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ASYMPTOTIC PROPERTIES OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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In this paper we are concerned with the oscillatory and asymptotic behavior of the solutions of the delay differential equation

$$L_n u(t) - p(t)u(g(t)) = 0 \tag{1}$$

and its special case, the ordinary differential equation

$$L_n u(t) - p(t)u(t) = 0, \tag{2}$$

where $n \geqslant 3$ and L_n denotes the general disconjugate differential operator of the form

$$L_n \cdot = \frac{1}{r_n(t)} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{r_{n-1}(t)} \frac{\mathrm{d}}{\mathrm{d}t} \cdots \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{r_1(t)} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\cdot}{r_0(t)}. \tag{3}$$

It is always assumed that

(i) $r_i(t)$, $0 \leqslant i \leqslant n$, $p(t)$ and $g(t)$ are continuous, $r_i(t) > 0$, $p(t) > 0$, $g(t) \leqslant t$ and $g(t) \to \infty$ as $t \to \infty$;

(ii)
$$\int^{\infty} r_i(s)\,\mathrm{d}s = \infty \quad \text{for} \quad 1 \leqslant i \leqslant n-1.$$

The operator L_n satisfying (ii) is said to be in canonical form. It is known that any differential operator of the form (3) can always be represented in a canonical form in an essentially unique way (see Trench [11]). In the sequel we will assume that the operator L_n is in canonical form.

We introduce the following notation:

$$D_0 u(t) = \frac{u(t)}{r_0(t)},$$
$$D_i u(t) = \frac{1}{r_i(t)} \frac{\mathrm{d}}{\mathrm{d}t} D_{i-1} u(t), \quad 1 \leqslant i \leqslant n.$$

Equation (1) can then be rewritten as

$$D_n u(t) - p(t) u(g(t)) = 0.$$

The domain $D(L_n)$ of L_n is defined to be the set of all functions $u\colon [T_u, \infty) \to \mathbb{R}$ such that $D_i u(t)$, $0 \leqslant i \leqslant n$ exist and are continuous on $[T_u, \infty)$. A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

If $u(t)$ is a nonoscillatory solution of (1) then according to a generalization of a lemma of Kiguradze [4, Lemma 3] there is an integer ℓ, $0 \leqslant \ell \leqslant n$ such that $\ell \equiv n \pmod 2$ and

$$\begin{aligned} u(t) D_i u(t) &> 0, \quad 0 \leqslant i \leqslant \ell, \\ (-1)^{i-\ell} u(t) D_i u(t) &> 0, \quad \ell \leqslant i \leqslant n \end{aligned} \tag{4}$$

for all sufficiently large t. A function $u(t)$ satisfying (4) is said to be a function of degree ℓ. The set of all nonoscillatory solutions of degree ℓ of (1) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (1), then

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \ldots \cup \mathcal{N}_n \quad \text{if } n \text{ is odd,}$$

and

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \ldots \cup \mathcal{N}_n \quad \text{if } n \text{ is even.}$$

It is known (see [3] and [4]) that the ordinary equation (2) has always a nonoscillatory solution of degree n ($\mathcal{N}_n \neq \emptyset$) regardless of the parity of n and that in the case n even equation (2) also has a nonoscillatory solution of degree 0 ($\mathcal{N}_0 \neq \emptyset$). Therefore, investigating the delay equation (1) which is of particular interest is the extreme situation described in the following definition.

Definition 1. Equation (1) is said to have property (B) if for n even $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_n$ and for n odd $\mathcal{N} = \mathcal{N}_n$.

Now, we shall investigate property (B) of equation (2) and then we will extend our results to the more general equation (1). In a recent paper Kusano, Naito and Tanaka [9] have established sufficient conditions for equation (2) to have property (B). Their results generalize those of Lovelady [5] for equations of the form $y^{(n)} - p(t)y = 0$. Kusano, Naito and Tanaka have compared equation (2) with a set of second order equations of the form

$$\left(\frac{y'(t)}{r_i(t)}\right)' + \hat{q}_i(t)y(t) = 0.$$

The objective of this paper is to improve the above-mentioned results. We compare equation (2) with the set of second order delay equations

$$\left(\frac{z'(t)}{r_i(t)}\right)' + q_i(t)z(\tau_i(t)) = 0,$$

where $q_i(t)$ and $\tau_i(t)$ will be defined below.

