $\{f_n\}$ is any sequence of compact operators $f_n \in K$
tending $w'$ to $f$.

Proof. First we observe that if $f_n \xrightarrow{w'} f, f \in L$ and $f_n \in K$ then $\lim \Phi(f_n)$
exists for all $\Phi \in K^*$. Indeed, $\{\Phi(f_n)\}$ is bounded by the uniform boundedness principle and thus $\limsup \Phi(f_n) = \lim \Phi(f_{n_k})$ and $\liminf \Phi(f_n) = \lim \Phi(f_{m_k})$ for
suitable subsequences $\{n_k\}$ and $\{m_k\}$ of natural numbers. Thus $\limsup \Phi(f_n) - \liminf \Phi(f_n) = \lim \Phi(f_{n_k} - f_{m_k}) = 0$, because $f_{n_k} - f_{m_k} \rightharpoonup 0$ weakly by (K). Similarly
we show that if $f_n \xrightarrow{w'} f$ and $g_n \xrightarrow{w'} f$ with $\{f_n\} \subset K$ and $\{g_n\} \subset K$ then
$\lim \Phi(f_n) = \lim \Phi(g_n)$ for any $\Phi \in K^*$. Thus we may define $J(\Phi, f)$ by c). $J$ is
evidently bilinear and if $f \in K$ then $J(\Phi, f) = \lim \Phi(f_n) = \Phi(f)$ because $f_n = f \xrightarrow{w'} f$. To show b) let us assume

(i) there is $c > 0$ such that for any $f \in L$ there is $\{f_n\} \subset K$ with $f_n \xrightarrow{w'} f$ and
$\|f_n\| \leq c\|f\|$.
If (i) is satisfied and $\Phi \in K^*$ then

$$|J(\Phi, f)| = |\lim \Phi(f_n)| \leq \|\Phi\| \sup \|f_n\| \leq c\|\Phi\| \cdot \|f\|.$$ 

To complete the proof it is sufficient to show (i).

Lemma. Let $X, Y$ be such that for every $f \in L(X, Y)$ there is a sequence $\{f_n\} \subset K(X, Y)$
such that $f_n \xrightarrow{w'} f$. Then the condition (i) is satisfied. In deed, the norm
$\| \cdot \|$ 

$$\|f\| = \inf \{\sup \|f_n\|; \; f_n \subset K, \; f_n \xrightarrow{w'} f\} \quad \text{for} \; f \in L(X, Y)$$
is an equivalent norm on $L(X, Y)$.

Proof. The uniform boundedness theorem yields that if $f_n \xrightarrow{w'} f$ then $\{f_n\}$ is
bounded in the norm so that $\|f\|$ is finite. We observe that $\| \cdot \| \leq \| \cdot \|$ on $L$. In fact
for any $\varepsilon > 0$ let $\|x\| \leq 1$ and $\|y^*\| \leq 1$ be such that

$$\|f\| - \varepsilon \leq |y^*(f(x))| = \lim |y^*(f_n(x))| \leq \sup \|f_n\|.$$ 

Passing to the infimum gives the claim. Evidently $\| \cdot \|$ is a norm on $L$. Now we
observe that $(L, \| \cdot \|)$ is complete. To prove this it is sufficient to show that if $f_p \in L,
\sum_{p=1}^{\infty} \|f_p\| < \infty$ then $\sum_{p=1}^{\infty} f_p \in L$ exists in $L$ and $\| \sum f_p\| \leq \| \sum \|f_p\|$ (cf. Theorem 6.2.3

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To see this let \( f_{np} \in K \) be such that \( f_{np} \xrightarrow{w^*} f_p, \sup_n \|f_{np}\| \leq \|f_p\| + \frac{\varepsilon}{2^p} \). If \( \|x^{**}\| \leq 1, \|y^*\| \leq 1 \) then we have

\[
|x^{**}(f_{np}^*(y^*))| \leq \|f_p\| + \frac{\varepsilon}{2^p} \quad \text{for all } n.
\]

