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ON THE ORBITS OF AN OPERATOR WITH SPECTRAL RADIUS ONE

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0. INTRODUCTION

In [M], V. Müller proved the following theorem.

Theorem. *Let T be a bounded operator on a complex Banach space X with spectral radius $r(T) = 1$. Then for all $0 < \varepsilon < 1$ and $(\alpha_n) \in c_0$ of norm one there is a norm one vector $x \in X$ such that*

$$\|T^k x\| \geqslant (1 - \varepsilon)|\alpha_k|, \quad \forall k = 0, 1, 2, \ldots$$

For bounded operators on a Hilbert space, the above result was proved by Beauzamy [B, Thm. III.2.A.1]. He also shows that if there is no point spectrum on $\{|z| = 1\}$, such an x can be found in any ball of radius one.

For an application of the Theorem to stability theory of semigroups of operators, see [N].

The proof given in [M] relies on results from Fredholm theory. In fact, in case $r_e(T) < r(T) = 1$, where $r_e(T)$ is the essential spectral radius, there is an unimodular eigenvalue, and the theorem is trivial. The actual proof therefore concentrates on the case $r_e(T) = r(T)$.

For power bounded operators T, we will give a completely elementary proof of the Theorem. We do not use spectral theory, and our method works for both real and complex Banach spaces. In the case of a real Banach space, we define $r(T) = r(T_{\mathbb{C}})$, where $T_{\mathbb{C}}$ is the complexification of T; cf. [Ru].

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As usual, c_0 denotes the Banach space of all sequences $\alpha = (\alpha_n)_{n=0}^{\infty}$ that converge to zero, with norm $\|\alpha\| = \sup_n |\alpha_n|$.

1. Proof of the Theorem for power bounded operators

Lemma 1. *Suppose T is a bounded operator on a real or complex Banach space X with $r(T) = 1$. Then there exists a constant $C > 0$ with the following property. For each sequence $\alpha \in c_0$ of norm one there exists a norm one vector $x \in X$ and a subsequence (n_k) such that*

$$\|T^{n_k}x\| \geqslant C|\alpha_k|, \quad \forall k = 0, 1, 2, \dots$$

Proof. First note that we may assume without loss of generality that $T^n \to 0$ strongly. In particular, by the uniform boundedness theorem there is a constant M such that $\sup_n \|T^n\| = M < \infty$. Let $\alpha \in c_0$ be of norm one. Fix $0 < c < \frac{1}{2}M^{-1}$, fix $0 < \delta < c$ and choose m so large that

$$2^{-m+1} + M\sum_{i=1}^{\infty} 2^{-mi} < \delta \text{ and } \sum_{i=0}^{\infty} 2^{-mi} < 1 + \delta.$$

(In fact, the second is implied by the first).

Put $N_{-1} = -1$, $M_{-1} = -1$. Choose $N_0 \geqslant 0$ such that $|\alpha_i| \leqslant 2^{-m}$, $\forall i \geqslant N_0$.

In the complex case, $r(T) \geqslant 1$ implies that $\|T^n\| \geqslant 1$ for all $n \in \mathbb{N}$. In the real case, we use that $\|T_{\mathbb{C}}\| \leqslant 2\|T\|$ to conclude that $r(T) \geqslant 1$ implies $\|T^n\| \geqslant \frac{1}{2}$ for all $n \in \mathbb{N}$. In either case, the choice of c implies that there is a norm one vector $x_0 \in X$ such that $\|T^{N_0}x_0\| \geqslant cM$. For all $n = 0, 1, \dots, N_0$ we have $\|T^n x_0\| \geqslant M^{-1}\|T^{N_0}x_0\| \geqslant c$. Put $n_j := j$, $j = 0, \dots, N_0$. Since $\lim_n \|T^n x_0\| = 0$, we may choose M_0 such that $\|T^n x_0\| \leqslant 2^{-m}$, for all $n \geqslant M_0$.

