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ON COMPLETE MV -ALGEBRAS

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Though the number of published papers on MV -algebras is rather large (the fundamental source are Chang's articles [1] and [2]), the terminology and notation in this field seem to be far from being unified. We will apply the terminology from [5], [6].

It is well-known that MV -algebras are term equivalent to Wajsberg algebras (called also W -algebras); cf., e.g., Cignoli [3]. Further, MV -algebras are categorically equivalent to bounded commutative BCK -algebras (cf. Mundici [8]); such BCK -algebras were studied by Traczyk [10].

Cignoli [3] studied the structure of MV -algebras which are complete and atomic. His main result is the following theorem:

- (*) ([3], Theorem 2.6.) An MV -algebra is complete and atomic if and only if it is a direct product of finite linearly ordered MV -algebras.

An MV -algebra \mathcal{A} which is a direct product of MV -algebras \mathcal{A}_i ($i \in I$) is complete if and only if all \mathcal{A}_i are complete. Further, a complete linearly ordered MV -algebra is atomic if and only if it is finite (cf. 1.3 below). Thus (*) can be expressed as follows:

- (**) An MV -algebra is complete and atomic if and only if it is a direct product of complete atomic linearly ordered algebras.

Let $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$ be an MV -algebra. We can introduce lattice operations \vee, \wedge , and hence also the corresponding partial order \leq on A (cf. Section 1 below). Let $0 < x \in A$ and let $\alpha > 1$ be a cardinal. The element x will be called an α -atom of \mathcal{A} if the interval $[0, x]$ is a chain having cardinality α . Hence the notion of the 2-atom coincides with the usual notion of the atom. The MV -algebra \mathcal{A} is said to be α -atomic if for each $0 < y \in A$ there exists an α -atom x of \mathcal{A} with $x \leq y$.

Let R be the additive group of all reals with the natural linear order. For each MV -algebra \mathcal{A} there exists a lattice ordered group G with a strong unit such that \mathcal{A} can be constructed by means of G (cf. $(*_2)$ and $(*_3)$ in Section 1 below). If G is isomorphic to R , then \mathcal{A} will be said to be of type R .

By applying the results of [6] the following will be proved in the present paper:

- (A) Let \mathcal{A} be an MV -algebra and let α be a cardinal.
- (i) \mathcal{A} is complete and α -atomic if and only if it is isomorphic to a direct product of complete α -atomic linearly ordered MV -algebras.
 - (ii) Let $\alpha > 2$. An MV -algebra is complete, α -atomic and linearly ordered if and only if it is of type R .
 - (iii) If \mathcal{A} is a complete α -atomic MV -algebra with $A \neq \{0\}$, then either $\alpha = 2$ or $\alpha = c$ (the cardinality of the continuum).
- (B) Let \mathcal{A} be a complete MV -algebra. Then \mathcal{A} is isomorphic to a direct product $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ such that
- (i) \mathcal{A}_1 is atomic;
 - (ii) \mathcal{A}_2 is c -atomic;
 - (iii) for each cardinal α , there are no α -atoms in \mathcal{A}_3 .

Let us remark that for each infinite cardinal α there exists a non-complete MV -algebra \mathcal{A} such that, whenever x is a nonzero element of A , then x is an α -atom of \mathcal{A} .

1. PRELIMINARIES AND AUXILIARY RESULTS

For the notion of the MV -algebra we introduce the following definition (cf. [5] and [6]):

(**) An MV -algebra is a system $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$ (where $\oplus, *$ are binary operations, \neg is a unary operation and $0, 1$ are nullary operations) such that the following identities are satisfied:

- (m₁) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (m₂) $x \oplus 0 = x$;
- (m₃) $x \oplus y = y \oplus x$;
- (m₄) $x \oplus 1 = 1$;
- (m₅) $\neg\neg x = x$;
- (m₆) $\neg 0 = 1$;
- (m₇) $x \oplus \neg x = 1$;
- (m₈) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$;
- (m₉) $x * y = \neg(\neg x \oplus \neg y)$.

We recall the following results $(*_i)$ ($i = 1, 2, 3$) (for $(*_1)$ cf. [5]; for $(*_2)$ and $(*_3)$ cf. [7] 2.5 and 3.8; cf. also [6], 1.2, 1.3 and 1.4).

$(*_1)$ Let \mathcal{A} be an *MV*-algebra. For each $x, y \in A$ put $x \vee y = (x * \neg y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. Then $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge)$ is a distributive lattice with the least element 0 and the greatest element 1.