In passing, the term "property (A)", refers to the equation

$$L_n u(t) + p(t)u(g(t)) = 0, \tag{1^+}$$

and is used to describe the situation in which for n even (1^+) has no nonoscillatory solution and for n odd every nonoscillatory solution of (1^+) is of degree 0. For analogous results concerning property (A) of (1^+) the reader is referred to [2], [7] and [10].

We begin by formulating some preparatory results which are needed in proving the main theorems.

Let $i_k \in \{1, \ldots, n-1\}$, $1 \leqslant k \leqslant n-1$ and $t, s \in [t_0, \infty)$, We define

$$I_0 = 1,$$
$$I_k(t, s; r_{i_k}, \ldots, r_{i_1}) = \int_s^t r_{i_k}(x) I_{k-1}(x, s; r_{i_{k-1}}, \ldots, r_{i_1})\, \mathrm{d}x.$$

It is easy to verify that for $1 \leqslant k \leqslant n-1$

$$\begin{aligned} I_k(t, s; r_{i_k}, \ldots, r_{i_1}) &= (-1)^k I_k(s, t; r_{i_1}, \ldots, r_{i_k}), \\ I_k(t, s; r_{i_k}, \ldots, r_{i_1}) &= \int_s^t r_{i_1}(x) I_{k-1}(t, x; r_{i_k}, \ldots, r_{i_2})\, \mathrm{d}x. \end{aligned} \tag{5}$$

For simplicity of notation we put

$$\begin{aligned} J_i(t, s) &= r_0(t) I_i(t, s; r_1, \ldots, r_i), & J_i(t) &= J_i(t, t_0), \\ K_i(t, s) &= r_n(t) I_i(t, s; r_{n-1}, \ldots, r_{n-i}), & K_i(t) &= K_i(t, t_0). \end{aligned}$$

Lemma 1. *Let $r(t) > 0$ be a continuous function. Assume that $p(t)$ and $g(t)$ satisfy* (i). *Then the second order delay equation*

$$\left(\frac{1}{r(t)}y'(t)\right)' + p(t)y(g(t)) = 0$$

has a positive solution if and only if so does the corresponding differential inequality

$$\left(\frac{1}{r(t)}y'(t)\right)' + p(t)y(g(t)) \leqslant 0.$$

For the proof of this lemma see e.g. Kusano and Naito [8].

Lemma 2. *Let i, $0 \leqslant i \leqslant n-1$ be fixed. Equation* (2) *has a solution $u(t)$ satisfying*

$$\lim_{t\to\infty} D_i u(t) = c_i \in \mathbb{R} - \{0\}$$

for some c_i if and only if

$$\int^{\infty} K_{n-i-1}(t)J_i(t)p(t)\,\mathrm{d}t < \infty.$$

This lemma is an analog of Theorem 1 of Kitamura and Kusano [6].

Lemma 3. *If $u \in D(L_n)$ then the following formula holds for $0 \leqslant i \leqslant k \leqslant n-1$ and $t, s \in [T_u, \infty)$:*

$$\begin{aligned} D_i u(t) = &\sum_{j=i}^{k} (-1)^{j-i} D_j u(s) I_{j-i}(s,t;r_j,\ldots,r_{i+1}) \\ &+ (-1)^{k-i+1} \int_t^s I_{k-i}(x,t;r_k,\ldots,r_{i+1}) r_{k+1}(x) D_{k+1}u(x)\,\mathrm{d}x. \end{aligned}$$

This lemma is a generalization of Taylor's formula with remainder encountered in calculus. The proof is immediate.

The following theorem can be found in [9].

Theorem 1. *Let*

$$\int^{\infty} K_{n-i-1}(t)J_{i-1}(t)p(t)\,\mathrm{d}t = \infty \tag{6}$$

for $i = 2,4,\ldots,n-2$ if n is even and for $i = 1,3,\ldots,n-2$ if n is odd. Then equation (2) *has property* (B).

