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CONNECTIONS ON SOME FUNCTIONAL BUNDLES

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INTRODUCTION

Our starting point was the idea of the Schrödinger connection on a double fibered manifold by Jadczyk and Modugno, [4], [5]. We discuss the "pure case" of two classical fiber bundles $E_1$ and $E_2$ over the same base and define a connection $\Gamma$ on the bundle $\mathcal{F}(E_1, E_2)$ of all smooth maps from a fiber of $E_1$ into the fiber of $E_2$ over the same base point. We study systematically the geometry of the iterated tangent bundle of the infinite dimensional space $\mathcal{F}(E_1, E_2)$ as well as the jet prolongations of $\mathcal{F}(E_1, E_2)$ by means of the ideas introduced by the second author in [9]. Since we deal with functional bundles, our vector fields and connections represent a kind of differential operators. That is why we pay special attention to the case of finite order operators, in which we are able to deduce a very concrete description of the objects and operations in question.

In such a situation we found the simplest way for introducing the curvature of $\Gamma$ in a construction by Ehresmann, [2], which is based on the notion of semiholonomic 2-jets. In the new context we were obliged to rearrange some results, deduced in the finite dimension by direct evaluation, into a more geometrical setting, which could be generalized to our infinite dimensional case. Only then we study the bracket of two vector fields on $\mathcal{F}(E_1, E_2)$. This is a modification of the bracket of two vertical prolongation operators on a classical fibered manifold by Kosmann-Schwarzbach, [11], and the second author, [8]. In Proposition 14 we deduce a satisfactory bracket formula for the curvature of $\Gamma$. We also discuss the absolute differentiation with respect to $\Gamma$ and the special case $E_2$ is a vector bundle.

If we deal with two finite dimensional manifolds and a map between them, we always assume they are of class $C^\infty$, i.e. smooth in the classical sense. On the other

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hand, the idea of smoothness in the infinite dimension is taken from the theory of smooth structures by Frölicher, [3].

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1. THE TANGENT BUNDLE OF $\mathcal{F}(E_1, E_2)$

Let $p_1 : E_1 \to M$ and $p_2 : E_2 \to M$ be two classical fiber bundles (i.e. locally trivial fibered manifolds) over the same base. Consider the set of all fiber maps

$$\mathcal{F}(E_1, E_2) = \bigcup_{x \in M} C^\infty(E_{1x}, E_{2x})$$

and denote by $p : \mathcal{F}(E_1, E_2) \to M$ the canonical projection. We define no topology on $\mathcal{F}(E_1, E_2)$, but we introduce the concept of a smooth map from a classical manifold $Q$ into $\mathcal{F}(E_1, E_2)$.

**Definition 1.** A map $f : Q \to \mathcal{F}(E_1, E_2)$ is called smooth, if

(i) $p \circ f : Q \to M$ is smooth and
(ii) the induced map $\tilde{f} : (p \circ f)^*E_1 \to E_2$,

$$\tilde{f}(q, y) = f(q)(y), \quad (q, y) \in (p \circ f)^*E_1$$

is also smooth.

As usual, $(p \circ f)^*E_1 \to Q$ denotes the bundle induced from $E_1$ by means of $p \circ f$, i.e.

$$(p \circ f)^*E_1 = \{(q, y) \in Q \times E_1 \mid (p \circ f)(q) = p_1(y)\}.$$ 

Thus, $\mathcal{F}(E_1, E_2)$ is endowed with a smooth structure in the sense of Frölicher, [3].

For every smooth curve $f : \mathbb{R} \to \mathcal{F}(E_1, E_2)$ we first construct the tangent vector $X = \frac{\partial}{\partial t}\big|_0 (p \circ f) \in TM$ of its base map at $t = 0$. Write

$$T_XE_1 = (Tp_1)^{-1}(X) \subset TE_1 \quad \text{or} \quad T_XE_2 = (Tp_2)^{-1}(X) \subset TE_2,$$

so that $T_XE_1$ or $T_XE_2$ is an affine bundle over $E_{1x}$ or $E_{2x}$, $x = p(f(0))$, with the derived vector bundle $T(E_{1x}) := V_xE_1$ or $T(E_{2x}) := V_xE_2$, respectively. Then $f$ defines a map $T_0f : T_XE_1 \to T_XE_2$ by

$$T_0f\left(\frac{\partial}{\partial t}\big|_0 h(t)\right) = \frac{\partial}{\partial t}\big|_0 f(t)(h(t))$$

where we may assume that $h : \mathbb{R} \to E_1$ satisfies $p \circ f = p_1 \circ h$. 

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Definition 2. We say that two smooth curves \( f, g : U \to \mathcal{F}(E_1, E_2) \) satisfying
\[
\frac{\partial}{\partial t} \big|_0 \nu \circ f = \frac{\partial}{\partial t} \big|_0 \nu \circ g = X
\]
determine the same tangent vector at \( f(0) = g(0) = \varphi \), if
\[
T_0f = T_0g : TXE_1 \to TXE_2.
\]
The set \( T\mathcal{F}(E_1, E_2) \) of all equivalence classes will be called the tangent bundle of \( \mathcal{F}(E_1, E_2) \).

We write \( \frac{\partial}{\partial t} \big|_0 f(t) \in T\mathcal{F}(E_1, E_2) \) for the tangent vector determined by \( f \) and
\[
\pi : T\mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2) \text{ and } Tp : T\mathcal{F}(E_1, E_2) \to TM \text{ for the canonical projections.}
\]
If \( A \in T\mathcal{F}(E_1, E_2) \), then we denote by \( \tilde{A} : T_{Tp(A)}E_1 \to T_{Tp(A)}E_2 \) the associated map (1).

Remark 1. Let \( \varepsilon \subset \mathcal{F}(E_1, E_2) \) be any subset. Then we define \( T\varepsilon \subset T\mathcal{F}(E_1, E_2) \) by restricting ourselves to the smooth curves with values in \( \varepsilon \).

