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A HAKE-TYPE PROPERTY FOR THE ν_1 -INTEGRAL
AND ITS RELATION TO OTHER INTEGRATION PROCESSES

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INTRODUCTION

Specializing our abstract concept of non-absolutely convergent integrals (cf. [Ju-No 1]), we introduced in [Ju-No 2] the relatively simple ν_1 -integral over n -dimensional compact intervals. This integral not only shows all the usual properties but also yields a very general divergence theorem including points of unboundedness of the involved vector function. In [Ju-No 3] this result is used as a basic part for a geometrically improved version of the divergence theorem.

The studies in this paper are devoted to a further development of the ν_1 -theory. In Section 1 we extend the notion of ν_1 -integrability to point functions f defined on a bounded measurable set $A \subseteq \mathbb{R}^n$, and we then establish a Hake-type theorem involving both a point function f and an interval function F (the associated indefinite integral). In particular, it is shown how the integrability on A can be deduced from the integrability on any interval contained in the interior of A . Of course here the main difficulties arise at the boundary of A , and we found a characteristic null condition for F to be the relevant property. Indeed we do not require this condition along the topological but only along a 'reduced' boundary of A which will be important for further applications.

In [Ju-No 2] the ν_1 -integral was shown to extend the M_1 -integral (cf. [JKS]), and in Section 2 we prove that it also extends the variational integral as defined in [Pf].

0. PRELIMINARIES

\mathbb{R} and \mathbb{R}^+ denote the set of all real and all positive real numbers respectively, n is a fixed positive integer, and we work in \mathbb{R}^n with the usual inner product and the associated norm. For $x = (x_i) \in \mathbb{R}^n$ and $r > 0$ we set $B(x, r) = \{y = (y_i) \in \mathbb{R}^n : |x_i - y_i| < r, 1 \leq i \leq n\}$.

Given a set $E \subseteq \mathbb{R}^n$ we denote by E° , $\text{cl} E$, ∂E and $d(E)$ the interior, closure, boundary and the diameter of E , respectively.

The n -dimensional outer Lebesgue measure in \mathbb{R}^n is denoted by $|\cdot|_n$, and terms like measurable and almost everywhere (a.e.) always refer to this measure if the contrary is not stated explicitly. By $|\cdot|_{n-1}$ we denote the $(n-1)$ -dimensional outer Hausdorff measure in \mathbb{R}^n which coincides on \mathbb{R}^{n-1} ($\subseteq \mathbb{R}^n$) with the $(n-1)$ -dimensional outer Lebesgue measure ($|\cdot|_0$ being the counting measure). A set $E \subseteq \mathbb{R}^n$ is said to be σ_{n-1} -finite if it can be expressed as a countable union of sets with finite $(n-1)$ -dimensional outer Hausdorff measure.

Let $E \subseteq \mathbb{R}^n$ be measurable, $x \in \mathbb{R}^n$. Then we call x a density or a dispersion point of E if, respectively,

$$\liminf_{r \rightarrow 0} \frac{|E \cap B(x, r)|_n}{(2r)^n} = 1 \quad \text{or} \quad \limsup_{r \rightarrow 0} \frac{|E \cap B(x, r)|_n}{(2r)^n} = 0.$$

We denote the set of all density points of E by $\text{int}_e E$, and $\text{cl}_e E$ denotes the complement of the set of all dispersion points of E . By [Saks] the sets E , $\text{int}_e E$, $\text{cl}_e E$ differ at most by sets of $|\cdot|_n$ -measure zero, and we obviously have the inclusions $E^\circ \subseteq \text{int}_e E \subseteq \text{cl}_e E \subseteq \text{cl} E$. We set $\partial_e E = \text{cl}_e E - \text{int}_e E$, $\text{cl}_r E = \text{cl} \text{cl}_e E$, $\partial_r E = \text{cl}_r E - E^\circ$, and we see that $\partial_e E \subseteq \partial_r E \subseteq \partial E$.

An interval in \mathbb{R}^n is always assumed to be compact and non-degenerate, and a family of intervals in \mathbb{R}^n is said to be non-overlapping if they have pairwise disjoint interiors. A cube is an interval with all sides having equal length, and the support of a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is the closure of the set of points where f is different from zero.