Thus \( \sum_p x^{**}(f_{np}^*(y^*)) \) converges uniformly in \( n \) and

\[
\lim_n \sum_{p=1}^{\infty} x^{**}(f_{np}^*(y^*)) = \sum_{p=1}^{\infty} \lim x^{**}(f_{np}^*(y^*)) = \sum_{p=1}^{\infty} x^{**}(f_p^*(y^*)).
\]

Observe now that \( \sum_p f_p \in L \) exists because \( \|f_p\| \leq \|f_p\| \) and similarly also \( \sum_{p=1}^{\infty} f_{np} \in K \) exists because \( K \) is \( \| \| \)-complete. Thus (1) implies that

\[
\sum_p f_{np} \xrightarrow{w^*} \sum_p f_p.
\]

Then \( \| \sum_p f_p \| \leq \sup_n \| \sum f_{np} \| \leq \sup_n \| f_{np} \| \leq \varepsilon + \sum_p \| f_p \| \), showing that \( \| \sum_p f_p \| \leq \sum_p \| f_p \| \). Finally, the open mapping theorem gives that \( \| \cdot \| \leq \frac{1}{2} \| \cdot \| \) which implies (i). \( \square \)

**Proposition 2.** Suppose that every \( f \in L(X,Y) \) is factorable through a Banach space \( Z \), \( f = f_1f_2 \) (\( Z \) depending on \( f \)) such that \( Z^* \) is separable and has the approximation property. Then for every \( f \in L \) there is a sequence \( \{f_n\} \subset K \) with \( f_n \xrightarrow{w^*} f \), i.e. the assumptions of Proposition 1 are satisfied.

**Proof.** Under the assumptions \( Z^* \) has the metric approximation property. Let \( f = f_1f_2 \) be any factorization of \( f \in L \) through the Banach space \( Z \), let \( p_n(z^*) \to z^* \) for every \( z^* \in Z^* \). We may suppose that \( p_n = P_n \) where \( P_n \in K(Z), \|P_n\| \leq 1 \) are finite-dimensional operators [5]. Let us define \( f_n = f_1P_nf_2 \in K \). Then \( f_n \xrightarrow{w^*} f \).

\( \square \)

**Remark 1.** \( J \) gives rise to two isomorphic imbeddings:

\[
J_K: K^* \to L^* \quad J_K\Phi(f) = J(\Phi, f)
\]

and

\[
J_L: L \to K^{**} \quad J_Lf(\Phi) = J(\Phi, f), \quad J_L = J_K^*/L.
\]

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Evidently $J_L f = f$ if $f \in K$.
Moreover, 
$$\|\Phi\| \leq \|J_K \Phi\| \leq c \|\Phi\| \quad \text{for all } \Phi \in K^*$$
and 
$$\|f\| \leq \|J_L f\| \leq c \|f\| \quad \text{for all } f \in L.$$ 
Thus $J_K$ and $J_L$ are $c$ isomorphisms and $\|J_K\| \leq c$, $\|J_L\| \leq c$.

**Proof.** Indeed, given $f \in L$ and $\varepsilon > 0$ we have for suitable $\|x\| = 1$, $\|y^*\| = 1$

$$\|f\| - \varepsilon \leq \left| y^*(f(x)) \right| = \left| \lim x\left( f_n^*(y^*) \right) \right|$$

$$= \left| J(x \otimes y^*, f) \right| = \left| J_L f(x \otimes y^*) \right|$$

$$\leq \sup \left\{ \|J_L f(\Phi)\| ; \|\Phi\| \leq 1 \right\} = \|J_L f\|.$$

Similarly $\|\Phi\| = \sup \left\{ \|\Phi(f)\| ; f \in K ; \|f\| \leq 1 \right\}$. But $\Phi(f) = J(\Phi, f) = J_K(\Phi, f)$. Thus 

$$\|\Phi\| \leq \sup \left\{ \|J_K(\Phi(f))\| ; f \in L ; \|f\| \leq 1 \right\} = \|J_K \Phi\|.$$

QED

**Remark 2.** If $\text{Re} : L^* \to K^*$ is the restriction operator then $P = J_K \text{Re}$ is a projection in $L^*$ whose range is $c$-isomorphic to $K^*$ and $\text{Ker} P = K^0$.