The following theorem generalizes Theorem 2 in [9] and covers the case when condition (6) is violated. For convenience we make use of the following notation:

$$q_i(t) = r_{i+1}(t)\int_t^\infty K_{n-i-2}(x,t)J_{i-1}(x,\tau_i(t))p(x)\,\mathrm{d}x, \qquad i = 1,2,\ldots,n-2, \tag{7}$$

where $\tau_i(t)\colon [t_0,\infty)\to\mathbb{R}$, $1\leqslant i\leqslant n-2$, are continuous and satisfy

$$\tau_i(t)\to\infty \text{ as } t\to\infty,\quad \tau_i(t)\leqslant t,\quad \text{for } i=1,2,\ldots,n-2. \tag{8}$$

Theorem 2. *Suppose that* (8) *holds and the integrals in* (6) *converge. Assume that the second order delay equations*

$$\left(\frac{z'(t)}{r_i(t)}\right)' + q_i(t)z(\tau_i(t)) = 0 \tag{E_i}$$

are oscillatory for $i = 2,4,\ldots,n-2$ if n is even and for $i = 1,3,\ldots,n-2$ if n is odd. Then equation (2) *has property* (B).

P r o o f. We assume that $u(t)$ is a positive solution of (2). Then there exists an integer $\ell\in\{0,1,\ldots,n\}$ such that $n+\ell$ is even, and a number t_1 such that (4) holds for $t\geqslant t_1$. For the sake of contradiction we assume that $1\leqslant\ell\leqslant n-2$. By Lemma 3 with $i=\ell+1$, $k=n-1$, $s\geqslant t\geqslant t_1$, using (2), (5) and letting $s\to\infty$ we obtain for $t\geqslant t_1$

$$-D_{\ell+1}u(t)\geqslant\int_t^\infty r_n(s)I_{n-\ell-2}(s,t;r_{n-1},\ldots,r_{\ell+2})p(s)u(s)\,\mathrm{d}s, \tag{9}$$

and if $\ell\geqslant 2$, putting $i=0$ and $k=\ell-2$ in Lemma 2 we get

$$D_0u(t)\geqslant\int_{t_1}^t I_{\ell-2}(t,s;r_1,\ldots,r_{\ell-2})r_{\ell-1}(s)D_{\ell-1}u(s)\,\mathrm{d}s. \tag{10}$$

For details the reader is referred to [9]. Combining (9) with (10) we have

$$\begin{aligned}
-D_{\ell+1}u(t) &\geqslant \int_t^\infty r_n(x)I_{n-\ell-2}(x,t;r_{n-1},\ldots,r_{\ell+2})p(x)r_0(x)\\
&\quad\times\int_{t_1}^x I_{\ell-2}(x,s;r_1,\ldots,r_{\ell-2})r_{\ell-1}(s)D_{\ell-1}u(s)\,\mathrm{d}s\,\mathrm{d}x\\
&\geqslant \int_t^\infty r_n(x)I_{n-\ell-2}(x,t;r_{n-1},\ldots,r_{\ell+2})p(x)r_0(x)\\
&\quad\times\int_{\tau_\ell(t)}^x I_{\ell-2}(x,s;r_1,\ldots,r_{\ell-2})r_{\ell-1}(s)D_{\ell-1}u(s)\,\mathrm{d}s\,\mathrm{d}x
\end{aligned}$$

for all $t \geqslant t_2$, where $t_2 \geqslant t_1$ is chosen so that $\tau_\ell(t) \geqslant t_1$ for $t \geqslant t_2$. Since $D_{\ell-1}u(t)$ is increasing, we conclude from above that

$$-D_{\ell+1}u(t) \geqslant D_{\ell-1}(\tau_\ell(t)) \int_t^\infty r_n(x) I_{n-\ell-2}(x,t;r_{n-1},\dots,r_{\ell+2})p(x)r_0(x) \\ \times \int_{\tau_\ell(t)}^x I_{\ell-2}(x,s;r_1,\dots,r_{\ell-2})r_{\ell-1}(s)\,\mathrm{d}s\,\mathrm{d}x.$$