One sees easily that \( T_0f = T_0g : TXE_1 \to TXE_2 \) is an affine bundle morphism over the base map \( \varphi : E_{1x} \to E_{2x} \) with the derived linear morphism \( T\varphi : T(E_{1x}) \to T(E_{2x}) \). Indeed, let \( x^i \) be some local coordinates on \( M \), \( y^p \) or \( z^a \) be some additional coordinates on \( E_1 \) or \( E_2 \) and
\[
(2) \quad x^i = f^i(t), \quad z^a = f^a(y^p, t)
\]
be the coordinate expression of \( f(t) \). Write
\[
Y^p = dy^p, \quad Z^a = dz^a, \quad \varphi^a(y) = f^a(y^p, 0), \quad \Phi^a(y) = \frac{\partial f^a(y^p, 0)}{\partial t}.
\]
Then the coordinate form of (1) is
\[
(3) \quad Z^a = \frac{\partial \varphi^a(y)}{\partial y^p} Y^p + \Phi^a(y).
\]
Hence the tangent vector to (2) is locally characterized by two systems of numbers and two systems of functions
\[
(4) \quad x^i = f^i(0), \quad X^i = \frac{\partial f^i(0)}{\partial t}, \quad \varphi^a(y^p), \quad \Phi^a(y^p).
\]
The following lemma gives a global assertion of such a type.

Lemma 1. Let \( F : TXE_1 \to TXE_2 \) be an affine bundle morphism over \( \varphi : E_{1x} \to E_{2x} \) with the derived linear morphism \( T\varphi : T(E_{1x}) \to T(E_{2x}) \). Then there exists a smooth curve \( f : \mathbb{R} \to \mathcal{F}(E_1, E_2) \) such that \( F = \tilde{A} \) for the tangent vector \( A = \frac{\partial}{\partial t} \big|_0 f(t) \).
Proof. Consider some local trivializations \( U \times S_1 \) and \( U \times S_2 \) of \( E_1 \) and \( E_2 \) over a neighborhood \( U \subset M \) of \( x \). Then \( \mathcal{F}(U \times S_1, U \times S_2) = U \times C^\infty(S_1, S_2) \). The restriction of \( F \) to \( Y^p = 0 \) represents a map \( \bar{F} : S_1 \to TS_2 \) along \( \varphi \). By Proposition 5 from [16] there exists a smooth curve \( \gamma : \mathbb{R} \to C^\infty(S_1, S_2) \) such that \( \bar{F}(y) = \frac{\partial z(y,0)}{\partial t} \), where \( \tilde{\gamma} : \mathbb{R} \times S_1 \to S_2 \) is defined by \( \tilde{\gamma}(y, t) = \gamma(t)(y) \). If \( \delta : \mathbb{R} \to U \) is any curve with \( \frac{\partial \delta}{\partial t} \big|_0 \delta = X \), then the curve \( (\delta, \gamma) : \mathbb{R} \to U \times C^\infty(S_1, S_2) \) has the required property. \( \square \)

Now we show that each fiber of \( T\mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2) \) is a vector space. Consider \( \tilde{A}_1 : T_{X_1} E_1 \to T_{X_1} E_2 \) and \( \tilde{A}_2 : T_{X_2} E_1 \to T_{X_2} E_2 \) over the same \( \varphi \). Given \( Y \in (T_{X_1} + x_2 E_1)_y, y \in E_1 \), we take any \( W \in (T_{X_2} E_2)_y \), so that \( Y - W \in (T_{X_2} E_2)_y \), and we define

\[
\tilde{A}_1 + \tilde{A}_2(Y) = \tilde{A}_1(W) + \tilde{A}_2(Y - W).
\]

If we select another \( \tilde{W} \in (T_{X_1} E_1)_y \), then \( W - \tilde{W} \) is a vertical vector. Hence

\[
\tilde{A}_1(\tilde{W}) = \tilde{A}_1(W) + T\varphi(\tilde{W} - W), \quad \tilde{A}_2(\tilde{Y} - \tilde{W}) = \tilde{A}_2(Y - W) + T\varphi(W - \tilde{W}),
\]

so that our definition is correct. Further, for \( 0 \neq k \in \mathbb{R} \) we define

\[
\tilde{kA} : T_{kX} E_1 \to T_{kX} E_2 \quad \text{by} \quad \tilde{kA}(Y) = k\tilde{A}\left(\frac{1}{k}Y\right)
\]

while for \( k = 0 \) we prescribe \( \tilde{0A} \) to be \( T\varphi : T_0 E_1 \to T_0 E_2 \). In coordinates, if \( A_1 = (x^i, X_1^a, \varphi^a, \Phi_1^a) \) and \( A_2 = (x^i, X_2^a, \varphi^a, \Phi_2^a) \), then

\[
A_1 + A_2 = (x^i, X_1^i + X_2^i, \varphi^a, \Phi_1^a + \Phi_2^a), \quad kA_1 = (x^i, kX_1^i, \varphi^a, k\Phi_1^a).
\]

This proves that each \( \pi^{-1}(\varphi) \) is a vector space.

In general, consider another pair \( E_3 \to N, E_4 \to N \) of fiber bundles over the same base and subset \( \varepsilon \subset \mathcal{F}(E_1, E_2) \).

**Definition 3.** A map \( f : \varepsilon \to \mathcal{F}(E_3, E_4) \) is called smooth, if \( f \circ g : Q \to \mathcal{F}(E_3, E_4) \) is smooth for every smooth map \( g : Q \to \varepsilon \).

**Definition 4.** A vector field on \( \mathcal{F}(E_1, E_2) \) is a smooth map \( A : \mathcal{F}(E_1, E_2) \to T\mathcal{F}(E_1, E_2) \) satisfying \( \pi \circ A = \text{id} \). We say that \( A \) is projectable, if there exists a classical smooth vector field \( A^0 : M \to TM \) such that \( A^0 \circ \pi = Tp \circ A \).

