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GENERALIZED COUPLINGS

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The notion of generalized couplings was introduced and investigated, not long ago, in a joint paper of P. Vrbová and the present author [4]. The idea of combining two spaces, each with an operator given on it, appears first—in the particular case of semiunitary operators—in a paper of Adamyan and Arov. The ideas of Adamyan and Arov [1] were further developed by a number of authors; in particular, the connections of this notion with dilations of mappings of positive type form the subject of related investigations of Arocena and Cotlar [2]. In its full generality, as considered in [4], the problem may be formulated as follows.

Given two bounded linear operators A_1 and A_2 acting on the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and a contraction $X\colon \mathscr{H}_1 \to \mathscr{H}_2$, what are the conditions for the existence of a Hilbert space $\mathscr{K}$ and an operator $U \in B(\mathscr{K})$ such that

(1) *$\mathscr{K}$ contains $\mathscr{H}_1$ and $\mathscr{H}_2$,*

(2) *$\mathscr{H}_1$ is U invariant and $U|\mathscr{H}_1 = A_1$,*

(3) *$\mathscr{H}_2$ is U^* invariant and $U^*|\mathscr{H}_2 = A_2$,*

(4) *$P(\mathscr{H}_2)|\mathscr{H}_1 = X$.*

By $P(\mathscr{H})$ we denote the orthogonal projection onto $\mathscr{H}$. An operator satisfying the four properties listed above is called a coupling of A_1 and A_2. It is not difficult [4] to show that, for the existence of U, the following intertwining relation is necessary: the existence of U implies the relation $XA_1 = A_2^*X$. This necessary condition for the existence of couplings is also sufficient; it turns out that contractive U exist if A_1 and A_2 are both contractions. In [4] a parametrization of all contractive couplings was given together with a characterization of the triples A_1, A_2, X for which an isometric coupling exists. The approach used in [4], though straightforward enough, does not permit to describe the parametrization in terms of the original spaces and operators in a simple manner; in particular, the verification of the existence conditions for isometric solutions is not simple. In view of the fact that many questions in dilation theory may be reformulated in terms of couplings of suitable spaces and operators it seems that the notion of couplings deserves a more careful study, in particular, a treatment that simplifies the technicalities. The present note is the result of an attempt to eliminate the technical complications connected with parametrizing he set of all solutions of the coupling problem (proposition (3.3) of [4]). The approach used in this note differs from the one described in [4] in that the operator to be constructed is interpreted as a mapping between two different representations of the coupling space. This makes it possible to apply methods related to those used in the proof of the Davis-Kahane-Weinberger theorem and to parametrize the set of all solutions in a different manner, in terms of the given data. At the same time, the whole treatment is considerably simpler than that used in [4].

1. Notation and preliminaries

If $\mathscr{H}$ is a Hilbert space, we denote by $B(\mathscr{H})$ the algebra of all bounded linear operators in $\mathscr{H}$. By $P(\mathscr{H}_0)$ we denote the orthogonal projection onto a subspace $\mathscr{H}_0$. A contraction is an element of $B(\mathscr{H})$ of norm at most one. If T is a contraction we set $D(T) = (1 - T^*T)^{1/2}$ and $D(T^*) = (1 - TT^*)^{1/2}$; the closure of the set $D(T)\mathscr{H}$ is denoted by $\mathscr{D}(T)$; similarly, $\mathscr{D}(T^*)$ stands for the closure of $D(T^*)\mathscr{H}$. When there is no danger of misunderstanding, we abbreviate $D(T)$ and $D(T^*)$ to D and D_* respectively.

2. Coupling spaces

We begin by formulating the intertwining relation which represents the basic necessary condition for the existence of couplings.