Let $y(t)$ be given by

$$y(t) = D_{\ell-1}u(t).$$

Note that $y(t) > 0$ and in view of the above inequalities

$$(11) \qquad -D_{\ell+1}u(t) \geqslant y(\tau_\ell(t)) \int_t^\infty K_{n-\ell-2}(x,t) J_{\ell-1}(x,\tau_\ell(t))p(x)\,\mathrm{d}x.$$

That (11) also holds for $\ell = 1$ follows from (9) and the fact that $D_0u(t) \geqslant D_0u(\tau_1(t))$. Noting that

$$\left(\frac{y'(t)}{r_\ell(t)}\right)' = r_{\ell+1}(t)D_{\ell+1}u(t),$$

we obtain from (11) that $y(t)$ is a positive solution of the differential inequality

$$\left(\frac{y'(t)}{r_\ell(t)}\right)' + q_\ell(t)y(\tau_\ell(t)) \leqslant 0 \quad \text{for} \quad t \geqslant t_2.$$

Lemma 1 then implies that equation (E_ℓ) has an eventually positive solution. But this contradicts our assumptions. The proof is complete now. □

We show that the conclusions of Theorems 1 and 2 can be strengthened as follows:

Theorem 3. *Assume that equation* (2) *has property* (B). *Then every nonoscillatory solution $u(t)$ of degree* 0 *of* (2) *satisfies*

$$\lim_{t\to\infty} D_0u(t) = 0$$

if and only if

$$\int^\infty J_0(t)K_{n-1}(t)p(t)\,\mathrm{d}t = \infty,$$

and every nonoscillatory solution $u(t)$ of degree n of (2) *satisfies*

$$\lim_{t\to\infty} |D_{n-1}u(t)| = \lim_{t\to\infty} \frac{|D_0u(t)|}{J_{n-1}(t)} = \infty$$

if and only if

$$\int^\infty J_{n-1}(t)K_0(t)p(t)\,\mathrm{d}t = \infty.$$

The proof of this theorem immediately follows from Lemma 3. □

For the special case of equation (2), namely, for the equation

$$\left(\frac{1}{r_3(t)}\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}u'(t)\right)'\right)'\right)' + p(t)u(g(t)) = 0 \tag{12}$$

we have the following result:

Corollary 1. *Let* $\tau_2(t)$ *satisfy* (8). *Further suppose that the second order delay equation*

$$\left(\frac{z'(t)}{r_2(t)}\right)' + \left(r_3(t)\int_t^\infty p(x)\int_{\tau_2(t)}^x r_1(s)\,\mathrm{d}s\,\mathrm{d}x\right) z(\tau_2(t)) = 0$$

is oscillatory. Then equation (12) *has property* (B).

Example 1. Let us consider the fourth order Euler equation

$$(t^{1/2}y''')' - \frac{a}{t^{7/2}}y = 0, \quad t > 1, \quad a > 0. \tag{13}$$

We put for this equation $\tau_2(t) = \lambda t$ for some $\lambda \in (0,1]$. Then by Corollary 1 equation (13) has property (B) if the second order delay equation

$$y''(t) + \frac{a}{t^2}\left(\frac{2}{3} - \frac{2}{5}\lambda\right) y(\lambda t) = 0$$

is oscillatory. By a generalization of the well-known criterion of Hille (see [1]), this occurs if

$$2a\lambda\left(\frac{1}{3} - \frac{\lambda}{5}\right) > \frac{1}{4}. \tag{14}$$

If we put $\lambda = \frac{5}{6}$ then (14) reduces to $a > \frac{9}{10}$ and moreover by Theorem 3 if $a > \frac{9}{10}$ then every nonoscillatory solution $y(t)$ of (18) satisfies either $\lim\limits_{t\to\infty} y(t) = 0$ or $\lim\limits_{t\to\infty} \frac{|u(t)|}{t^{5/2}} = \infty$. One should note that for the choice $\lambda = 1$ in (14) we obtain Kusano, Naito and Tanaka's criterion for (12) to have property (B), which is evidently weaker.

To describe better the situation in which not all second order equations (E_i) are oscillatory we use the following definition and in the sequel we suppose that $k_1, k_2, \ldots, k_m \in \{1, 2, \ldots, n-2\}$, where $m \geqslant 1$, are all mutually different and such that $n \equiv k_i \pmod 2$ for $1 \leqslant i \leqslant m$.