Write \( V \mathcal{F}(E_1, E_2) \) for the kernel of \( Tp : T\mathcal{F}(E_1, E_2) \to TM \), which will be called the vertical tangent bundle of \( \mathcal{F}(E_1, E_2) \). Then we have an exact sequence

\[
0 \to V \mathcal{F}(E_1, E_2) \to T\mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2) \times TM_M \to 0
\]
Consider a linear splitting $\Gamma: \mathcal{F}(E_1, E_2) \times TM \to TM$, i.e. $\pi \circ \Gamma = \text{pr}_1, Tp \circ \Gamma = \text{pr}_2$ and $\Gamma(\varphi, :) : T_x M \to T_{\varphi} \mathcal{F}(E_1, E_2)$ is a linear map for each $\varphi \in \mathcal{F}(E_1, E_2), x = \pi(\varphi)$. Then for every vector field $X: M \to TM$ we have defined its $\Gamma$-lift $\Gamma X: \mathcal{F}(E_1, E_2) \to T \mathcal{F}(E_1, E_2)$. We say that $\Gamma$ is smooth, if $\Gamma X$ is smooth for every classical smooth vector field $X: M \to TM$.

**Definition 5.** A connection (in tangent form) on $\mathcal{F}(E_1, E_2)$ is a smooth linear splitting $\Gamma: \mathcal{F}(E_1, E_2) \times TM \to TM$.

**Remark 2.** If $E_1$ is the trivial fibering $M \to M$, then $\mathcal{F}(E_1, E_2) = E_2$ and we obtain the standard connection on $E_2 \to M$.

### 2. Jet prolongations of $\mathcal{F}(E_1, E_2)$

The simplest way how to define the $r$-th jet prolongation of $\mathcal{F}(E_1, E_2)$ is based on the concept of the fiber $r$-jet, [9], [10]. In general, given a fiber bundle $E \to M$ and a manifold $N$, two maps $f, g: E \to N$ are said to determine the same fiber $r$-jet $j^r_x f = j^r_x g$ at $x \in M$, if $j^r_y f = j^r_y g$ for all $y \in E_x$. Every smooth section $s$ of $\mathcal{F}(E_1, E_2)$ determines the associated base-preserving morphism $\tilde{s}: E_1 \to E_2$, $\tilde{s}(y) = s(p_1y)(y)$.

**Definition 6.** Two sections $s_1, s_2: M \to \mathcal{F}(E_1, E_2)$ determine the same $r$-jet $j^r_x s_1 = j^r_x s_2$ at $x \in M$, if $j^r_x \tilde{s}_1 = j^r_x \tilde{s}_2$. The set $J^r \mathcal{F}(E_1, E_2)$ of all $r$-jets of the local sections of $\mathcal{F}(E_1, E_2)$ is called the $r$-jet prolongation of $\mathcal{F}(E_1, E_2)$.

However, it will be useful to discuss another approach as well. Since $\tilde{s}: E_1 \to E_2$ is a base-preserving morphism, we can construct its $r$-th jet prolongation $J^r \tilde{s}: J^r E_1 \to J^r E_2$. Write $J^r \tilde{s} = J^r \tilde{s} | J^1 E_1, x \in M$. By direct evaluation, one easily verifies.

**Proposition 1.** We have $j^r_x s_1 = j^r_x s_2$ iff $J^r_x \tilde{s}_1 = J^r_x \tilde{s}_2$.

Let $z^a = f^a(x^i, y^p)$ be the coordinate expression of $\tilde{s}$. Then the additional coordinate expression of $J^1_x \tilde{s}$ is

$$z^a_i = \frac{\partial f^a}{\partial x^i} + \frac{\partial f^a}{\partial y^p} y^p_i$$

where $y^p_i$ or $z^a_i$ are the induced coordinates on $J^1 E_1$ or $J^1 E_2$. For $x = 0$, the functions $\varphi^a(y^p) := f^a(0, y^p)$ are the coordinates of the target $s(0)$ of $j^0_0 s_1$ and $J^1_0 \tilde{s}$ has the form

$$z^a_i = \frac{\partial \varphi^a(y)}{\partial y^p} y^p_i + \varphi^a_i(y), \quad \varphi^a_i(y) = \frac{\partial f^a(0, y)}{\partial x^i}$$

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It is well-known that $J^1E_1$ or $J^1E_2$ is an affine bundle over $E_{1x}$ or $E_{2x}$, whose derived vector bundle is $V_xE_1 \otimes T_x^*M$ or $V_xE_2 \otimes T_x^*M$, respectively. Obviously, (8) is an affine bundle morphism over $\varphi$ with the derived linear morphism $T\varphi \otimes \text{id}_{T_x^*M}$. Similarly to §1, we denote by $\tilde{j}_x^s$ the associated map $J^1\tilde{s}: J^1_xE_1 \to J^1_xE_2$. Analogously to Lemma 1, one can prove

**Lemma 2.** Let $S: J^1_xE_1 \to J^1_xE_2$ be an affine bundle morphism over $\varphi: E_{1x} \to E_{2x}$ with the derived linear morphism $T\varphi \otimes \text{id}_{T_x^*M}$. Then there exists a local section $s$ of $\mathcal{F}(E_1, E_2)$ such that $s(x) = \varphi$ and $j^1_x s = S$.

By (8), every $X = j^1_x|_{E_x}T^xM$ and every $S = j^1_x s$ define a vector

$$S(X) = \left. \frac{\partial}{\partial t} \right|_0 (s \circ f) \in T_{s(x)} \mathcal{F}(E_1, E_2)$$

such that $T \varphi(S(X)) = X$.

**Definition 7.** A connection in the jet form on $\mathcal{F}(E_1, E_2)$ is a smooth section $\Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$ of the target jet projection.

**Proposition 2.** The map (9) establishes a bijection between the jet form and the tangent form of connections on $\mathcal{F}(E_1, E_2)$.

**Proof.** Using (8) we find directly that (9) defines a bijection between the linear splittings $T_xM \to T_x \mathcal{F}(E_1, E_2)$ of $T \varphi$ and the elements of $J^1 \mathcal{F}(E_1, E_2)$. Assume the jet form of $\Gamma$ is smooth and $f: Q \to \mathcal{F}(E_1, E_2)$ is a smooth map, so that $\Gamma \circ f: Q \to J^1 \mathcal{F}(E_1, E_2)$ is smooth. For every smooth vector field $X: M \to T^xM$, the map $(\Gamma \circ f)(X \circ p \circ f)$ is also smooth, so that the tangent form of $\Gamma$ is smooth. Conversely, take a local basis $X_1, \ldots, X_m$ of vector fields on $T^xM$. Then $(\Gamma X_1) \circ f, \ldots, (\Gamma X_m) \circ f$ are smooth maps $Q \to T^x \mathcal{F}(E_1, E_2)$. By (8) we deduce that $\Gamma \circ f: Q \to J^1 \mathcal{F}(E_1, E_2)$ is smooth. 