Lemma 2.1. *Let $\mathscr{K}$ be a Hilbert space, $\mathscr{H}$ be a subspace of $\mathscr{K}$, $U \in B(\mathscr{K})$. Suppose $\mathscr{H}$ is invariant with respect to U^*; then $P(\mathscr{H})U = (U^*|\mathscr{H})^*P(\mathscr{H})$.*

Proof. Consider an arbitrary $x \in \mathscr{K}$ and an arbitrary $h \in \mathscr{H}$. Then

$$\begin{aligned}\big(P(\mathscr{H})Ux, h\big) &= (Ux, h) = \big(x, (U^*|\mathscr{H})h\big) = \big(P(\mathscr{H})x, (U^*|\mathscr{H})h\big) \\ &= \big((U^*|\mathscr{H})P(\mathscr{H})x, h\big).\end{aligned}$$

Since this equality holds for arbitrary $x \in \mathscr{K}$ and $h \in \mathscr{H}$, the identity follows. Using the terminology of dilation theory, the identity says that U is a lifting of $(U^*|\mathscr{H})^*$. □

Corollary 2.1. *Let $\mathscr{K}$ be a Hilbert space, $\mathscr{H}_1$ and $\mathscr{H}_2$ two subspaces of $\mathscr{K}$. Furthermore, let $U \in B(\mathscr{K})$ and suppose that $\mathscr{H}_1$ is invariant with respect to U and $\mathscr{H}_2$ invariant with respect to U^*. Set*

$$A_1 = U|\mathscr{H}_1,$$
$$A_2 = U^*|\mathscr{H}_2.$$

Then $P(\mathscr{H}_2)U = A_2^* P(\mathscr{H}_2)$.

If X stands for the restriction $P(\mathscr{H}_2)|\mathscr{H}_1$ then

$$XA_1 = A_2^* X.$$

Proof. An immediate consequence of the lemma. □

Suppose $\mathscr{H}_1$ and $\mathscr{H}_2$ are closed subspaces of a Hilbert space $\mathscr{K}$ such that their sum $\mathscr{K}_0 = \mathscr{H}_1 + \mathscr{H}_2$ is dense in $\mathscr{K}$. Write X for the restriction of $P(\mathscr{H}_2)$ to $\mathscr{H}_1$. Given two elements x, $x' \in \mathscr{K}_0$

$$x = h_1 + h_2, \quad x' = h_1' + h_2'$$

their scalar product is given by the expression

$$(x, x') = (Xh_1 + h_2, h_2') + (h_1 + X^* h_2, h_1').$$

In particular

$$\begin{aligned}(x, x) &= |h_1|^2 + |h_2|^2 + 2\operatorname{Re}(Xh_1, h_2) \\ &= |Xh_1 + h_2|^2 + |Dh_1|^2 = |h_1 + X^* h_2|^2 + |D_* h_2|^2.\end{aligned}$$

Consider the space $\mathscr{H}_1 \oplus \mathscr{H}_2$; its elements will be column vectors of the form $v = (h_1, h_2)^T$. We define the operator $A\colon \mathscr{H}_1 \oplus \mathscr{H}_2 \to \mathscr{K}_0$ by the formula $Av = h_1 + h_2$. Using the above formula for $x = Av$, we obtain an explicit description of $\operatorname{Ker} A$ as follows

$$\operatorname{Ker} A = \left\{ \begin{pmatrix} h_1 \\ -Xh_1 \end{pmatrix}, Dh_1 = 0 \right\} = \left\{ \begin{pmatrix} -X^* h_2 \\ h_2 \end{pmatrix}, D_* h_2 = 0 \right\}.$$

Denote by W the operator

$$W = \begin{pmatrix} 1 & X^* \\ X & 1 \end{pmatrix}$$

acting on $\mathscr{H}_1 \oplus \mathscr{H}_2$. It is easy to verify the identity

$$(Wx, x') = (Ax, Ax').$$

It follows that $W \geqslant 0$ and

$$\operatorname{Ker} W = \{x; (Wx, y) = 0 \ \forall y\} = \{x; (Wx, x) = 0\} = \operatorname{Ker} A.$$

Denote by $\mathscr{H}_0$ the quotient $\mathscr{H}_1 \oplus \mathscr{H}_2$ modulo $\operatorname{Ker} W$ equipped with the scalar product (Wx, y). The relation

$$(Wx, x') = (Ax, Ax')$$

shows that A establishes an isometric isomorphism mapping $\mathscr{H}_0$ onto $\mathscr{K}_0$.