1. A HAKE-TYPE PROPERTY FOR THE ν_1 -INTEGRAL

We begin this section by recalling the basic definitions concerning ν_1 -integration, cf. [Ju-No 1,2].

An *interval function* F (on \mathbb{R}^n) associates with every interval I ($\subseteq \mathbb{R}^n$) a real number $F(I)$. The interval function F is said to be *additive* if for every interval I and any *decomposition* $\{I_k\}$ of I (i.e. a finite sequence of non-overlapping intervals I_k whose union is I) the equality $F(I) = \sum F(I_k)$ holds true.

We call an interval function F *differentiable at* $x \in \mathbb{R}^n$ if F is derivable in the ordinary sense at x (according to [Saks]), and in that case $F'(x)$ denotes the ordinary derivative of F at x .

Let F be additive and suppose I to be an interval such that $F(J) = 0$ for each interval $J \subseteq \mathbb{R}^n - I^\circ$ (we say that F has *compact support*). Then a standard argument yields that the real number $F(I)$ is independent of I and in what follows this unique number will be denoted by $F(\mathbb{R}^n)$.

Remark 1.1. By \mathcal{B} we denote the system of all compact subsets I of \mathbb{R}^n which are of the form $I = \times_{i=1}^n [a_i, b_i]$ where $a_i, b_i \in \mathbb{R}$, and with the understanding that $I = \emptyset$ if $a_i > b_i$ for one i . Whenever an interval function F is given, we can extend F to the whole of \mathcal{B} by setting $F(I) = 0$ if I is not an interval. If F is an additive interval function then its extension is additive in the sense of [Ju-No 1, Sec. 3].

A *control condition* C associates with any positive numbers K and Δ a class $C(K, \Delta)$ of finite sequences $\{I_k\}$ with $I_k \in \mathcal{B}$. Furthermore, with C we associate a system $\mathcal{E}(C)$ of subsets of \mathbb{R}^n , and the control conditions $C_{1,2}^\alpha$ ($0 \leq \alpha < n$), C^n we use in the concept of ν_1 -integration are explicitly defined in [Ju-No 2, Sec. 1]. We set $\Gamma = \{C_i^\alpha: 0 \leq \alpha \leq n-1, i = 1, 2\}$, $\dot{\Gamma} = \{C^n\} \cup \{C_i^\alpha: n-1 < \alpha < n, i = 1, 2\}$.

Given $E \subseteq \mathbb{R}^n$ and $\delta: E \rightarrow \mathbb{R}^+$, a finite sequence of pairs $\{(x_k, I_k)\}$ is called (E, δ) -*fine* if the I_k are non-overlapping intervals, $x_k \in E \cap I_k$ and $d(I_k) < \delta(x_k)$.

Let F be an interval function, $C \in \Gamma \cup \dot{\Gamma}$ and $E \subseteq \mathbb{R}^n$. Then F satisfies the *null condition corresponding to* C on E , in short F satisfies $\mathcal{N}(C, E)$, if the following is true: $\forall \varepsilon > 0, K > 0 \exists \Delta > 0, \delta: E \rightarrow \mathbb{R}^+$ such that $\sum |F(I_k)| \leq \varepsilon$ holds for any (E, δ) -fine sequence $\{(x_k, I_k)\}$ with $\{I_k\} \in C(K, \Delta)$.

Remark 1.2. Let $E \subseteq \mathbb{R}^n$ and let F be an additive interval function which is differentiable at each point $x \in E$. Then F satisfies $\mathcal{N}(C^*, E)$ with $C^* = C_1^{n-1}$. Indeed, if $\varepsilon > 0$ and $K > 0$ are given then set $\Delta = 1$ and let $x \in E$. By the differentiability of F at x there exist positive numbers $K(x)$ and $\delta(x)$ such that $|F(I)| \leq K(x)d(I)^n$ holds for any interval I containing x and having diameter less than $\delta(x)$ (cf., e.g., [Ku-Jar, Cor. 1]). We obviously may assume $\delta(x) \leq \varepsilon/KK(x)$ for $x \in E$, and thus we conclude for any (E, δ) -fine sequence $\{(x_k, I_k)\}$ with $\{I_k\} \in C^*(K, \Delta)$ (i.e. $\sum d(I_k)^{n-1} \leq K$):

$$\sum |F(I_k)| \leq \sum K(x_k)\delta(x_k)d(I_k)^{n-1} \leq \varepsilon$$

as desired.