Inductively, suppose norm one vectors $x_0, x_1, \dots, x_{l-1} \in X$, and numbers $N_0 < N_1 < \dots < N_{l-1}$ and $n_1 < n_2 < \dots < n_{N_{l-1}}$ and $M_0, \dots, M_{l-1}$ have been chosen subject to the following conditions:

(a) $|\alpha_i| \leqslant 2^{-m(j+1)}$, $\forall i \geqslant N_j$; $j = 0, 1, \dots, l-1$;
(b) $n_{N_{j-1}+1} \geqslant M_{j-1}$, $\forall j = 0, 1, \dots, l-1$;
(c) $\|T^{n_k}x_j\| \geqslant c$, $\forall k = N_{j-1}+1, \dots, N_j$; $j = 0, \dots, l-1$.
(d) $\|T^n x_i\| \leqslant 2^{-m(j+2)}$, $\forall 0 \leqslant i \leqslant j$ and $n \geqslant M_j$; $j = 0, 1, \dots, l-1$.

Choose $N_l \geqslant N_{l-1} + 1$ such that $|\alpha_i| \leqslant 2^{-m(l+1)}$, $\forall i \geqslant N_l$. Then (a) holds for the induction variable l. Choose a norm one vector $x_l \in X$ and numbers $n_{N_{l-1}+1} < \dots < n_{N_l}$ such that $n_{N_{l-1}+1} > n_{N_{l-1}}$, $n_{N_{l-1}+1} \geqslant M_{l-1}$ (this is (b)) and

$$\|T^{n_k}x_l\| \geqslant c, \quad k = N_{l-1} + 1, \dots, N_l.$$

Then (c) is satisfied. Finally, choose M_l such that

$$\|T^n x_i\| \leqslant 2^{-m(l+2)}, \quad \forall 0 \leqslant i \leqslant l \text{ and } n \geqslant M_l.$$

Then again (a)–(d) hold for the value l. Continue this process by induction. Put

$$x := \sum_{j=0}^{\infty} 2^{-mj} x_j.$$

Now let k be a fixed integer and choose $j \geqslant 0$ such that $N_{j-1} + 1 \leqslant k \leqslant N_j$. If $j \geqslant 1$, then by (a) and the fact that $k \geqslant N_{j-1}$ we have,

$$2^{-mj} = 2^{-m[(j-1)+1]} \geqslant |\alpha_k|.$$

In case $j = 0$, note that this inequality holds trivially. By (b) we have $n_k \geqslant n_{N_{j-1}+1} \geqslant M_{j-1}$ and consequently, by (d), for all $0 \leqslant i \leqslant j-1$ we have $\|T^{n_k} x_i\| \leqslant 2^{-m(j+1)}$. Therefore,

$$\sum_{i=0}^{j-1} 2^{-mi} \|T^{n_k} x_i\| \leqslant 2^{-m(j+1)+1}.$$

Also, we have the trivial estimate

$$\sum_{i=j+1}^{\infty} 2^{-mi} \|T^{n_k} x_i\| \leqslant 2^{-mj} M \sum_{i=1}^{\infty} 2^{-mi}.$$

Therefore,

$$\|T^{n_k} x\| \geqslant 2^{-mj} \Big(c - 2^{-m+1} - M \sum_{i=1}^{\infty} 2^{-mi} \Big) \geqslant 2^{-mj}(c - \delta) \geqslant |\alpha_k|(c - \delta).$$

Finally, observe that x has norm $\leqslant \sum_{j=0}^{\infty} 2^{-mj} \leqslant 1 + \delta$. Hence, by rescaling x to a norm one vector, for the rescaled x we obtain

$$\|T^{n_k} x\| \geqslant \frac{c - \delta}{1 + \delta} |\alpha_k|.$$

This proves the theorem, with $C = (c - \delta)/(1 + \delta)$. □

Theorem 1.2. *Let T be a power bounded operator on a real or complex Banach space X with $r(T) = 1$. Then for all $\varepsilon > 0$ and all $\alpha \in c_0$ of norm one, there exists a norm one vector $x \in X$ such that*

$$\|T^k x\| \geqslant (1 - \varepsilon)|\alpha_k|, \quad \forall k = 0, 1, 2, \ldots$$