Summing up, we have proved the following proposition.

Proposition 2.2. *Given a Hilbert space $\mathscr{K}$ generated by a pair of subspaces $\mathscr{H}_1$, $\mathscr{H}_2$ then $\mathscr{K}$ is isometrically isomorphic to the completion of $\mathscr{H}_0$.*

On the other hand, suppose we are given two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ together with a contraction $X\colon \mathscr{H}_1 \to \mathscr{H}_2$.

Consider the space $\mathscr{H}_1 \oplus \mathscr{H}_2$ and the operator W on $\mathscr{H}_1 \oplus \mathscr{H}_2$ given by the matrix

$$W = \begin{pmatrix} 1 & X^* \\ X & 1 \end{pmatrix}.$$

Denote by $\mathscr{M}$ the closed subspace $\operatorname{Ker} W$. It is not difficult to show that the following two alternative descriptions of $\mathscr{M}$ are possible

$$\mathscr{M} = \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}; Xh_1 + h_2 = 0, Dh_1 = 0 \right\} = \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}; h_1 + X^* h_2 = 0, D_* h_2 = 0 \right\}.$$

Now consider the space $\mathscr{H}_1 \oplus \mathscr{H}_2$ and its subspace $\mathscr{D}(X) \oplus \mathscr{H}_2$. Now $\mathscr{H}_1 = \mathscr{D} \oplus \operatorname{Ker} D$ the corresponding orthogonal projectors being denoted by $P(\mathscr{D})$ and $P(\operatorname{Ker} D)$. Consider the mapping

$$E\colon \mathscr{H}_1 \oplus \mathscr{H}_2 \to \mathscr{D} \oplus \mathscr{H}_2$$

defined as follows

$$E \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} P(\mathscr{D}) h_1 \\ h_2 + XP(\operatorname{Ker} D) h_1 \end{pmatrix} = \begin{pmatrix} P(\mathscr{D}) & 0 \\ XP(\operatorname{Ker} D) & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

It is easy to see that $\operatorname{Ker} E = \mathscr{M}$ and $E^2 = E$.

It follows that E is a projection operator of $\mathscr{H}_1 \oplus \mathscr{H}_2$ onto $\mathscr{D} \oplus \mathscr{H}_2$ with kernel $\mathscr{M}$.

In this manner we may identify the quotient of $\mathscr{H}_1 \oplus \mathscr{H}_2$ modulo $\mathscr{M}$ with $\mathscr{D} \oplus \mathscr{H}_2$.

Consider, on $\mathscr{H}_1 \oplus \mathscr{H}_2$, the two operators

$$P_2 = \begin{pmatrix} 0 & 0 \\ X & 1 \end{pmatrix} \quad \text{and} \quad P_2^{\perp} = \begin{pmatrix} 1 & 0 \\ -X & 0 \end{pmatrix}.$$

Since $P_2\binom{h_1}{-Xh_1} = 0$ for any $h_1 \in \mathscr{H}_1$ we have, in particular, $P_2\mathscr{M} = 0$. It follows that both P_2 and $P_2^{\perp}$ may be considered as operators on the quotient of $\mathscr{H}_1 \oplus \mathscr{H}_2$ modulo $\mathscr{M}$.

If $x = \binom{h_1}{h_2}$ then $P_2 x = \binom{0}{Xh_1+h_2}$. So that $|P_2 x|_W^2 = |Xh_1 + h_2|^2$. For $P_2^{\perp}x$ we obtain $P_2^{\perp}x = \binom{h_1}{-Xh_1}$ and $|P_2^{\perp}x|_W^2 = |Dh_1|^2$ thus $|x|_W^2 = |P_2x|_W^2 + |P_2^{\perp}x|_W^2$.