A *division of a set* $M \subseteq \mathbb{R}^n$ consists of a set E' and a sequence of pairs $(E_i, C_i)_{i \in \mathbb{N}}$ such that M is the disjoint union of all the sets E_i and E' , $|M - E'|_n = 0$, $C_i \in \Gamma \cup \dot{\Gamma}$ and $E_i \in \mathcal{E}(C_i)$.

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a function with compact support. Then f is said to be ν_1 -integrable if there exists an additive interval function F with $F' = f$ a.e. and a division $E', (E_i, C_i)_{i \in \mathbb{N}}$ of \mathbb{R}^n such that F is differentiable on E' and satisfies $\mathcal{N}(C_i, E_i)$ for $i \in \mathbb{N}$ as well as $\mathcal{N}(C^*, E_i)$ if $C_i \in \dot{\Gamma}$. In this case F is uniquely determined, has compact support, and we write $\nu \int f = F(\mathbb{R}^n)$, cf. Remark 1.1, 1.2 and [Ju-No 1, Sec. 5].

Suppose A to be a bounded measurable subset of \mathbb{R}^n and $f: A \rightarrow \mathbb{R}$. Then we define the function $f_A: \mathbb{R}^n \rightarrow \mathbb{R}$ by $f_A(x) = f(x)$ if $x \in A$ and zero elsewhere. We say that f is ν_1 -integrable on A if f_A is ν_1 -integrable, and in this case we write $\nu \int_A f = \nu \int f_A$.

Remark 1.3. For the properties of the ν_1 -integral we refer the reader to [Ju-No 1, Sec. 5] and [Ju-No 2]. In particular, the ν_1 -integral extends the Lebesgue integral, and therefore the definition of ν_1 -integrability also applies to functions which are defined only a.e. Furthermore, if f is ν_1 -integrable and if F denotes the corresponding interval function then f is ν_1 -integrable on any interval I and $F(I) = \nu \int_I f$.

Remark 1.4. Let I be an interval in \mathbb{R}^n , $x \in I^\circ$, $C \in \Gamma \cup \dot{\Gamma}$, $E \subseteq \mathbb{R}^n$. Then it is clear what we mean by an (additive) interval function on I , which is differentiable at x or which satisfies $\mathcal{N}(C, E)$ (just require all intervals occurring in the definitions given above to lie in I). By [Ju-No 1, Remark 5.1 (iii)] we see that a function $f: I \rightarrow \mathbb{R}$ is ν_1 -integrable on I iff there exists an additive interval function F on I with $F' = f$ a.e. on I and a division $E', (E_i, C_i)_{i \in \mathbb{N}}$ of I with $E' \subseteq I^\circ$ and such that F is differentiable on E' and satisfies $\mathcal{N}(C_i, E_i)$ for $i \in \mathbb{N}$ as well as $\mathcal{N}(C^*, E_i)$ if $C_i \in \dot{\Gamma}$. Furthermore, in that case F is uniquely determined and $F(J) = \nu \int_J f$ for each subinterval J of I , cf. also [Ju-No 2, Sec. 1].

Suppose G to be an open subset of \mathbb{R}^n , $f: G \rightarrow \mathbb{R}$ and let F be an interval function on \mathbb{R}^n . Then F and f are said to be ν_1 -associated in G if f is ν_1 -integrable on any interval I contained in G and $F(I) = \nu \int_I f$.

Let M be an arbitrary subset of \mathbb{R}^n and F an interval function on \mathbb{R}^n . We say that F satisfies a condition (N) on M if there exists a division $E', (E_i, C_i)_{i \in \mathbb{N}}$ of M such that F is differentiable on E' and satisfies $\mathcal{N}(C_i, E_i)$ for $i \in \mathbb{N}$ as well as $\mathcal{N}(C^*, E_i)$ if $C_i \in \dot{\Gamma}$.