To define the curvature of a connection of $\mathcal{F}(E_1, E_2)$ in §5, we shall use the second semiholonomic prolongation of $\mathcal{F}(E_1, E_2)$. We recall that $J^1(J^1E_1 \to M) := J^2E_1$ is the classical second nonholonomic prolongation of $E_1 \to M$. If $x^i, y^p, y^p_i$ are the above local coordinates of $J^1E_1$, then the induced coordinates on $J^2E_1$ are $y^p_i = \frac{\partial y^p}{\partial x^i}$ and $y^p_{ij} = \frac{\partial y^p}{\partial x^i}$. We have the target jet projection $\beta_1: J^2E_1 \to J^1E_1$ and the induced map $J^1\beta: J^2E_1 \to J^1E_1$ of the target jet projection $\beta: J^1E_1 \to E_1$. An element $Y \in J^2E_1$ is said to be semiholonomic if $\beta_1(Y) = J^1\beta(Y)$. In coordinates this is characterized by $y^p_i = y^p_{0i}$. All semiholonomic elements form a subbundle $\bar{J}^2E_1 \subset J^2E_1$, and the second holonomic prolongation $J^2E$ is a subbundle of $J^2E$.

Since we have interpreted $J^1 \mathcal{F}(E_1, E_2)$ as a subset of $\mathcal{F}(J^1E_1, J^1E_2)$, we have defined $j^1_x \sigma$ for a local smooth section $\sigma$ of $J^1 \mathcal{F}(E_1, E_2) \to M$ by $j^1_x \sigma$. In this way we
introduce the second nonholonomic prolongation $\tilde{J}^2 \mathcal{F}(E_1, E_2)$ of $\mathcal{F}(E_1, E_2)$. An element $j^1_x \sigma$ is said to be semiholonomic, if $\sigma(x) = j^1_x (\beta \circ \sigma)$, where $\beta: J^1 \mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2)$ is the target jet projection. This defines $\tilde{J}^2 \mathcal{F}(E_1, E_2) \subset \tilde{J}^2 \mathcal{F}(E_1, E_2)$. The inclusion $J^2 \mathcal{F}(E_1, E_2) \subset \tilde{J}^2 \mathcal{F}(E_1, E_2)$ is given by $j^2_x \sigma \mapsto j^1_x (j^1 \sigma)$. Analogously to the first order case, $j^1_x \sigma$ determines a map $\tilde{j}^1_x \sigma: \tilde{J}^2 E_1 \to \tilde{J}^2 E_2$. In coordinates, if $\sigma = (f^a(x, y), f^i(x, y))$, then $s$ is of the form

$$z^a = f^a(x, y), \quad z^a_i = \frac{\partial f^a(x, y)}{\partial y^i} y^i + f^i(x, y).$$

Hence

$$\varphi^a(y) = f^a(0, y), \quad \varphi^a_i = f^a_i(0, y), \quad \varphi^a_0 = \frac{\partial f^a(0, y)}{\partial x^0}, \quad \varphi^a_{ij} = \frac{\partial f^a_i(0, y)}{\partial x^j}$$

are the coordinates of $j^1_x \sigma$. From (10) we obtain the coordinate expression of $\tilde{j}^1_x \sigma$ in the form $z^a = \varphi^a(y)$ and

$$z^a_i = \frac{\partial \varphi^a}{\partial y^i} y^i + \varphi^a_i, \quad z^a_0 = \frac{\partial \varphi^a}{\partial y^0} y^0 + \varphi^a_0,$$

$$z^a_{ij} = \varphi^a_{ij} + \frac{\partial \varphi^a_i}{\partial y^0} y^0_j + \frac{\partial \varphi^a_0}{\partial y^i} y^0_j + \frac{\partial^2 \varphi^a}{\partial y^0 \partial y^j} y^0_i y^0_j + \frac{\partial \varphi^a}{\partial y^j} y^j_i.$$

Using (12) we deduce directly the following assertion.

**Proposition 3.** $j^1_x \sigma$ is semiholonomic or holonomic iff $\tilde{j}^1_x \sigma$ maps $\tilde{J}^2 E_1$ into $\tilde{J}^2 E_2$ or $J^2 \mathcal{F}(E_1, E_2)$ is characterized by $\varphi^a_i = \varphi^a_0$ and the additional condition for a holonomic element is $\varphi^a_{ij} = \varphi^a_{ji}$.

We remark that the higher order nonholonomic and semiholonomic prolongations of $\mathcal{F}(E_1, E_2)$ can be defined in a quite similar way.

### 3. The Finite Order Case

Since both vector fields from §1 and the connections from §2 are defined on a functional bundle, they represent a kind of differential operators. We are going to describe the simplest case of finite order operators.

**Definition 8.** A projectable vector field $A: \mathcal{F}(E_1, E_2) \to T \mathcal{F}(E_1, E_2)$ over $A^0: M \to TM$ is of order $r$, if the condition $j^r_y \varphi = j^r_y \psi, \varphi, \psi \in C^\infty(E_{1x}, E_{2x}), y \in E_{1x}$ implies that the restrictions of $A(\varphi)$ and $A(\psi)$ over $y$ coincide, i.e.

$$\widehat{A(\varphi)}|(T_{A^0(y)}E_1)_y = \widehat{A(\psi)}|(T_{A^0(y)}E_1)_y.$$
Let $S(TE_1, TE_2)$ be the set of all affine morphism $(T_x E_1)_y \rightarrow (T_x E_2)_z$, $p_1 y = p_2 z = \pi_M X$, where $\pi_M: TM \rightarrow M$ is the bundle projection. This is a fibered manifold over $E_1 \times E_2 \times TM$. Write

$$\mathcal{F} J^r(E_1, E_2) = \bigcup_{x \in M} J^r(E_{1x}, E_{2x}).$$

This is a classical manifold as well.