Denote by $\mathscr{P}(X)$ the completion of the space $\mathscr{D} \oplus \mathscr{H}_2$ taken in the metric given by W

$$\left|\begin{pmatrix} d \\ h_2 \end{pmatrix}\right|_W^2 = |Dd|^2 + |Xd + h_2|^2.$$

In this manner $\mathscr{P}(X)$ may be considered as the Hilbert space generated from $\mathscr{H}_1 \oplus \mathscr{H}_2$ by the weighted metric W.

The expression for the norm in $\mathscr{P}(X)$ suggests a straightforward representation of $\mathscr{P}(X)$ avoiding the use of weights.

Consider the matrix

$$\psi = \begin{pmatrix} D & 0 \\ X & 1 \end{pmatrix}.$$

Taken as an operator on $\mathscr{H}_1 \oplus \mathscr{H}_2$ it satisfies the identity $\psi^*\psi = W$ so that $\operatorname{Ker}\psi = \operatorname{Ker} W = \mathscr{M}$. Since $\psi\mathscr{M} = 0$, ψ may be considered as a mapping of $\mathscr{P}(X)$ into $\mathscr{D} \oplus \mathscr{H}_2$. We intend to show that ψ establishes an isometry between $\mathscr{P}(X)$ and $\mathscr{D} \oplus \mathscr{H}_2$, the space $\mathscr{D} \oplus \mathscr{H}_2$ being taken in its standard metric.

Indeed, decomposing ψ into the sum

$$\begin{aligned}\psi = \psi P_2 + \psi P_2^{\perp} &= \begin{pmatrix} D & 0 \\ X & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ X & 1 \end{pmatrix} + \begin{pmatrix} D & 0 \\ X & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -X & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ X & 1 \end{pmatrix} + \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},\end{aligned}$$

we obtain $|\psi(x)|^2 = |Xh_1 + h_2|^2 + |Dh_1|^2 = |x|_W^2$.

Performing the obvious identifications we may formulate the following

Proposition 2.3. *Given two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and a contraction $X: \mathscr{H}_1 \to \mathscr{H}_2$ there exists exactly one Hilbert space $\mathscr{P}(X)$ with the following properties*

1° $\mathscr{P}(X)$ *contains both* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *and is generated by their sum*
2° $P(\mathscr{H}_2)|\mathscr{H}_1 = X$

We shall need more precise information on the structure of $\mathscr{P}(X)$.

Proposition 2.4. *Consider the mappings* F_1 *and* F_2 *defined by*

$$F_1 \colon \mathscr{H}_1 \to \mathscr{D} \oplus \mathscr{H}_2 \qquad F_1 = \begin{pmatrix} D \\ X \end{pmatrix},$$

$$F_2 \colon \mathscr{H}_2 \to \mathscr{D} \oplus \mathscr{H}_2 \qquad F_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then F_1 *and* F_2 *are isometric mappings respectively of* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *into* $\mathscr{D} \oplus \mathscr{H}_2$.

The range of F_1 *equals the kernel of* $(X, -D_*)$. *On the range of* F_1, *the inverse of* F_1 *equals* (D, X^*).

Proof. If $\binom{d}{k} = F_1 h$ then $(X, -D_*)\binom{d}{k} = (X, -D_*)\binom{Dh}{Xh} = (XD - D_* X)h = 0$. On the other hand, suppose $Xd - D_* k = 0$; set $h = Dd + X^* k$ and let us show that $\binom{d}{k} = F_1 h$. Indeed,

$$\begin{aligned} D(Dd + X^*k) &= d - X^*Xd + DX^*k = d - X^*Xd + X^*D_*k \\ &= d - X^*(Xd - D_*k) = d, \\ X(Dd + X^*k) &= XDd + XX^*k = D_*Xd + XX^*k \\ &= k - D_*^2 k + D_*Xd = k. \end{aligned}$$

Let us show that (D, X^*) is isometric on the kernel of $(X, -D_*)$.

$$\begin{aligned} |Dd + X^*k|^2 &= |Dd|^2 + |X^*k|^2 + 2\,\mathrm{Re}\,(Dd, X^*k) \\ &= |d|^2 - |Xd|^2 + |k|^2 - |D_*k|^2 + 2\,\mathrm{Re}\,(Xd, D_*k) \\ &= |d|^2 + |k|^2 - |Xd - D_*k|^2 = |d|^2 + |k|^2. \end{aligned}$$

□

Expressed in the form of matrix identities the first statement is a consequence of

(1) $$(X, -D_*)\begin{pmatrix} D \\ X \end{pmatrix} = 0,$$

(2) $$\begin{pmatrix} D \\ X \end{pmatrix}(D, X^*) = 1 - \begin{pmatrix} X^* \\ -D_* \end{pmatrix}(X, -D_*).$$

The second statement follows from (2) and

(3) $$(D, X^*)\begin{pmatrix} D \\ X \end{pmatrix} = 1.$$

Proposition 2.5. *In a similar manner we define the mapping*

$$G_2 \colon \mathscr{D}_* \to \mathscr{D} \oplus \mathscr{H}_2 \quad G_2 = \begin{pmatrix} -X^* \\ D_* \end{pmatrix}.$$

Then G_2 is an isometric mapping of $\mathscr{D}_*$ onto the orthogonal complement of $F_1\mathscr{H}_1$; the range of G_2 equals the kernel of (D, X^*). On the range of G_2, the inverse of G_2 equals $(-X, D_*)$.

Proof. First of all, we observe that $\binom{u_1}{u_2} \in (F_1\mathscr{H}_1)^\perp$ if and only if $Du_1 + X^*u_2 = 0$.

The fact that the range of G_2 equals $\operatorname{Ker}(D, X^*)$ follows from the identities

$$(1^*) \qquad (D, X^*)\begin{pmatrix} -X^* \\ D_* \end{pmatrix} = 0$$

and

$$(2^*) \qquad \begin{pmatrix} -X^* \\ D_* \end{pmatrix}(-X, D_*) = 1 - \begin{pmatrix} D \\ X \end{pmatrix}(D, X^*),$$

the second statement follows from (2^*) and

$$(4) \qquad (-X, D_*)\begin{pmatrix} -X^* \\ D_* \end{pmatrix} = 1.$$

□

It follows from the preceding two propositions that the mapping $\varphi = F_1 \oplus G_2$

$$\varphi = \begin{pmatrix} D & -X^* \\ X & D_* \end{pmatrix}$$

of $\mathscr{H}_1 \oplus \mathscr{D}_*$ onto $\mathscr{D} \oplus \mathscr{H}_2$ is an isometry.

3. Coupling operators

In this section we intend to consider the following situation. We are given two Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$ and two contractions $A_1 \in B(\mathscr{H}_1)$, $A_2 \in B(\mathscr{H}_2)$. Furthermore, let $X \colon \mathscr{H}_1 \to \mathscr{H}_2$ be a contraction.

The problem is to construct and parametrize the family of all Hilbert spaces $\mathscr{K}$ with the following properties

1° $\mathscr{K}$ contains $\mathscr{H}_1$ and $\mathscr{H}_2$ and is generated by their sum

2° $P(\mathscr{H}_2)|\mathscr{H}_1 = X$

3° there exists an operator $U \in B(\mathscr{K})$ such that $U|\mathscr{H}_1 = A_1$ and $U^*|\mathscr{H}_2 = A_2$

4° the operator U is a contraction.

Our approach is based on treating U as a mapping between two different realizations of $\mathscr{P}(X)$. This makes it possible to parametrize the family of all solutions directly in terms of operators on the spaces $\mathscr{H}_1$ and $\mathscr{H}_2$.