Definition 2. We say that equation (1) has property $B_{k_1,\ldots,k_m}$ if

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_{k_1} \cup \ldots \cup \mathcal{N}_{k_m} \cup \mathcal{N}_n \quad \text{if } n \text{ is even,}$$

and

$$\mathcal{N} = \mathcal{N}_{k_1} \cup \ldots \cup \mathcal{N}_{k_m} \cup \mathcal{N}_n \quad \text{if } n \text{ is odd.}$$

Theorem 4. *Assume that* (8) *holds. Let* (6) *be satisfied for* $i \in \{1,3,\ldots,n-1\} - \{k_1,\ldots,k_m\}$ *if* n *is odd and for* $i \in \{2,4,\ldots,n-1\} - \{k_1,\ldots,k_m\}$ *if* n *is even. Then equation* (2) *has property* $B_{k_1,\ldots,k_m}$.

Theorem 5. *Assume that* (8) *holds and the integrals in* (6) *converge. Let* $q_i(t)$ *and* $\tau_i(t)$, $1 \leqslant i \leqslant n-2$ *be defined as in* (7) *and* (8). *Then equation* (2) *has property* $B_{k_1,\ldots,k_m}$ *if equations* (E_i) *are oscillatory for* $i \in \{1,3,\ldots,n-1\} - \{k_1,\ldots,k_m\}$ *if* n *is odd and for* $i \in \{2,4,\ldots,n-1\} - \{k_1,\ldots,k_m\}$ *if* n *is even.*

The proofs of Theorems 4 and 5 follow from Theorems 1 and 2, taking Theorem 1 in [1] into account.

Example 2. Let us consider the fifth order delay equation

$$(t^{-1}y^{(4)}(t))' - \frac{a}{t^{\frac{7}{2}}}y(\sqrt{t}) = 0, \quad t > 1, \quad a > 0. \tag{15}$$

We put $\tau_1(t) = t$. Then, by Theorem 2 equation (15) has no solution of degree 1 if the second order equation

$$y''(t) + \frac{a}{15t^2}y(t) = 0 \tag{16}$$

is oscillatory, which is true if $a > \frac{15}{4}$. If we let $\tau_3(t) = \lambda t$ for some $\lambda \in (0,1]$ then by Theorem 2 equation (15) has no solution of degree 3 if the second order delay equation

$$y''(t) + \frac{a}{t^2}\left(\frac{1}{3} - \frac{\lambda}{2} + \frac{\lambda^2}{5}\right)y(\lambda t) = 0$$

is oscillatory, which by a generalization of Hille's criterion [1] occurs if

$$a\lambda\left(\frac{1}{3} - \frac{\lambda}{2} + \frac{\lambda^2}{5}\right) > \frac{1}{4}. \tag{17}$$

If we put $\lambda = \frac{5-\sqrt{5}}{6}$ then (17) reduces to $a > \frac{27}{2\sqrt{5}}$. Finally, by Theorems 2 and 5

if $a > \frac{27}{2\sqrt{5}}$ then equation (15) has property (B),
if $\frac{15}{4} < a < \frac{27}{2\sqrt{5}}$ then equation (15) has property B_3,
if $a > 0$ then equation (15) has property $B_{1,3}$.

Note that we have obtained a better result than the criterion in [9] provides.

Now we are prepared to extend our previous results to equation (1). Our extension is based on the following theorem which is due to Kusano and Naito [8].

Theorem 6. *Assume that*

$$g(t) \in C^1([t_0\infty)), \quad g'(t) > 0, \quad g(t) \leqslant t. \tag{18}$$

Let $g^{-1}(t)$ *be the inverse function to* $g(t)$. *Then equation* (1) *has property* (B) *if the equation*

$$D_n u(t) - \frac{p(g^{-1}(t))r_n(g^{-1}(t))}{g'(g^{-1}(t))r_n(t)} u(t) = 0 \tag{19}$$

has property (B).

Applying our previous results to equation (19) we obtain sufficient conditions for (1) to have property (B).

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