A projectable $r$-th order vector field $A: \mathcal{F}(E_1, E_2) \rightarrow T \mathcal{F}(E_1, E_2)$ over $A^0$ defines the associated map $\mathcal{A}: \mathcal{F} J^r(E_1, E_2) \rightarrow S(TE_1, TE_2)$ by

$$\mathcal{A}(j^r_y \varphi) = A(\varphi)|(T_{A^0(x)} E_1)_y.$$  

**Proposition 4.** The associated map of a projectable $r$-th order vector field on $\mathcal{F}(E_1, E_2)$ is a classical $C^\infty$-map.

**Proof.** This follows from the fact that $A$ is smooth in the sense of Definition 3 quite analogously to [6].

The local coordinates on $\mathcal{F} J^r(E_1, E_2)$ induced by $x^i, y^p$ and $z^a$, $1 \leq |\alpha| \leq r$, where $\alpha$ is a multiindex, the range of which is the fiber dimension of $E_1$. Hence the coordinate form of $\mathcal{A}$ is $X^i(x^j)$ and

$$\Phi^a = \Phi^a(x^i, y^p, z^a), \quad 0 \leq |\alpha| \leq r.$$  

The derived linear map of each element of $S(TE_1, TE_2)$ is identified with an element of $\mathcal{F} J^1(E_1, E_2)$. This defines a map $D: S(TE_1, TE_2) \rightarrow \mathcal{F} J^1(E_1, E_2)$ and the following diagram commutes:

\[\begin{array}{ccc}
\mathcal{F} J^1(E_1, E_2) & \xrightarrow{D} & S(TE_1, TE_2) \\
\downarrow \beta & \swarrow & \downarrow \\
\mathcal{F} J^r(E_1, E_2) & \xrightarrow{\mathcal{A}} & S(TE_1, TE_2) \\
E_1 \times E_2 & \xrightarrow{id \times A^0} & E_1 \times E_2 \times TM \\
\end{array}\]

where $\beta$ is the jet projection. Conversely, let $\mathcal{A}: \mathcal{F} J^r(E_1, E_2) \rightarrow S(TE_1, TE_2)$ be a smooth map with an underlying vector field $A^0: M \rightarrow TM$ such that (16) commutes. Then the rule

$$A(\varphi) = \bigcup_{y \in E_1 \times E_2} \mathcal{A}(j^r_y \varphi)$$  

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defines a projectable $r$-th order vector field $A$ on $\mathcal{F}(E_1, E_2)$.

Since $T\mathcal{F}(E_1, E_2)$ is a subset of $\mathcal{F}(TE_1, TE_2)$, we can define the second tangent bundle $T(T\mathcal{F}(E_1, E_2))$. This will be described in more detail in §6. Here we restrict ourselves to a general remark, which is related to our study of the order of connections.

**Definition 9.** A vector field $A: \mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2)$ is called differentiable if the formula

\[ TA\left( \frac{\partial}{\partial t}\big|_0 f \right) = \frac{\partial}{\partial t}\big|_0 A \circ f \]

defines a smooth map $TA: T\mathcal{F}(E_1, E_2) \to TT\mathcal{F}(E_1, E_2)$.

From (16) we easily deduce (see the coordinate formula in §6) the following assertion.

**Proposition 5.** Every $r$-th order vector field on $\mathcal{F}(E_1, E_2)$ is differentiable.

**Definition 10.** A connection $\Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$ is of order $r$ if the condition $j_y^r \varphi = j_y^r \psi$, $\varphi, \psi \in C^\infty(E_{1x}, E_{2x})$, $y \in E_{1x}$, implies

\[ \overline{\Gamma(\varphi)}|J^1_y E_1 = \overline{\Gamma(\psi)}|J^1_y E_1. \]

Let $S(J^1E_1, J^1E_2)$ be the set of all affine maps $(J^1E_1)_y \to (J^1E_2)_z$ with the derived linear map of the form

\[ B \otimes \text{id}_{T^*_y M} \quad B \in \mathcal{L} \in (V_y E_1, V_z E_2). \]

An $r$-th order connection $\Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$ defines the associated map $\mathcal{G}: \mathcal{F} J^r (E_1, E_2) \to S(J^1E_1, J^1E_2)$ by

\[ \mathcal{G}(j_y^r \varphi) = \overline{\Gamma(\varphi)}|J^1_y E_1. \]

The coordinate form of $\mathcal{G}$ is

\[ \Phi_i^a = \Phi_i^a(x^i, y^p, z^a_\alpha), \quad 0 \leq |\alpha| \leq r. \]

Analogously to Proposition 4, one proves

**Proposition 6.** The associated map of an $r$-th order connection $\mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$ is a classical $C^\infty$-map.
Let $D: S(J^1E_1, J^1E_2) \to J^1\mathcal{F}(E_1, E_2)$ be the map defined by (20). Then the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{F}J^1(E_1, E_2) & \xrightarrow{D} & S(J^1E_1, J^1E_2) \\
\beta & & \\
\mathcal{F}J^r(E_1, E_2) & \xrightarrow{\mathcal{G}} & S(J^1E_1, J^1E_2)
\end{array}
$$

(23)

Conversely, let $\mathcal{G}: \mathcal{F}J^r(E_1, E_2) \to S(J^1E_1, J^1E_2)$ be a smooth morphism over the identity of $E_1 \times E_2$ such that (23) commutes. Then the rule

$$\Gamma(\varphi) = \bigcup_{y \in E_1} \mathcal{G}(j^r_y \varphi)$$

(24)

defines an $r$-th order connection on $\mathcal{F}(E_1, E_2)$.

Analogously to Definition 9, we introduce

**Definition 11.** A connection $\Gamma: \mathcal{F}(E_1, E_2) \to J^1\mathcal{F}(E_1, E_2)$ is called differentiable if the formula

$$J^1\Gamma(j^1_x s) = j^1_x (\Gamma \circ s)$$

(25)

defines a smooth map $J^1\mathcal{F}(E_1, E_2) \to \tilde{J}^2\mathcal{F}(E_1, E_2)$.

