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SPACES WITH A BOREL-COMPLETE STONE-ČECH COMPACTIFICATION

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A topological space is called *Borel-complete* if every two-valued Borel measure is a Dirac measure. In this short note we want to investigate the Tychonoff spaces X whose Stone-Čech compactification βX is Borel-complete. In [3] it was proved that pseudocompactness is a necessary condition; moreover a conjecture was stated that X is even compact. In other words, the conjecture claims that the following class

$$\mathscr{B} := \{\beta X \setminus X;\ \beta X \text{ is Borel-complete}\} \tag{1}$$

consists only of the empty set. Our main result shows that the conjecture is not true; e.g. it shows that every separable metric space is in $\mathscr{B}$.

Theorem. *A Tychonoff space is in the class $\mathscr{B}$ (up to homeomorphy) iff it is a subset of a Borel-complete compact space.*

Let Ψ be the space described in [6, p. 272]. As pointed out in [3] Ψ is a Borel-complete pseudocompact space. Moreover one can assume that $\Psi^* := \beta\Psi \setminus \Psi$ consists exactly of one element, cf. [4]. Consequently $\beta\Psi$ is the union of two Borel-complete [illegible]el subsets; by Theorem 2.4 in [3] $\beta\Psi$ is Borel-complete. This example already shows that the above conjecture is not valid.

Proof of Theorem. The necessity is clear since Borel-completeness is hereditary. Let now Y be a subset of a Borel-complete compact Hausdorff space K. Define $S := \beta\Psi \times K \setminus (\Psi^* \times Y)$. At first we show that $\beta S = \beta\Psi \times K$. Note that $\Psi \times K$ is a pseudocompact space. By Glicksberg's Theorem (see [6, p. 199]) $\Psi \times K$ is C^*-embedded in $\beta\Psi \times K$. Since S contains $\Psi \times K$ the space S is C^* embedded in $\beta\Psi \times K$ and therefore $\beta S = \beta\Psi \times K$. As a product of two Borel-complete spaces βS is Borel-complete. On the other side the space $\beta S \setminus S = \Psi^* \times Y$ is homeomorphic to Y. The proof is complete. □

As pointed out in [3] it is unclear which topological spaces X admit a Borel-complete compact extension. By Theorem 3.1 in [3] every space with a perfectly normal compactification and by Corollary 2.5 every discrete space of non-measurable cardinality have this property.

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