Proof. *Step 1.* Put $\sup_n \|T^n\| = M < \infty$. Define the equivalent norm $|||\cdot|||$ on X by $|||x||| = \sup_n \|T^n x\|$. Then $\|x\| \leqslant |||x||| \leqslant M\|x\|$ and $|||T||| \leqslant 1$. Let (β_n) be a norm one sequence in c_0 such that $\beta_n \downarrow 0$ and $\beta_n \geqslant |\alpha_n|$ for all n. By the Lemma, there exists a vector x of $|||\cdot|||$-norm one and a subsequence (n_k) such that $|||T^{n_k}x||| \geqslant C\beta_k$. Set $c := CM^{-1}$. We have $\|x\| \leqslant 1$, and for all k we have

$$\|T^k x\| \geqslant M^{-1}|||T^k x||| \geqslant M^{-1}|||T^{n_k}x||| \geqslant c\beta_k \geqslant c|\alpha_k|.$$

Step 2. We will now show that the constant c can actually be replaced by $1-\varepsilon$. Let $0 < \varepsilon < 1$ be arbitrary and fix a norm one $(\alpha_n) \in c_0$. Fix some $\delta > 0$ such that $(1-\delta)(1+\delta)^{-1} \geqslant 1-\varepsilon$. We start by choosing integers $0 = M_0 < M_1 < \ldots$ such that $|\alpha_k| \leqslant (1+\delta)^{-n}$ whenever $k \geqslant M_n$. Next, choose integers $0 = N_0 < N_1 < \ldots$ in such a way that $N_n \geqslant M_n$ for each n and $N_m + N_n \leqslant N_{m+n}$ for all n, m. Define the norm one element $(\beta_n) \in c_0$ by $\beta_k = (1+\delta)^{-n}$ whenever $N_n \leqslant k < N_{n+1}$. Note that $\beta \geqslant |\alpha|$.

We claim that $\beta_{m+n} \geqslant (1+\delta)^{-1}\beta_m\beta_n$. Indeed, choose k_m and k_n such that $N_{k_m} \leqslant m \leqslant N_{k_m+1}$ and $N_{k_n} \leqslant n \leqslant N_{k_n+1}$. Then $\beta_m = (1+\delta)^{-k_m}$ and $\beta_n = (1+\delta)^{-k_n}$, whereas from $m+n < N_{k_m+1} + N_{k_n+1} \leqslant N_{k_m+k_n+2}$ we have $\beta_{m+n} \geqslant (1+\delta)^{-k_m-k_n-1}$. This proves the claim.

Now choose a norm one vector $y \in X$ such that $\|T^k y\| \geqslant c\beta_k$ for all k, where c is the constant of Step 1. Let

$$\gamma := \inf_k \frac{\|T^k y\|}{\beta_k}.$$

Note that $\gamma \geqslant c$; moreover, for all k we have $\|T^k y\| \geqslant \gamma\beta_k$. Choose an index k_0 such that

$$\frac{\gamma\beta_{k_0}}{\|T^{k_0}y\|} \geqslant 1-\delta$$

and put $x = \|T^{k_0}y\|^{-1}T^{k_0}y$. Then for all n we have

$$\|T^n x\| = \frac{\|T^{k_0+n}y\|}{\|T^{k_0}y\|} \geqslant \frac{\gamma\beta_{k_0+n}}{\|T^{k_0}y\|} \geqslant (1-\delta)\frac{\beta_n}{1+\delta} \geqslant (1-\varepsilon)|\alpha_n|.$$

□

2. The weak case

In this section, we will give some partial answers as to whether every operator T with $r(T) \geqslant 1$ has weak orbits that converge to zero arbitrarily slowly.