$(*_2)$ Let G be an abelian lattice ordered group with a strong unit u . Let A be the interval $[0, u]$ of G . For each a and b in A we put

$$a \oplus b = (a + b) \wedge u, \quad \neg a = u - a, \quad 1 = u, \quad a * b = \neg(\neg a \oplus \neg b).$$

Then $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$ is an *MV*-algebra.

If G and \mathcal{A} are as in $(*_2)$, then we put $\mathcal{A} = \mathcal{A}_0(G, u)$.

$(*_3)$ Let \mathcal{A} be an *MV*-algebra. Then there exists an abelian lattice ordered group G with a strong unit u such that $\mathcal{A} = \mathcal{A}_0(G, u)$.

In what follows, \mathcal{A} and G are as in $(*_2)$ and $(*_3)$.

1.1. Lemma. *\mathcal{A} is complete if and only if G is complete.*

Proof. Let \mathcal{A} be complete. Hence the interval $[0, u]$ is complete. The fact that u is a strong unit of G implies that for proving the completeness of G it suffices to verify that for each positive integer n the lattice $[0, nu]$ is complete.

We proceed by induction on n . The case $n = 1$ is trivial. Suppose that $n > 1$ and that the interval $[0, (n - 1)u]$ is complete. Since $[(n - 1)u, nu]$ is isomorphic to $[0, u]$, we obtain that $[(n - 1)u, nu]$ is complete as well.

Let $X = \{x_i\}_{i \in I}$ be a nonempty subset of $[0, nu]$. For each $i \in I$ we put

$$x_i^1 = x_i \wedge (n - 1)u, \quad x_i^2 = x_i \vee (n - 1)u.$$

In view of the assumption the elements

$$x^1 = \bigvee_{i \in I} x_i^1, \quad x^2 = \bigvee_{i \in I} x_i^2$$

exist. For each $i \in I$ the relation

$$x_i = x_i^1 + (x_i^2 - a)$$

is valid, where $a = (n - 1)u$. Put

$$x = x^1 + (x^2 - a).$$

Then $x \geq x_i$ for each $i \in I$. Let $y \in [0, nu], y \geq x_i$ for each $i \in I$. Put $y^1 = y \wedge (n - 1)u, y^2 = y \vee (n - 1)u$. Then $y^1 \geq x_i^1$ and $y^2 \geq x_i^2$ for each $i \in I$. At the same time we have

$$y = y^1 + (y^2 - a).$$

Therefore $y \geq x$. Thus $x = \sup X$ is valid in $[0, nu]$. Similarly we can verify that $\inf X$ does exist in $[0, nu]$. Hence $[0, nu]$ is complete.

The converse implication is obvious. □

1.2. Lemma. *\mathcal{A} is linearly ordered if and only if G is linearly ordered.*

Proof. If G is linearly ordered, then clearly \mathcal{A} is linearly ordered as well. Suppose that G fails to be linearly ordered. Then there are $g_i \in G$ with $0 < g_i$ ($i = 1, 2$), $g_1 \wedge g_2 = 0$. Since u is a strong unit in G we infer that $u_i = g_i \wedge u > 0$ ($i = 1, 2$). We have $u_1 \wedge u_2 = 0$ and $u_1, u_2 \in A$. Hence \mathcal{A} is not linearly ordered. □

Let Z be the additive group of all integers with the usual linear order. It is well-known that if $H \neq \{0\}$ is a complete linearly ordered group, then H is isomorphic either to Z or to R ; hence if $0 < h \in H$, then the interval $[0, h]$ is atomic if and only if $[0, h]$ is finite. Hence $(*_1), (*_2), 1.1$ and 1.2 yield

1.3.1. Corollary. *Let \mathcal{A} be an MV-algebra, $A \neq \{0\}$. Suppose that \mathcal{A} is linearly ordered and complete. Then (i) \mathcal{A} is finite if and only if it is atomic, and (ii) \mathcal{A} is infinite if and only if it is c-atomic.*

1.3.2. Corollary. *Let \mathcal{A} be as in 1.3.1. Then (i) \mathcal{A} is atomic if and only if G is isomorphic to Z ; (ii) \mathcal{A} is c-atomic if and only if it is of type R .*

For each nonempty subset X of a lattice ordered group H we denote

$$X^\delta = \{y \in H : |y| \wedge |x| = 0 \quad \text{for each } x \in X\}.$$

X^δ will be said to be the polar in H generated by the set X . For a thorough theory of polars in lattice ordered groups cf. Šik [9]. Each polar is a convex ℓ -subgroup of H ; if $0 \in X$ and X is a linearly ordered convex subset of H , then $X^{\delta\delta}$ is linearly ordered as well.

1.4. Lemma. *Let \mathcal{A} be a complete MV-algebra. Let $0 < x \in G$ and suppose that the interval $[0, x]$ is linearly ordered. Then either $[0, x]$ is finite or $[0, x]$ has cardinality c .*

Proof. Put $X = [0, x]$. Then X is, at the same time, an interval of G . Thus $X^{\delta\delta}$ is linearly ordered. According to 1.1, G is complete. Hence by the Riesz Theorem (cf., e.g., Fuchs [4], Chap. V), $X^{\delta\delta}$ is a direct factor of G . Therefore in view of [6], 3.2, $X_1 = X^{\delta\delta} \cap [0, u]$ is a direct factor of \mathcal{A} . Moreover, $X^{\delta\delta}$ is linearly ordered and hence X_1 is linearly ordered as well. Each direct factor of a complete MV -algebra must be complete. Now it suffices to apply 1.3.2. \square

1.5. Corollary. *Let α be a cardinal and let \mathcal{A} be a complete MV -algebra, $A \neq \{0\}$. If \mathcal{A} is α -atomic, then either $\alpha = 2$ or $\alpha = c$.*

The notion of an α -atom of a lattice ordered group can be defined in the same way as in the case of MV -algebras. A lattice ordered group G is said to be α -atomic if for each $g \in G$ with $0 < g$ there exists an α -atom g_1 in G such that $g_1 \leq g$.

By a similar argument as above we obtain

1.5'. Lemma. *Let α be a cardinal and let G be a complete nonzero lattice ordered group. If G is α -atomic, then either $\alpha = 2$ or $\alpha = c$.*

1.6. Example. Let α be an infinite cardinal. Next, let I be a linearly ordered set which is isomorphic to the first ordinal having the power α . For each $i \in I$ let G_i be a linearly ordered group isomorphic to Z . Put $G' = \Gamma_{i \in I} G_i$, where Γ denotes the operation of lexicographic product (cf., e.g., [4]). For $g' \in G'$ and $i \in I$ let g'_i be the component of g' in G_i . Denote $I(g') = \{i \in I: g'_i \neq 0\}$. Let G'' be the subgroup of G consisting of all $g' \in G'$ for which the set $I(g')$ is finite; G'' is linearly ordered by the inherited order. G'' is a non-complete linearly ordered group such that whenever $x, y \in G$ and $x < y$, then the power of the interval $[x, y]$ in G is α . Choose $u \in G''$ with $0 < u$ and let G be the convex ℓ -subgroup of G'' generated by the element u . Hence u is a strong unit in G . Let \mathcal{A} be as in $(*_2)$. Thus each strictly positive element of A is an α -atom in \mathcal{A} .

2. PROOFS OF (A) AND (B)

The assertion (ii) and (iii) of (A) were already proved (cf. 1.3.2 and 1.5); the remaining part of (A) will be proved as follows. The case $A = \{0\}$ being trivial we can suppose that $A \neq \{0\}$.

a) Suppose that an MV -algebra \mathcal{A} is a direct product of MV -algebras \mathcal{A}_i ($i \in I$). Without loss of generality we can suppose that the direct decomposition under consideration is internal (in the sense of [5]). Assume that all \mathcal{A}_i are linearly ordered, complete and α -atomic. For each $x \in A$ and $i \in I$ we denote by x_i the component

of x in \mathcal{A}_i . Let $x > 0$. Then there exists $i \in I$ such that $x_i > 0$. There is an α -atom y^i of \mathcal{A}_i with $y^i \leq x_i$. The element y^i is, at the same time, an α -atom in \mathcal{A} and $y^i \leq x$. Thus \mathcal{A} is α -atomic.

b) Suppose that \mathcal{A} is a complete and α -atomic MV-algebra. Since $A \neq \{0\}$, the set of all α -atoms of \mathcal{A} is nonempty. For each α -atom x of \mathcal{A} let X be as in the proof of 1.4. Hence $X^{\delta\delta}$ is a direct factor of G ; let $\{G_i\}_{i \in I}$ be the set of all $X^{\delta\delta}$ which can be constructed in this way. Each G_i is linearly ordered, complete and α -atomic.

For each $y \in G$ and $i \in I$ let y_i be the component of y in G_i . It is well-known that if $y \geq 0$, then y_i is the greatest element of the set $G_i \cap [0, y]$.