Theorem 1. Suppose A to be a bounded measurable subset of \mathbb{R}^n , $f: A \rightarrow \mathbb{R}$ and let F be an interval function on \mathbb{R}^n . Then f is ν_1 -integrable on A and $F(I) = \nu \int_I f_A$ for each interval I in \mathbb{R}^n iff the following conditions are satisfied:

- (i) F is additive and $F(I) = 0$ for each interval $I \subseteq \mathbb{R}^n - \text{cl}_r A$,
- (ii) F and f are ν_1 -associated in A° ,
- (iii) F satisfies a condition (N) on $\partial_r A$ and $F' = f_A$ a.e. on $\partial_r A$.

In any case F' exists a.e. on A , F' is ν_1 -integrable on A and

$$F(\mathbb{R}^n) = \int_A^{\nu_1} F' = \int_A^{\nu_1} f.$$

PROOF. Assume first f to be ν_1 -integrable on A and $F(I) = \int_I^{\nu_1} f_A$ for each interval I . By definition there is an additive interval function G with $G' = f_A$ a.e. and a corresponding division. Consequently, by Remark 1.3 we see that $F' = G$, and F and f are ν_1 -associated in A° . Let $I \subseteq \mathbb{R}^n - \text{cl}_r A$ be an interval and observe that $f_A = 0$ a.e. on I . Thus, again by Remark 1.3, $\int_I^{\nu_1} f_A = 0$. To see that F satisfies a condition (N) on $\partial_r A$ one only has to intersect the division corresponding to G with $\partial_r A$ (note that $E \in \mathcal{E}(C)$ implies $\tilde{E} \in \mathcal{E}(C)$ for any $\tilde{E} \subseteq E$, $C \in \Gamma \cup \tilde{\Gamma}$). Furthermore, since $F' = f$ a.e. on A the ν_1 -integrability of F' on A follows and $\int_A^{\nu_1} F' = \int_A^{\nu_1} f = \int_A^{\nu_1} f_A = F(\mathbb{R}^n)$.

Conversely, suppose the conditions (i)–(iii) to be satisfied. We express A° as an at most countable union of non-overlapping cubes I_i ($i \geq 1$). By (ii) and Remark 1.4 f is ν_1 -integrable on I_i and F satisfies a condition (N) on I_i° (intersect the division of I_i according to Remark 1.4 with the interior of I_i). Note that in particular $F' = f$ a.e. on I_i° , hence a.e. on A° , and since $F' = 0$ on $\mathbb{R}^n - \text{cl}_r A$ by (i) we have $F' = f_A$ a.e. Since any I_i is contained in the interior of A we can slightly enlarge I_i to a cube J_i still lying in the interior of A and containing I_i in its interior. The same argument as before yields that F satisfies a condition (N) on ∂I_i and consequently on $\partial I_i - \bigcup_{j=1}^{i-1} \partial I_j$, $i \geq 1$. Now, taking into account the set $\mathbb{R}^n - \text{cl}_r A$ and all divisions according to the conditions (N) satisfied by F on $\partial_r A$, I_i° and $\partial I_i - \bigcup_{j=1}^{i-1} \partial I_j$ ($i \geq 1$), we see that F satisfies a condition (N) on \mathbb{R}^n , and thus by definition f_A is ν_1 -integrable. Again by Remark 1.3 we have $F(I) = \int_I^{\nu_1} f_A$ for any interval I in \mathbb{R}^n . \square

2. RELATIONS TO OTHER INTEGRALS

In this section we assume I to be a fixed interval in \mathbb{R}^n and f to be a fixed real-valued function defined on I . We will prove that if f is *variationally integrable* on I in the sense of [Pf] then f is ν_1 -integrable on I and both integrals coincide.

A bounded measurable set $A \subseteq \mathbb{R}^n$ is called a *BV set* if $|\partial_e A|_{n-1}$ is finite (see [Pf], [Fed]), and for any *BV set* A we define its regularity by $r(A) = |A|_n/d(A)|\partial_e A|_{n-1}$ if $d(A)|\partial_e A|_{n-1} > 0$ and by $r(A) = 0$ else. We denote by BV_I the system of all *BV sets* contained in I , and a function $F: BV_I \rightarrow \mathbb{R}$ is called *continuous* if for every

$\varepsilon > 0$ there is a $\delta > 0$ such that $|F(B)| < \varepsilon$ for each $B \in BV_I$ with $|B|_n < \delta$ and $|\partial_\varepsilon B|_{n-1} < 1/\varepsilon$.