**Proposition 7.** Every $r$-th order connection is differentiable.

**Proof.** We deduce from (22) the coordinate form of $J^1\Gamma$ in some coordinates $x^i, \varphi^a, \psi^a_i$ on $J^1\mathcal{F}(E_1, E_2)$ and $x^i, \varphi^a, \varphi^a_i, \varphi^a_{0i}, \varphi^a_{ij}$ on $\tilde{J}^2\mathcal{F}(E_1, E_2)$. Take a section $\sigma$

$$z^a = \Psi^a(x^i, y^p)$$

(26)

so that $\varphi^a = \psi^a(0, y)$ and $\psi^a_i = \frac{\partial \varphi^a(0, y)}{\partial x^i}$. Then we obtain for $\Gamma \circ \sigma$

$$z^a_i = \frac{\partial \varphi^a(x, y)}{\partial y^p} y^p_i + \Phi^a_i(x, y, \partial_\alpha \psi^a(x, y)).$$

(27)

Now (26) yields

$$z^a_{0i} = \frac{\partial \psi^a(0, y)}{\partial y^p} y^p_{0i} + \frac{\partial \psi^a(0, y)}{\partial x^i}, \text{ i.e. } \varphi^a_{0i} = \psi^a_i$$

(28)

and (27) implies

$$z^a_{ij} = \frac{\partial \psi^a_j}{\partial y^p} y^p_i + \frac{\partial^2 \varphi^a}{\partial y^p \partial y^q} y^p_i y^q_j + \frac{\partial \varphi^a}{\partial y^p} y^p_{ij} + \frac{\partial \varphi^a}{\partial y^p} y^p_{0j}$$

$$+ \frac{\partial \Phi^a_i}{\partial x^j} + \frac{\partial \Phi^a_j}{\partial z^b} \partial_j \psi^b + \ldots + \frac{\partial \Phi^a_{ij}}{\partial z^\alpha} \partial_\alpha \partial_j \psi^b.$$ 

(29)

In particular, (29) shows that $J^1\Gamma$ is well-defined and smooth. \qed

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Following Virsik, [17], if \( \Gamma \) is differentiable and \( \Delta \) is another connection \( \mathcal{F}(E_1, E_2) \rightarrow J^1 \mathcal{F}(E_1, E_2) \), we define a section

\[
(30) \quad \Gamma \ast \Delta = J^1 \Gamma \circ \Delta : \mathcal{F}(E_1, E_2) \rightarrow J^2 \mathcal{F}(E_1, E_2).
\]

The order of such a section can be introduced similarly to Definition 10.

**Proposition 8.** If \( \Gamma \) and \( \Delta \) are connections of orders \( r \) and \( s \), respectively, then \( \Gamma \ast \Delta \) has the order \( r + s \).

**Proof.** We substitute the associated map of \( \Delta \) into (28) and (29). \( \square \)

To obtain an explicit formula for the associated map of \( \Gamma \ast \Delta \), we introduce the following concept. Having a smooth function \( f : \mathcal{F} J^r(E_1, E_2) \rightarrow \mathbb{R} \), we define its formal differential \( Df \) by

\[
(31) \quad Df : \mathcal{F} J^{r+1}(E_1, E_2) \rightarrow V^* E_1, \quad Df(j^{r+1}_y \varphi) = d_y f(j^r \varphi).
\]

Then every vertical vector field \( \mu \) on \( V^* E_1 \) determines \( (Df, \mu) : \mathcal{F} J^{r+1}(E_1, E_2) \rightarrow \mathbb{R} \). For the coordinate vector fields \( \frac{\partial}{\partial y^p} \) we obtain the formal derivatives

\[
(32) \quad D_p f = \frac{\partial f}{\partial y^p} + \frac{\partial f}{\partial z^a} z^a_p + \ldots + \frac{\partial f}{\partial z^a} z^a_{a+p}.
\]

By iteration, we introduce \( D_\beta f : \mathcal{F} J^{r+|\beta|}(E_1, E_2) \rightarrow \mathbb{R} \). Let \( \Psi_i^a(x^i, y^p, z^a_\beta) \), \( 0 \leq |\beta| \leq s \), be associated map of \( \Delta \). Then the coordinate form of the main term of (29) is

\[
(33) \quad \varphi_{ij}^a = \frac{\partial \Phi_i^a}{\partial x^j} + \frac{\partial \Phi_i^a}{\partial z^b} \Psi_j^b + \frac{\partial \Phi_i^a}{\partial z^b_\beta} \Psi_j^\beta + \ldots + \frac{\partial \Phi_i^a}{\partial z^b_\alpha} \Psi_j^\alpha.
\]

**Remark 3.** In both cases of connections in the jet form and of projectable vector fields we have a situation somewhat similar to the vertical prolongation operators on classical fibered manifolds studied by Kosmann-Schwarzbach, [11], and the second author, [8]. In [10] Slovák deduced that every vertical prolongation operator is differentiable in the sense of our Definitions 9 and 11. However, his proof is based on quite sophisticated procedures in mathematical analysis, so that we have the feeling that such a problem in our setting is beyond the scope of the present paper.
We describe some properties of connections on a classical fibered manifold \( p: E \to M \) in a way which can be generalized to \( \mathcal{F}(E_1, E_2) \). Given \( A \in J^1_y E \) and \( B \in T_x M, x = py \), we denote by \( A(B) \in T_y E \) the \( A \)-lift of \( B \). We show that every \( A \in \bar{J}^2_y E \) induces similarly a lifting \( \lambda A: TT_x M \to TT_y E \). If \( A = J^1_y \sigma \) and \( B = \frac{\partial}{\partial t} |_0 f(t) \in TT_x M \), then we construct \( \sigma(\pi(f(t)))(f(t)): \mathbb{R} \to TE \) and set

\[
\lambda A(B) = \frac{\partial}{\partial t} |_0 \sigma(\pi(f(t)))(f(t))
\]

where \( \pi: TM \to M \) is the bundle projection. Given some local fiber coordinates \( x^i, y^p \) on \( E \), we have the induced coordinates \( y^p_i, y^p_{0i}, y^p_{ij} \) on \( \bar{J}^2 E \), the induced coordinates \( X^i, Y^p \) on \( TE \) and the additional coordinates on \( TEE \) denoted by a dot. Then one finds easily the following coordinate form of (34):

\[
Y^p = y^p_{ij} X^i, \quad \dot{y}^p = y^p_{0i} \dot{x}^i, \quad \dot{Y}^p = y^p_{ij} X^i \dot{x}^j + y^p_i \dot{X}^i.
\]