Lemma 2.1. [N, Cor. 2.5] *Let X be a real or complex Banach space. Let $\beta_n \geqslant 0$, $n \in \mathbb{N}$, and assume that $\sum_{n=0}^{\infty} \beta_n = \infty$. If $1 \leqslant p < \infty$ and T is a bounded operator such that*

$$\sum_{n=0}^{\infty} \beta_n |\langle x^*, T^n x\rangle|^p < \infty, \quad \forall x \in X < x^* \in X^*,$$

then $r(T) < 1$.

Theorem 2.2. *Let T be a bounded operator on a real or compex Banach space X with $r(T) = 1$. Let $\alpha \in c_0$ be of norm one. Then each sequence (n_k) has a subsequence (n_{k_j}) with the property that there exist norm one vectors $x \in X$, $x^* \in X^*$ such that*

$$|\langle x^*, T^{n_{k_j}} x\rangle| \geqslant |\alpha_{k_j}|, \quad j = 0, 1, \ldots$$

Proof. By replacing α_n by $\sup_{k \geqslant n} |\alpha_k|$, we may assume that $\alpha_0 = 1$ and $\alpha_n \downarrow 0$. Put $N_0 := -1$ and for $k = 1, 2, \ldots$ put

$$N_k := \max\{n \in \mathbb{N}\colon \alpha_n \geqslant k^{-1}\}.$$

Then for $0 \leqslant n \leqslant N_1$ we have $\alpha_n = 1$ and for $k \geqslant 1$ and $N_k + 1 \leqslant n \leqslant N_{k+1}$ we have $(k+1)^{-1} \leqslant \alpha_n < k^{-1}$. Define the sequence (β_n) by $\beta_n = 1$, $n = 0, \ldots, N_1$, and

$$\beta_n := k^{-1}(N_{k+1} - N_k)^{-1}, \quad n = N_k + 1, \ldots, N_{k+1}; \ k = 1, 2, \ldots$$

Then $\sum_{n=0}^{\infty} \beta_n = \infty$, and

$$\sum_{n=0}^{\infty} \alpha_n \beta_n \leqslant N_1 + 1 + \sum_{k=1}^{\infty} (N_{k+1} - N_k) \cdot k^{-1} \cdot k^{-1} (N_{k+1} - N_k)^{-1} < \infty.$$

Let (n_k) be any given sequence, and define $(\tilde{\beta}_n)$ by

$$\tilde{\beta}_j := \begin{cases} \beta_k, & \text{if } j = n_k \text{ for some } k; \\ 0, & \text{else.} \end{cases}$$

Then $\sum\limits_{j=0}^{\infty} \tilde{\beta}_j = \sum\limits_{n=0}^{\infty} \beta_n = \infty$. By Lemma 2.1, there exist $x \in X$ and $x^* \in X^*$ such that

$$\sum_{j=0}^{\infty} \tilde{\beta}_j |\langle x^*, T^j x\rangle| = \sum_{k=0}^{\infty} \beta_k |\langle x^*, T^{n_k} x\rangle| = \infty.$$

Since $\sum\limits_{n=0}^{\infty} \alpha_n \beta_n < \infty$, there have to be infinitely many indices n_k for which

$$|\langle x^*, T^{n_k} x\rangle| \geqslant \alpha_k.$$

This proves the theorem. □

In the case of a positive operator on a Banach lattice the full weak analogue of the Theorem holds. This is the content of our next result.

Theorem 2.3. *Let T be a positive operator on a real or complex Banach lattice with $r(T) = 1$. Then for each $\varepsilon > 0$ and $\alpha \in c_0$ of norm one, there exist norm one vectors $0 \leqslant x \in X$ and $0 \leqslant x^* \in X^*$ such that*

$$\langle x^*, T^n x\rangle \geqslant (1-\varepsilon)|\alpha_n|, \quad n = 0, 1, 2, \ldots$$

Proof. We may assume that $\alpha_n \downarrow 0$. Also, we may assume that X is complex. Indeed, if X is real we consider the complexification $T_{\mathbb{C}}$ on $X_{\mathbb{C}}$, and observe that positive vectors in $X_{\mathbb{C}}$ in fact belong to the real part X.

Choose $\delta > 0$ such that $(1+\delta)^{-2}(1-\delta) \geqslant 1-\varepsilon$. By considering approximate eigenvectors, it is easy to see (cf. [N, Lemma 2.1]) that for each $N \in \mathbb{N}$, there exist norm one vectors $0 \leqslant x_N \in X$ and $0 \leqslant x_N^* \in X^*$ such that

$$\langle x_N^*, T^n x_N\rangle \geqslant 1-\delta, \quad n = 0, 1, \ldots, N.$$

The proof can now be given along the lines of Lemma 1.1; the positivity simplifies the argument.

Choose m such that $\sum\limits_{n=0}^{\infty} 2^{-mn} \leqslant 1+\delta$. For each $k = 0, 1, \ldots$, let

$$N_k = \max\{n \in \mathbb{N} \colon \alpha_n \geqslant 2^{-2mk}\},$$

and choose norm one vectors $0 \leqslant x_k \in X$ and $0 \leqslant x_k^* \in X^*$ such that

$$\langle x_k^*, T^n x_k\rangle \geqslant 1-\delta, \quad n = 0, 1, \ldots, N_{k+1}.$$

Set $x = (1+\delta)^{-1} \sum\limits_{k=0}^{\infty} 2^{-mk} x_k$ and $x^* = (1+\delta)^{-1} \sum\limits_{k=0}^{\infty} 2^{-mk} x_k^*$. Then both x and x^* are positive vectors of norm $\leqslant 1$. Fix $n \in \mathbb{N}$. If $0 \leqslant n \leqslant N_0$, then

$$\langle x^*, T^n x\rangle \geqslant (1+\delta)^{-2}\langle x_0^*, T^n x_0\rangle \geqslant (1+\delta)^{-2}(1-\delta) \geqslant 1-\varepsilon = (1-\varepsilon)\alpha_n.$$

We used that $\alpha_n = 1$ for $n = 0, \ldots, N_0$. If $n \geqslant N_0 + 1$, say $N_j + 1 \leqslant n \leqslant N_{j+1}$ for some j, then $\alpha_n \leqslant \alpha_{N_j+1} < 2^{-2mj}$ and consequently,

$$\langle x^*, T^n x\rangle \geqslant 2^{-2mj}(1+\delta)^{-2}\langle x_j^*, T^n x_j\rangle \geqslant 2^{-2mj}(1-\varepsilon) \geqslant (1-\varepsilon)\alpha_n.$$

□

Theorem 2.3 fails for arbitrary operators, at least in the case of real scalars. Indeed, we have the following counterexample in $X = \mathbb{R}^2$.

Example 2.4. Let $\gamma \in [0, 2\pi)$ be a number such that $\gamma/(2\pi)$ is irrational. Let T_γ be rotation over γ in $X = \mathbb{R}^2$. Let $C > 0$ be an arbitrary real number. For $x, y \in \mathbb{R}^2$ on norm one, let $n(x, y)$ denote the first integer such that

$$|\langle T_\gamma^n x, y\rangle| < \frac{C}{2}.$$

Because the orbit $n \mapsto T_\gamma^n x$ is dense in the unit circle by the assumption on γ, the numbers $n(x, y)$ indeed exist. We claim that

$$N := \sup\{n(x,y) \colon \|x\| = \|y\| = 1\} < \infty.$$

Indeed, suppose not. Then for each $n \in \mathbb{N}$ there are x_n, y_n of norm one such that

$$|\langle T_\gamma^k x_n, y_n\rangle| \geqslant \frac{C}{2}, \quad 0 \leqslant k \leqslant n.$$