A function $F: BV_I \rightarrow \mathbb{R}$ is said to be *superadditive* if $\sum F(B_k) \leq F(B)$ for any $B \in BV_I$ and any finite sequence of disjoint BV sets B_k whose union is B . F is called *additive* if F and $-F$ are both superadditive.

Let $F: BV_I \rightarrow \mathbb{R}$, $\varepsilon > 0$ and a σ_{n-1} -finite set T be given. Then an ε -majorant of the pair (f, F) in $I \bmod T$ is a non-negative superadditive function $M: BV_I \rightarrow \mathbb{R}$ satisfying the following conditions: $M(I) < \varepsilon$, and for each $x \in I - T$ there exists a $\delta > 0$ such that $|F(B) - f(x)|_n|B|_n \leq M(B)$ for any $B \in BV_I$ with $x \in \text{cl} B$, $d(B) < \delta$, $r(B) > \varepsilon$.

We call f *v-integrable on I* if there is a continuous additive function $F: BV_I \rightarrow \mathbb{R}$ and a σ_{n-1} -finite set T such that for any $\varepsilon > 0$ there is an ε -majorant of (f, F) in $I \bmod T$. In this case F is uniquely determined, and we write $\int_I f = F(I)$, cf. [Pf, Def. 5.1, Cor. 5.5].

Proposition 1. *Suppose f to be v-integrable on I . Then f is ν_1 -integrable on I and $\int_I f = \int_I f$.*

Proof. We assume f to be v-integrable, and we denote by F the corresponding continuous additive function on BV_I and by T a corresponding σ_{n-1} -finite set. Note that $F(B) = 0$ for any $B \in BV_I$ with $|B|_n = 0$, hence F is an additive interval function on I (in the sense of Section 1).

First we show that F satisfies $\mathcal{N}(C_1^{n-1}, I)$: let $\varepsilon > 0$, $K > 0$ be given, set $\Delta = 1$, $\varepsilon' = \frac{1}{2} \min(\varepsilon, 1/2nK)$ and determine for ε' a $\delta' > 0$ in virtue of the continuity of F . We set $\delta(\cdot) = \delta'/2K$ on I , and we assume $\{(x_k, I_k)\}$ to be an (I, δ) -fine sequence with $I_k \subseteq I$ and $\{I_k\} \in C_1^{n-1}(K, \Delta)$ (i.e. $\sum d(I_k)^{n-1} \leq K$). Then $|\bigcup I_k|_n \leq \sum \delta(x_k) d(I_k)^{n-1} < \delta'$ and

$$\left| \bigcup \partial I_k \right|_{n-1} \leq \sum |\partial I_k|_{n-1} \leq 2nK < 1/\varepsilon',$$

thus

$$\sum |F(I_k)| = F\left(\bigcup_{F(I_k) \geq 0} I_k\right) - F\left(\bigcup_{F(I_k) < 0} I_k\right) \leq 2\varepsilon' \leq \varepsilon.$$

Let $E \subseteq I$ with $|E|_n = 0$ and $\varepsilon > 0$ be given. Then we can determine $\delta: E \rightarrow \mathbb{R}^+$ such that $\sum |f(x_k)|_n |I_k|_n \leq \varepsilon$ holds for any (E, δ) -fine sequence $\{(x_k, I_k)\}$ with $I_k \subseteq I$. Indeed, write $E = \bigcup_{j \in \mathbb{N}} E_j$ with $E_j = \{x \in E: j-1 \leq |f(x)| < j\}$, let $\varepsilon > 0$ be given and determine open sets $G_j \supseteq E_j$ with $|G_j|_n \leq \varepsilon/j2^j$. For $x \in E_j$ we choose a

$\delta(x) > 0$ such that $B(x, \delta(x)) \subseteq G_j$, which defines the function δ . Consequently, for any (E, δ) -fine sequence $\{(x_k, I_k)\}$ with $I_k \subseteq I$ we get

$$\sum |f(x_k)||I_k|_n \leq \sum_{j \in \mathbb{N}} \sum_{x_k \in E_j} j|I_k|_n \leq \sum_{j \in \mathbb{N}} j|G_j|_n \leq \varepsilon.$$