We are looking for a contraction

$$V : \mathscr{H}_1 \oplus \mathscr{D}_* \to \mathscr{D} \oplus \mathscr{H}_2$$

such that

$$V|\mathscr{H}_1 = A_1,$$
$$P(\mathscr{H}_2)V\begin{pmatrix} h_1 \\ d_* \end{pmatrix} = A_2^* P_2 \begin{pmatrix} h_1 \\ d_* \end{pmatrix} = A_2^*(Xh_1 + D_* d_*).$$

If we denote by M the operator

$$M : \mathscr{H}_1 \to \mathscr{H}_2, \qquad M = XA_1 = A_2^* X$$

the operator V shall assume the following form

$$V = \begin{pmatrix} D(X)A_1 & Y \\ M & A_2^* D(X^*) \end{pmatrix}$$

where $Y : \mathscr{D}_* \to \mathscr{D}$ is to be determined.

In our constructions we shall use two contractions the existence of which is a consequence of the following lemma.

Lemma 3.2. *There exist contractions*

$$C_1 : \mathscr{D}(M) \to \mathscr{D},$$
$$C_2 : \mathscr{D}(M^*) \to \mathscr{D}_*$$

such that

$$D(X)A_1 = C_1 D(M),$$
$$A_2^* D(X^*) = D(M^*) C_2^*.$$

P r o o f. The existence of C_1 is a consequence of the inequality

$$A_1^*(1 - X^*X)A_1 \leqslant 1 - A_1^* X^* X A_1.$$

In a similar manner, we have

$$A_2^*(1 - XX^*)A_2 \leqslant 1 - A_2^* XX^* A_2$$

so that there exists a contraction $C_2\colon \mathscr{D}(M^*) \to \mathscr{D}_*$ with $D(X^*)A_2 = C_2 D(M^*)$. □

Writing V in the form

$$V = T + Z = \begin{pmatrix} C_1 D(M) & 0 \\ M & D(M^*)C_2^* \end{pmatrix} + \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}.$$

V will be a contraction if and only if

$$1 - T^*T \geqslant T^*Z + Z^*T + Z^*Z;$$

the matrix on the right hand side may be rewritten in the form

$$\begin{pmatrix} 0 & D(M)C_1^*Y \\ Y^*C_1 D(M) & Y^*Y \end{pmatrix} = Q \begin{pmatrix} 0 & C_1^*Y \\ Y^*C_1 & Y^*Y \end{pmatrix} Q \tag{12}$$

where $Q = \begin{pmatrix} D(M) & 0 \\ 0 & 1 \end{pmatrix}$. Since

$$\begin{aligned} T^*T &= \begin{pmatrix} D(M)C_1^*C_1 D(M) + M^*M & M^*D(M^*)C_2^* \\ C_2 D(M^*)M & C_2 D(M^*)^2 C_2^* \end{pmatrix} \\ &= \begin{pmatrix} D(M)C_1^*C_1 D(M) + M^*M & D(M)M^*C_2^* \\ C_2 M D(M) & C_2C_2^* - C_2 M M^* C_2^* \end{pmatrix}, \end{aligned}$$

we obtain, for $1 - T^*T$, the following expression

$$\begin{aligned} I - T^*T &= \begin{pmatrix} D(M)D(C_1)^2 D(M) & -D(M)M^*C_1^* \\ -C_2 M D(M) & D(C_2^*)^2 + C_2 M M^* C_2^* \end{pmatrix} \\ &= Q \begin{pmatrix} D(C_1)^2 & -M^*C_2^* \\ -C_2 M & D(C_2^*)^2 + C_2 M M^* C_2^* \end{pmatrix} Q. \end{aligned} \tag{13}$$

Thus V will be a contraction iff

$$\begin{pmatrix} D(C_1)^2 & -M^*C_2^* \\ -C_2 M & D(C_2^*)^2 + C_2 M M^* C_2^* \end{pmatrix} \geqslant \begin{pmatrix} 0 & C_1^*Y \\ Y^*C_1 & Y^*Y \end{pmatrix}. \tag{14}$$