This is J. Johnson’s type of statement and it follows immediately from Remark 1.

**Remark 3.** Let every $f \in L(X, Y)$ be factorable as indicated in the assumption of Proposition 2. Let us put

$$p(f) = \inf \|f_1\| \cdot \|f_2\|$$

where the infimum is taken over all factorizations of $f$ through any $Z$ such that $Z^*$ has the bounded approximation property and is separable. Then

a) $p$ is an equivalent norm on $L(X, Y)$;
b) for every $\varepsilon > 0$ there are $f_n \in K$ such that

$$f_n \xrightarrow{w'} f \quad \text{and} \quad p(f_n) \leq (1 + \varepsilon)p(f).$$

Thus 

$$b_1) \quad |J(\Phi, f)| \leq p^*(\Phi)p(f) \quad \text{for } f \in L \text{ and } \Phi \in K^*.$$ 

Easy observations similar as in Remark 1 give that $J_K$ and $J_L$ are $p$-isometries and $p(P) = 1$. Thus $K$ is an ideal in $(L, p)$ in the terminology of [2]. The question when e.g. $(L, p)$ is a $u$-ideal or an $M$-ideal will be treated in a subsequent paper.

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Proof. We show e.g. a). As in the proof of Proposition 1 we have \( \| \cdot \| \leq p(\cdot) \) on \( L \). Evidently \( p \) is subadditive. In fact, let \( f_i = B_iA_i \) be factorizations of \( f_i \) through suitable \( Z_i \) so that \( \sum_i \|A_i\| \cdot \|B_i\| \leq \varepsilon + \sum_i p(f_i), \|A_i\| = \|B_i\| \). Let us put
\[
Z = (Z_i)_{t_2} \quad \text{and} \quad A = (A_i): X \to Z,
\]
\[
B: Z \to Y, \quad B(\{z_i\}) = \sum B_i(z_i). \quad \text{Then} \quad \|A\|^2 \leq \sum \|A_i\|^2 \quad \text{and} \quad \|B\|^2 = \|B^*\|^2 \leq \sum \|B_i\|^2. \quad \text{Thus}
\[
p\left( \sum f_i \right) \leq \|A\| \cdot \|B\| \leq \sum \|A_i\| \cdot \|B_i\| \leq \varepsilon + \sum p(f_i).
\]
To see that \((L,p)\) is complete it suffices as in the proof of the Lemma to show the following: Let \( f_i \in L \) be such that \( \sum p(f_i) < \infty \). Then \( \sum f_i \in L \) and \( p(\sum f_i) \leq \sum p(f_i) \). But this is exactly the above proof of the subadditivity of \( p \). \( \square \)

Remark 4. The isomorphism \( J_L: L \to K^{**} \) together with the local reflexivity of \( K \) gives: Under the assumptions of Proposition 1 the Banach space \( L \) is \((c+\varepsilon)\)-finitely representable in \( K \) so that the representations are the identity on \( K \).

Remark 5. It is not necessary to assume in Proposition 2 that \( Z^* \) is separable. In fact, the following is sufficient for the statement of Proposition 2 (and for Remark 3): Every \( f \in L \) is factorable through a Banach space \( Z \), \( f = f_1f_2 \) (\( Z \) depending on \( f \)) such that \( Z^* \) has the bounded approximation property and \( f_1^*(Y^*) \subset Z^* \) is separable.

Remark 6. Another modification of Proposition 2 is the following:
Suppose that every \( f \in L(X,Y) \) is factorable through a Banach space \( Z \), (\( Z \) depending on \( f \)) such that there is a sequence \( \{P_n\} \) in the unit ball of \( K(Z) \) such that \( P_n \to Id_Z \) in the weak operator topology and such that \( Z \) has the property (**) defined in [1, p. 678]. Then the assumptions of Proposition 1 are satisfied.
In order to have (**) it is sufficient that \( Z \) has the unique extension property in the sense of [1].

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References


Author's address: Mathematical Institute of the Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic.