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A NOTE ON GRAPH COLOURING

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Graphs considered here are undirected, finite and simple (without loops or multiple edges), and [1] is followed for terminology. Let $G = (V, E)$ be a graph, with $V$ the set of vertices and $E$ the set of edges. We denote by $\gamma(G)$ the chromatic number of $G$. A $k$-colouring of $G$ ($k \geq \gamma(G)$) is a partition of $V$ into $k$ classes such that no two vertices belonging to the same class are adjacent. For a graph $G$ with $1 < \gamma(G) \leq 3$ we denote by $K(3, G)$ the number of 3-colourings of $G$.

**Theorem 1.** If $G$ is a connected graph with $n \geq 3$ vertices and $\gamma(G) = 2$, then $K(3, G) \leq 2^{n-2} - 1$, and the only connected graphs for which this upper bound is attained are trees with $n$ vertices.

**Proof.** We prove this result by induction on $n$. Obviously, for $n = 3$ the result is true. Let now $G$ be a connected graph with $n+1$ ($n+1 \geq 3$) vertices and $\gamma(G) = 2$. According to [1], $G$ contains at least two vertices which are not cut-vertices. Let $v$ be such a vertex and $G - v$ the connected graph obtained from $G$ by removing $v$. According to [2], a connected graph with the chromatic number equal to two has a unique 2-colouring. Thus, by induction hypothesis, we have

$$K(3, G) \leq 1 + 2K(3, G - v) \leq 1 + 2(2^{n-2} - 1) = 2^{n-1} - 1$$

since, on the one hand, in view of the fact that $G - v$ has a unique 2-colouring, there exists a unique 3-colouring of $G$ which has a class consisting only of $v$, and, on the other hand, $v$ may be added to a 3-colouring of $G - v$ in at most two different ways (the graph being connected, $v$ is joined by an edge with at least one vertex of $G - v$ which belongs to a class of the 3-colouring of $G - v$). We observe that the equality holds only if $v$ is joined with a single vertex of $G - v$ which is a tree. Thus, $G$ is also a tree. $\square$
According to [3], the chromatic polynomial of a cycle $C_n$ with $n$ vertices is given by

$$P(C_n; \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1). \quad (1)$$

On the other hand, also by [3], if $G$ is a graph with $n$ vertices and $\gamma(G) = k$, then the number of $k$-colourings of $G$ is equal to

$$P(G; k)/k!. \quad (2)$$

Since an odd cycle has the chromatic number equal to three, then, by (1) and (2), we have

**Theorem 2.** If $C_{2k+1}$ is a cycle with $2k+1$ vertices, then $K(3, C_{2k+1}) = \frac{1}{3}(4^k - 1)$.

**Theorem 3.** If $G$ is a connected graph with $n \geq 3$ vertices and $\gamma(G) = 3$, then

$$K(3, G) \leq \begin{cases} \frac{1}{3}(2^{n-1} - 1) & \text{if } n \text{ is odd}, \\ \frac{2}{3}(2^{n-2} - 1) & \text{if } n \text{ is even}. \end{cases}$$

If $n$ is odd, the single connected graph for which this upper bound is attained is the cycle $C_n$, and if $n$ is even, the single connected graph for which this upper bound is attained consists of the cycle $C_{n-1}$ and another vertex which is joined by an edge with a vertex of $C_{n-1}$.

**Proof.** We denote

$$g(n) = \begin{cases} \frac{1}{3}(2^{n-1} - 1) & \text{if } n \text{ is odd}, \\ \frac{2}{3}(2^{n-2} - 1) & \text{if } n \text{ is even}. \end{cases}$$

We prove this result by induction on $n$. For $n = 3$, the single connected graph $G$ with three vertices and $\gamma(G) = 3$ is a 3-clique and this graph is uniquely 3-colourable, the result being true. So, suppose that the result is true for $n$ and let $G$ be a connected graph with $n + 1$ ($n + 1 \geq 3$) vertices and $\gamma(G) = 3$. As in the proof of Theorem 1, let $v$ be a vertex of $G$ such that $G - v$ is connected. If $\gamma(G - v) = 3$, then

$$K(3, G) \leq 2K(3, G - v) \leq 2g(n) \leq g(n + 1),$$

since $2g(n) = g(n + 1)$ if $n$ is odd, $2g(n) + 1 = g(n + 1)$ if $n$ is even and, $G$ being connected, $v$ is joined by an edge with at least one vertex which belongs to a class of every 3-colouring of $G - v$, that is, $v$ may be placed into at most two classes of a
3-colouring of $G - v$. By induction hypothesis, the equality of $K(3, G)$ and $g(n + 1)'$
holds only if $n$ is odd and $v$ is joined by an edge with a single vertex of the odd
cycle $C_n$.

If $\gamma(G - v) = 2$, then there exists a 3-colouring of $G$ of the form \{v\}, $I_1$, $I_2$, where
$I_1$ and $I_2$ are independent sets. If $G$ is the complementary graph of a perfect
graph, then, by definition, it contains a triangle, that is, a cycle $(v, u_1, v_1, v)$ such
that $u_1 \in I_1$ and $v_1 \in I_2$.

Otherwise, by Berge's Theorem [1], it contains an odd cycle $(v, u_1, v_1, u_2, v_2, \ldots, u_k, v, v)$
such that $u_i \in I_1$ and $v_i \in I_2$ for $1 \leq i \leq k$ and $k \geq 2$. Thus, in any
case, $G$ contains an odd cycle $C_{2k+1} = (v, u_1, v_1, u_2, v_2, \ldots, u_k, v, v)$ with $u_i \in I_1$,
$v_i \in I_2$, $1 \leq i \leq k$ and $k \geq 1$. Then we can remove the edges which are the diagonals
of this odd cycle, and the number of 3-colourings of the graph increases. Moreover,
the subgraph induced by an odd cycle has the chromatic number equal to three and, therefore, by this operation, the chromatic number of the graph does not decrease
and the graph remains connected. Since $G$ is connected, there exists at least one
vertex $w$ of $G$ which does not belong to $C_{2k+1}$ and which is joined by an edge with
at least one vertex of the cycle. If there are more edges joining $w$ with vertices of the
cycle, we may preserve a single edge and the resulting graph is also connected, with
the chromatic number equal to three, and the number of its 3-colourings increases.
If we find a vertex $z$ which is joined by an edge with at least one vertex from the set
\{v, w, u_1, v_1, u_2, v_2, \ldots, u_k, v\}, we repeat this construction, etc. Thus,

$$K(3, G) \leq 2^{n-2k} \cdot K(3, C_{2k+1}) \leq g(n + 1),$$

since $2g(n) = g(n + 1)$ for $n$ odd, $2g(n) < g(n + 1)$ for $n$ even, as the reader can
easily verify, and every new vertex may be added to at most two classes of the
3-colouring already obtained (since the graph is connected). Now, by using the
induction hypothesis and the definition of $g(n)$, it follows that the graph for which
this upper bound is attained is unique and has the form indicated in the theorem.

References

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