To prove that f is ν_1 -integrable on I with $\int_I f = F(I)$ ($= \int_I f$), we verify the constructive definition of our ν_1 -integral, see [Ju-No 2, Thm. 3.1]. Express $I \cap (T \cup \partial I)$ as a disjoint countable union of sets E_i with $|E_i|_{n-1} < \infty$ ($i \in \mathbb{N}$), and note that $E' = I - (T \cup \partial I)$, $(E_i, C_1^{n-1})_{i \in \mathbb{N}}$ is a division of I with $E' \subseteq I^\circ$. Now let, according to [Ju-No 2, Thm. 3.1], $\varepsilon > 0$, $K > 0$, $K_i > 0$ ($i \in \mathbb{N}$) be given, set $\Delta_i = 1$, $\varepsilon' = \frac{1}{5} \min(\varepsilon, 1/nK)$ and choose an ε' -majorant M of (f, F) in $I \bmod T$ which, by definition, yields a function $\delta: E' \rightarrow \mathbb{R}^+$. Since $|I - E'|_n = 0$ we can also determine a $\delta: I - E' \rightarrow \mathbb{R}^+$ such that $\sum |f(x_k)||I_k|_n \leq \varepsilon/5$ for any $(I - E', \delta)$ -fine sequence $\{(x_k, I_k)\}$ with $I_k \subseteq I$. Obviously we may assume $\delta(\cdot) \leq \varepsilon/5K(1 + |f(\cdot)|)$ on I , and since F satisfies $\mathcal{N}(C_1^{n-1}, E_i)$ resp. $\mathcal{N}(C_1^{n-1}, E')$ we can determine for $\varepsilon/5 \cdot 2^i$ and K_i resp. for $\varepsilon/5$ and K a corresponding function $\delta_i: E_i \rightarrow \mathbb{R}^+$ resp. $\delta': E' \rightarrow \mathbb{R}^+$, and we also may assume $\delta(\cdot) \leq \delta_i(\cdot)$ on E_i resp. $\delta(\cdot) \leq \delta'(\cdot)$ on E' . Thus a function δ is defined on I , and we denote by $\{(x_k, I_k)\} \cup \{(x'_k, I'_k)\}$ an (I, δ) -fine sequence with $I = \bigcup I_k \cup \bigcup I'_k$ fulfilling the conditions

- (i) if $x_k \in E'$ then $d(I_k)^n \leq K|I_k|_n$; $\{I_k: x_k \in E_i\} \in C_1^{n-1}(K_i)$ ($i \in \mathbb{N}$)
- (ii) $\{I'_k\} \in C_1^{n-1}(K)$ and $x'_k \in E'$ for all k .

Observing that $r(I_k^\circ) \geq 1/2nK > \varepsilon'$ for $x_k \in E'$ we conclude:

$$\begin{aligned} & \left| F(I) - \left(\sum f(x_k)|I_k|_n + \sum f(x'_k)|I'_k|_n \right) \right| \\ & \leq \sum_{x_k \in E'} |F(I_k^\circ) - f(x_k)|I_k^\circ|_n| + \sum_{i \in \mathbb{N}} \sum_{x_k \in E_i} |F(I_k)| \\ & \quad + \sum |F(I'_k)| + \sum_{x_k \in I - E'} |f(x_k)||I_k|_n + \sum |f(x'_k)|\delta(x'_k)d(I'_k)^{n-1} \\ & \leq \sum_{x_k \in E'} M(I_k^\circ) + \sum_{i \in \mathbb{N}} \frac{\varepsilon}{5 \cdot 2^i} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5K} \sum d(I'_k)^{n-1} \\ & \leq M(I) + \frac{4}{5}\varepsilon \leq \varepsilon, \end{aligned}$$

which completes the proof. □

Remark 2.1. In [Jar-Ku] a further n -dimensional non-absolutely convergent integral is introduced, the so called PU -integral. Assume the support of a function $g: \mathbb{R}^n \rightarrow \mathbb{R}$ to be contained in I , and suppose g to be PU -integrable. Then according to [Jar-Ku, Thm. 6.1], [JKS] g is M_1 -integrable on I , consequently by [Ju-No 2,

Prop. 4.1] g is ν_1 -integrable on I and all integrals coincide. For a comparison of related integration processes see [No].

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