Let \( \kappa \) be the canonical involution of the second tangent bundle. If \( A \in \bar{J}^2_y E \), then \( \kappa_E \circ \lambda A \circ \kappa_M: TT_x M \to TT_y E \) is the lifting of another element \( \kappa A \in \bar{J}^2_y E \), [15]. In coordinates, \( y^p_{ij}(\kappa A) = y^p_{ji}(A) \). Hence \( A \) is holonomic iff \( \kappa A = A \). Since \( \bar{J}^2_y E \to J^1_y E \) is an affine bundle with the derived vector bundle \( V E \otimes \wedge^2 T^*M \), the points \( \kappa A \) and \( A \) determine a vector \( \Delta(A) := (\kappa A) \widehat{A} \in V_y E \otimes \wedge^2 T_x^* M \), which is called the deviation (or difference tensor) of \( A \), [7], [12]. The coordinates of \( \Delta(A) \) are \( y^p_{ij} - y^p_{ji} \).

Let \( \pi_1 = \pi_{TM} = TT M \to TM \) and \( \pi_2 = T \pi_{TM} = TT M \to TM \) be the canonical projections. Consider \( C, D \in TT_x M \) satisfying

\[
\pi_1(C) = \pi_2(D) \quad \text{and} \quad \pi_1(D) = \pi_2(C).
\]

Since \( \kappa \) exchanges the two projections, \( C \) and \( \kappa D \) are in the same fiber of \( TT M \) with respect to \( \pi_1 \) and satisfy \( \pi_2(C - \kappa D) = 0 \). Hence \( C - \kappa D \) is a tangent vector to a fiber of \( TM \) and such a vector can be identified with an element of \( T_x M \), which will be denoted by \( C \vdash D \) and called the strong difference of \( C \) and \( D \). In coordinates, if

\[
C \equiv (a^i, b^i, c^i), D \equiv (b^i, a^i, d^i) \quad \text{then} \quad C \vdash D \equiv (c^i - d^i).
\]

In [8] it is deduced the the bracket \([X, Y]\) of two vector fields \( X, Y: M \to TM \) can be expressed by

\[
[X, Y] = TY \circ X \vdash TX \circ Y.
\]
Lemma 3. Let \( C, D \in T^2 T_x M \) satisfy the condition (36) for the strong difference and \( A \in J^2 E \). Then \( \lambda A(C), \lambda A(D) \) also satisfy (36) and

\[
\Delta A(\pi_1 C, \pi_2 C) = (\lambda A(C) - \lambda A(D)) - \beta_1(A)(C \mp D)
\]

where \( \beta_1: J^2 E \to J^1 E \) is the jet projection.

Proof. By (35) and (37) we have \( \lambda A(C) = (y_t^p a_i, y_t^p b_i, y_t^p a_i b^i + y_t^p c^i), \lambda A(D) = (y_t^p b^i, y_t^p a^i, y_t^p b^i a^j + y_t^p d^i) \). This implies our claim. \( \square \)

According to Remark 2, two connections \( \Gamma, \Delta: E \to J^1 E \) determine \( \Gamma \ast \Delta = \Delta \circ \Gamma: E \to J^2 E \). For \( \Gamma = \Delta \) the values of \( \Gamma \ast \Gamma \) lie in \( J^2_y E \). In this case we obtain a construction closely related to an idea by Ehresmann, [2].

Definition 12. The map \( \tilde{\Gamma} = J^1 \Gamma \circ \Gamma: E \to J^2 E \) is the Ehresmann prolongation of \( \Gamma \). The composition

\[
C \Gamma := -\Delta \circ \tilde{\Gamma}: E \to VE \otimes \Lambda^2 T^* M
\]

is the curvature of \( \Gamma \).

To deduce that \( C \Gamma \) coincides with the standard curvature of \( \Gamma \), we need a property of the lifting map

\[
\lambda \tilde{\Gamma}: E \times TTM \to TTE.
\]

Consider two vector fields \( X, Y: M \to TM \), so that \( TX \circ Y: M \to TTM \).

Lemma 4. We have

\[
\lambda \tilde{\Gamma}(TX \circ Y) = (\Gamma X) \circ \Gamma Y: E \to TTE.
\]

Proof. We have \( \tilde{\Gamma}(y) = j_x^1(\Gamma \circ s), j_x^1 s = \Gamma(y) \). If \( Y(x) = \frac{\partial}{\partial t} \bigg|_0 f(t) \), then

\[
TX(Y(x)) = \frac{\partial}{\partial t} \bigg|_0 (X \circ f).
\]

By (34),

\[
\lambda \tilde{\Gamma}(TX(Y(x))) = \frac{\partial}{\partial t} \bigg|_0 \Gamma(s(f(t)))(X(f(t))) = (\Gamma X \circ \Gamma Y)(y).
\]

\( \square \)
Proposition 9. For every vector fields $X, Y: M \to TM$, we have

$$C\Gamma(X, Y) = [\Gamma X, \Gamma Y] - \Gamma([X, Y]).$$

Proof. Consider $TX \circ Y, TY \circ X: M \to TTM$. By Lemma 4 we obtain

$$\lambda \tilde{\Gamma}(TX \circ Y) = T \Gamma X \circ \Gamma Y \quad \text{and} \quad \lambda \tilde{\Gamma}(TY \circ X) = TTY \circ \Gamma X.$$

Then Lemma 3 and (38) imply

$$\Delta \circ \tilde{\Gamma}(X, Y) = (\lambda \tilde{\Gamma}(TX \circ Y) - \lambda \tilde{\Gamma}(TY \circ X)) - \Gamma(TX \circ Y - TY \circ X) =$$

$$= -[\Gamma X, \Gamma Y] + \Gamma([X, Y]).$$

$$\square$$

5. The Curvature of a Connection on $\mathcal{F}(E_1, E_2)$

The deviation of an element $j^1_x \sigma \in \tilde{J}^2 \mathcal{F}(E_1, E_2)$ can be defined by means of the associated map $j^1_x \sigma: \tilde{J}^2_x E_1 \to \tilde{J}^2_x E_2$. In the semiholonomic case we have $\varphi_i^a = \varphi_{0i}^a$. So if we take a holonomic 2-jet $Y \in J^2_x E_1$, then the right-hand side of the second line in (12) is symmetric except the first term. Hence the deviation $\Delta(j^1_x \sigma(Y))$ is independent of $y_i^p$ and $y_j^p$. This defines a map $\Delta(j^1_x \sigma): E_{1x} \to V_x E_2 \otimes \Lambda^2 T^* M$ over $\phi$, i.e. an element of $\mathcal{F}(E_1, VE_2 \otimes \Lambda^2 T^* M)$.