To justify this equivalence it suffices to use the following facts. The matrix Q represents, in the formulae above, two different operators: on the right-hand side, Q is considered as an operator in $\mathscr{H}_1 \oplus \mathscr{D}_*$, on the left-hand side as an operator in $\mathscr{D} \oplus \mathscr{H}_2$. For the first row of the "inner" matrices, $D(C_1)^2$ is an operator $\mathscr{D}(M) \to \mathscr{D}(M)$ by definition, $C_1^* Y$ is $\mathscr{D}_* \to \mathscr{D}(M)$ by the definition of C_1. The fact that $-M^* C_2^*$ maps $\mathscr{D}_*$ into $\mathscr{D}(M)$ follows from the identity $M^* D(M^*) = D(M) M^*$.

Inequality (14) is equivalent to

$$\begin{pmatrix} 1 & -M^* C_2^* \\ -C_2 M & D(C_2^*)^2 + C_2 M M^* C_2^* \end{pmatrix} \geqslant \begin{pmatrix} C_1^* \\ Y^* \end{pmatrix} (C_1, Y).$$

Since the matrix on the left-hand side may be written in the form $K^* K$ for

$$K = \begin{pmatrix} 1 & -M^* C_2^* \\ 0 & D(C_2^*) \end{pmatrix},$$

V will be a contraction iff

$$(C_1, Y) = (K_1, K_2) K$$

for a suitable contraction (K_1, K_2). Since $K_1 = C_1$ we have $K_2 = D(C_1^*) C$ for a suitable contraction $C\colon \mathscr{D}(C_2^*) \to \mathscr{D}(C_1^*)$.

It follows that $Y = -C_1 M^* C_2^* + D(C_1^*) C D(C_2^*)$. The operator V assumes thus the following form

$$(15) \qquad V = \begin{pmatrix} C_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(M) & -M^* \\ M & D(M^*) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C_2^* \end{pmatrix} + \begin{pmatrix} 0 & D(C_1^*) C D(C_2^*) \\ 0 & 0 \end{pmatrix}.$$

The result may thus be formulated as follows.

Theorem 3.3. *The set of all contractions satisfying* 1°, 2° *and* 3° *in problem* (3.1) *is nonvoid. The solutions are given by* (15) *where* C *is an arbitrary contraction from* $\mathscr{D}(C_2^*)$ *into* $\mathscr{D}(C_1^*)$.

Let us turn now to the problem of isometric couplings.

4. Isometric couplings

Suppose that V is an isometry. Thus $1 - V^* V = 0$ and it follows from (12) and (13) that

$$D(M)(1 - C_1^* C_1) D(M) = 0.$$

Since C_1 is a contraction defined on $\mathscr{D}(M)$ with values in $\mathscr{D}$ this implies $1 - C_1^* C_1 = 0$, so that $\big(D(X) A_1\big)^* D(X) A_1 = D(M) C_1^* C_1 D(M) = D(M)^2$. Since

$$\begin{aligned} 1 - A_1^* A_1 &= 1 - A_1^* X^* X A_1 - A_1^* (1 - X^* X) A_1 \\ &= 1 - M^* M - \big(D(X) A_1\big)^* D(X) A_1 = 0 \end{aligned}$$

we have $A_1^*A_1 = 1$.

We have seen that C_1 is an isometry. In particular $D(C_1^*)C_1$ and $C_1^*D(C_1^*)$ are both zero—this follows from $D(C_1^*)C_1 = C_1D(C_1) = 0$. Since V is a contraction, we have

$$Y = -C_1M^*C_2^* + D(C_1^*)KD(C_2^*)$$

for some contraction $K\colon \mathscr{D}(C_2^*) \to \mathscr{D}(C_1^*)$. Using $D(C_1^*)C_1 = C_1^*D(C_1^*) = 0$, we obtain

$$D(C_1^*)Y = D(C_1^*)^2KD(C_2^*), \tag{16}$$

$$C_1^*Y = -M^*C_2^*. \tag{17}$$

Using (17) and (14) we obtain

$$Y^*D(C_1^*)^2Y = Y^*Y - C_2MM^*C_2^* = D(C_2^*)^2.$$

It follows that there exists an isometry W such that $D(C_1^*)Y = WD(C_2^*)$. Using (16) we obtain $WD(C_2^*) = D(C_1^*)Y = D(C_1^*)^2KD(C_2^*)$ so that $D(C_1^*)^2K = W$ is an isometry; thus K is an isometry.