Choose a subsequence (n_j) such that $x_{n_j} \to x$ and $y_{n_j} \to y$, and fix k. Then for all j such that $n_j \geqslant k$ we have

$$\begin{aligned}|\langle T_\gamma^k x, y\rangle| \geqslant\ & |\langle T_\gamma^k x_{n_j}, y_{n_j}\rangle| - |\langle T_\gamma^k x_{n_j}, y_{n_j}\rangle| \\ & - |\langle T_\gamma^k (x - x_{n_j}), y\rangle| - |\langle T_\gamma^k x_{n_j}, y - y_{n_j}\rangle|.\end{aligned}$$

Letting $j \to \infty$ we obtain

$$|\langle T_\gamma^k x, y\rangle| \geqslant \frac{C}{2}, \quad \forall k \in \mathbb{N}.$$

This contradicts the finiteness of $n(x, y)$. Now let $\alpha \in c_0$ be the vector

$$\alpha = (1, 1, \ldots, 1, 0, 0, \ldots),$$

where $\alpha_n = 1$ for $0 \leqslant n \leqslant N$ and $\alpha_n = 0$ for $n > N$. Then for *all* norm one vectors $x, y \in \mathbb{R}^2$ there is a $k = k(x,y) \in 0, \ldots, N$ such that

$$|\langle T_\gamma^k x, y\rangle| < C|\alpha_k|.$$

As it turns out, this example works because T_γ is unitary. To see why, we need some terminology Let H be a real or complex Hilbert space. An operator T on H is called an *isometry* if $\|Tx\| = \|x\|$ for all $x \in H$ or equivalently, if $T^*T = I$. The operator T is called an *unilateral shift* if there is an orthogonal decomposition $H = \bigoplus_{n\in\mathbb{N}} H_n$ such that $TH_n \subset H_{n+1}$ and the map $T\colon H_n \to H_{n+1}$ is an isometry for all $n \in \mathbb{N}$. We have the so-called *Wold decomposition*: If T is an isometry on a Hilbert space H, then there is an orthogonal decomposition $H = H_0 \oplus H_1$ with $TH_i \subset H_i$, $i = 0, 1$, such that $T_0 := T|_{H_0}$ is unitary and $T_1 := T|_{H_1}$ is an unilateral shift. For a proof we refer to [SF], Theorem 1.1.

Now we have the following result: *Let T be a non-unitary isometry on a real or complex Hilbert space H. Then for all $\varepsilon > 0$ and $\alpha \in c_0$ of norm one, there exist norm one vectors $x \in H$, $y \in H$, such that*

$$(*) \qquad |\langle T^n x, y\rangle| \geqslant (1-\varepsilon)|\alpha_n|, \quad \forall n \in \mathbb{N}.$$

Indeed, let $H = H_0 \oplus H_1$ be the Wold decomposition. Since T is not unitary, H_1 is non-empty. By considering the restriction of T to H_1, we therefore may assume that T is an unilateral shift on H.

Let $H = \bigoplus_{n\in\mathbb{N}} H_n$ be an orthogonal decomposition of H such that $T\colon H_n \to H_{n+1}$ is an isometry. Fix an arbitrary norm one vector $x_0 \in H_0$ and put $x_n := T^n x_0$. The closed linear span of $\{x_n\colon n \in \mathbb{N}\}$ is isometric to l^2 and the restriction of T to this span acts as the shift on l^2. Therefore, we can apply Theorem 2.3.

In fact, inspecting the proof of Theorem 2.3 for the shift operator on l^2, it is not hard to see that in fact we can find an $0 \leqslant x \in l^2$ of norm one such that $\langle T^n x, x\rangle \geqslant (1-\varepsilon)|\alpha_n|$ for all n. This implies that one can even achieve $x = y$ in $(*)$.

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