Let $y \in A$. We have $y_i \leq y$ for each $i \in I$; since \mathcal{A} is complete, there exists $y' = \bigvee_{i \in I} y_i$ in A . We shall verify that $y' = y$. By way of contradiction, suppose that $y'' = y - y' > 0$. Then $y'' \in A$ and hence there exists an α -atom x in A such that $x \leq y''$. Thus there is $i(1) \in I$ such that $G_{i(1)} = [0, x]^{\delta\delta}$. Clearly $y''_{i(1)} \geq x_{i(1)} = x > 0$. Since $0 \leq y_{i(1)} \in G_{i(1)} \cap [0, y']$ we obtain $y_{i(1)} \leq y'_{i(1)}$. On the other hand, the relation $y' < y$ gives $y'_{i(1)} \leq y_{i(1)}$. Thus $y_{i(1)} = y'_{i(1)}$. Therefore $y_{i(1)} = y'_{i(1)} + y''_{i(1)} = y_{i(1)} + x > y_{i(1)}$, which is a contradiction. Thus

$$(1) \quad y = \bigvee_{i \in I} y_i.$$

If $i(1)$ and $i(2)$ are distinct elements of I , then $G_{i(1)} \cap G_{i(2)} = \{0\}$. This implies that $y_{i(1)} \wedge y_{i(2)} = 0$.

Let φ be a mapping of A into the direct product $\prod_{i \in I} A_i$ (where $A_i = [0, u_i]$ for each $i \in I$) defined by

$$\varphi(y) = (y_i)_{i \in I}.$$

We consider A_i to be partially ordered by the inherited partial order. Let y and z be elements of A . If $y \leq z$, then clearly $y_i \leq z_i$ for each $i \in I$. Conversely, assume that $y_i \leq z_i$ for each $i \in I$; then we infer from (1) that $y \leq z$. Thus if y and z are distinct, then $\varphi(y)$ and $\varphi(z)$ are distinct as well. Further, let $(t^i) \in \prod_{i \in I} A_i$. There exists $t \in A$ with $t = \bigvee_{i \in I} t^i$. For each $i(1) \in I$ we have

$$t_{i(1)} = t_{i(1)} \wedge t = t_{i(1)} \wedge \left(\bigvee_{i \in I} t^i \right) = t_{i(1)} \wedge t^{i(1)}$$

(since $t_{i(1)} \wedge t^i = 0$ whenever $i \neq i(1)$). Thus $t_{i(1)} \leq t^{i(1)}$. On the other hand, $t^{i(1)} \in G_{i(1)} \cap [0, t]$ and hence $t^{i(1)} \leq t_{i(1)}$. We obtain $t^{i(1)} = t_{i(1)}$ and therefore $\varphi(t) = (t^i)_{i \in I}$. We have verified that φ is an isomorphism of the lattice A onto $\prod_{i \in I} A_i$.

For each $i \in I$ the element u_i is a strong unit in G_i , hence the MV -algebra $\mathcal{A}_i = (A_i; \oplus, *, \neg, 0, u_i)$ exists. From the construction of the isomorphism φ and from [6], 3.5 we infer that φ is, at the same time, an isomorphism of \mathcal{A} onto $\prod_{i \in I} \mathcal{A}_i$. Each \mathcal{A}_i is complete, linearly ordered and α -atomic. This completes the proof of (A).

PROOF OF (B). Let \mathcal{A} be a complete MV -algebra and let G be as above. We denote by X_1 and X_2 the system of all atoms of \mathcal{A} or the system of all c -atoms of \mathcal{A} , respectively. Put $G_i = X_i^{\delta}$ ($i = 1, 2$). By the Riesz Theorem, G_1 and G_2 are direct factors of G . For each $x_1 \in X_1$ and $x_2 \in X_2$ we have $x_1 \wedge x_2 = 0$. This yields that $G_1 \cap G_2 = \{0\}$. Therefore

$$(2) \quad G = G_1 \times G_2 \times G_3,$$

where

$$(3) \quad G_3 = (G_1 \cup G_2)^{\delta}.$$

All G_i ($i = 1, 2, 3$) are complete. It follows from the definition of G_1 that it is atomic; analogously, G_2 is c -atomic. The relation (3) yields that for each cardinal α no α -atom exists in G_3 .

For $i \in \{1, 2, 3\}$ let u_i be the component of u in G_i . We can construct the MV -algebras $\mathcal{A}_i = (A_i; \oplus, *, \neg, 0, u_i)$ for $i = 1, 2, 3$, where $A_i = [0, u_i]$. Then all \mathcal{A}_i are complete, \mathcal{A}_1 is atomic, \mathcal{A}_2 is c -atomic, and for each cardinal α , \mathcal{A}_3 has no α -atoms. Now we can apply [6], Lemma 3.2 (this lemma deals with direct decompositions having two factors, but by an obvious induction we can extend the validity of the lemma to direct decompositions having a finite number of direct factors); from (2) we infer that \mathcal{A} is a direct product of \mathcal{A}_1 , \mathcal{A}_2 and \mathcal{A}_3 . \square

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