On the other hand, suppose that A_1 is an isometry and let us prove that C_1 is an isometry. Since $D(X)A_1 = C_1D(M)$ we have

$$D(M)C_1^*C_1D(M) = A_1^*(1 - X^*X)A_1 = A_1^*A_1 - M^*M = 1 - M^*M = D(M)^2.$$

It follows that $C_1^*C_1 = 1$.

Suppose further that K is an isometry from $\mathscr{D}(C_2^*)$ into $\mathscr{D}(C_1^*)$. Define V by taking

$$Y = -C_1M^*C_2^* + D(C_1^*)KD(C_2^*).$$

We intend to show that $V^*V = 1$. First observe that $C_1^*D(C_1^*) = 0$ whence $C_1^*K = 0$.

It suffices to show that equality is attained in (14). This is a consequence of the following two identities.

$$C_1^*Y = -M^*C_2^* + C_1^*D(C_1^*)KD(C_2^*) = -M^*C_2^*,$$

$$\begin{aligned} Y^*Y &= \big(-C_2MC_1^* + D(C_2^*)K^*D(C_1^*)\big)\big(-C_1M^*C_2^* + D(C_1^*)KD(C_2^*)\big) \\ &= C_2MM^*C_2^* + D(C_2^*)K^*(1 - C_1C_1^*)KD(C_2^*) \\ &= C_2MM^*C_2^* + D(C_2^*)^2. \end{aligned}$$

Summing up, we have proved

Proposition 4.1. *Isometric couplings exist if and only if A_1 is an isometry and* $\dim \mathscr{D}(C_1^*) \geqslant \dim \mathscr{D}(C_2^*)$. *In that case, they are given by* (15) *where C is an arbitrary isometry from $\mathscr{D}(C_2^*)$ into $\mathscr{D}(C_1^*)$.*

Example 4.2. To show that isometry of A_1 alone is not sufficient for the existence of an isometric coupling it suffices, in view of (4.1), to produce an example of a triple A_1, A_2, X for which $D(C_2^*) = 1$ and $D(C_1^*) = 0$.

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two nontrivial Hilbert spaces, let A_1 be unitary and $A_2 = X = 0$. It follows that $M = 0$ so that $D(M) = D(M^*) = 1$. The equalities defining C_1 and C_2, $C_1 D(M) = D(X)A_1$ and $C_2 D(M^*) = D(X^*)A_2$ imply $C_1 = A_1$ and $C_2 = 0$. Thus $D(C_1^*) = 0$ and $D(C_2^*) = 1$.

It is much easier to see this directly. Indeed, suppose U is an isometry on $\mathscr{H}_1 \oplus \mathscr{H}_2$ leaving $\mathscr{H}_1$ invariant and such that $U|\mathscr{H}_1$ is unitary and $U^*|\mathscr{H}_2 = 0$. We prove first that $U\mathscr{H}_2 \subset \mathscr{H}_2$. Given $x \in \mathscr{H}_1$, we have $x = Ux'$ for a suitable $x' \in \mathscr{H}_1$ so that $(Uh_2, x) = (Uh_2, Ux') = (h_2, x') = 0$ for every $h_2 \in \mathscr{H}_2$ and $x \in \mathscr{H}_1$. Now $U^*U = 1$ since U is isometric. For $h_2 \in \mathscr{H}_2$ we have $Uh_2 \in \mathscr{H}_2$ whence $U^*Uh_2 = 0$. It follows that $\mathscr{H}_2 = 0$.

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