Definition 13. $\Delta(j^1_x \sigma)$ is called the deviation of $j^1_x \sigma$. The coordinate form of $\Delta(j^1_x \sigma)$ is $\varphi_{ij}^a - \varphi_{ji}^a$.

Definition 14. For a differentiable connection $\Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$, the map $\tilde{\Gamma} := J^1 \Gamma \circ \Gamma: \mathcal{F}(E_1, E_2) \to \tilde{J}^2 \mathcal{F}(E_1, E_2)$ is the Ehresmann prolongation of $\Gamma$.

Definition 15. The composition

$$C \Gamma := -\Delta \circ \tilde{\Gamma}: \mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, VE_2 \otimes \Lambda^2 T^* M)$$

is the curvature of a differentiable connection $\Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$.

Clearly, $C \Gamma$ is a section of the canonical projection $\mathcal{F}(E_1, VE_2 \otimes \Lambda^2 T^* M) \to \mathcal{F}(E_1, E_2)$.

Let $\Gamma$ be an r-th order connection with the associated map $\Phi_i^a(x^i, y^j, z^a)$. Then we obtain the associated map of $C \Gamma$ by setting $\Psi_i^a = \Phi_i^a$ in (33) and by antisymmetrizing in $i$ and $j$. This implies

Proposition 10. The curvature of an r-th order connection has the order $2r$. 

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6. THE BRACKET FORMULA FOR CURVATURE

As remarked in §3, the inclusion $T\mathcal{F}(E_1, E_2) \subset \mathcal{F}(TE_1 \to TM, TE_2 \to TM)$ defines the second tangent bundle $T(T\mathcal{F}(E_1, E_2)) = TT\mathcal{F}(E_1, E_2)$. We have a projection $TTp: TT\mathcal{F}(E_1, E_2) \to TTM$ and two projections $\pi_T, T\pi: TT\mathcal{F}(E_1, E_2) \to T\mathcal{F}(E_1, E_2)$. In the above coordinates, consider an element $F \in TT\mathcal{F}(E_1, E_2)$ tangent to a curve $x^j(t), X^i(t), f^a(y, t)$ and

$$Z^a = \frac{\partial f^a(y, t)}{\partial y^p} Y^p + \Phi^a(y, t).$$

Then its associated map $\tilde{F}: TT_x E_1 \to TT_x E_2, X = TTp(F)$, is of the form

$$Z^a = \frac{\partial \varphi^a}{\partial y^p} Y^p + \Phi^a(y), \tilde{z}^a = \frac{\partial \varphi^a}{\partial y^p} Y^p + f^a(y).$$

$$\dot{Z}^a = F^a(y) + \frac{\partial \Phi^a}{\partial y^p} \dot{Y}^p + \frac{\partial f^a}{\partial y^p} \dot{Y}^p + \frac{\partial^2 \varphi^a}{\partial y^p \partial y^q} Y^p \dot{y}^q + \frac{\partial \varphi^a}{\partial y^p} \dot{Y}^p.$$

So $\varphi^a, \Phi^a, f^a, F^a$ are the functional coordinates of $F$, which are completed by the coordinates $x^i, X^i, \dot{x}^i, \dot{X}^i$ of $X \in TTM$. The coordinate form of $\pi_T$ or $T\pi$ is

$$\pi_T(x^i, X^i, \dot{x}^i, \dot{X}^i, \varphi^a, \Phi^a, f^a, F^a) = (x^i, X^i, \varphi^a, \Phi^a),$$

$$T\pi(x^i, X^i, \dot{x}^i, \dot{X}^i, \varphi^a, \Phi^a, f^a, F^a) = (x^i, \dot{x}^i, \varphi^a, f^a).$$

Consider the canonical involution $\kappa_{E_1}$ or $\kappa_{E_1}$ of the second tangent bundle.

**Proposition 11.** For every $F \in TT\mathcal{F}(E_1, E_2)$ over $X \in TTM$ there exists a unique element $\kappa F \in TT\mathcal{F}(E_1, E_2)$ such that its associated map $\kappa \tilde{F}: TT_{\kappa M} X E_1 \to TT_{\kappa M} X E_2$ is $\kappa \tilde{F} = \kappa_{E_2} \circ \tilde{F} \circ \kappa_{E_1}$.

**Proof.** This follows from (40). \qed

Obviously, the coordinate form of $\kappa$ is

$$\kappa(x, X, \dot{x}, \dot{X}, \varphi, \Phi, f, F) = (x, \dot{x}, X, \dot{X}, \varphi, f, \Phi, F).$$

Consider $C, \bar{C} \in TT\mathcal{F}(E_1, E_2)$ over $X, \bar{X} \in TTM$ satisfying

$$\pi_T(C) = T\pi(\bar{C}) \quad \text{and} \quad \pi_T(\bar{C}) = T\pi(C).$$

Then we define the strong difference $C \rhd \bar{C} \in T\mathcal{F}(E_1, E_2), Tp(C \rhd \bar{C}) = X \rhd \bar{X}$, as follows. For every $B \in (T_{X \rhd \bar{X}} E_1)_y$ we take any $Y, \bar{Y} \in (TT E_1)_y$ over $X, \bar{X}$
such that \( Y \sim \bar{Y} = B \). Then one easily verifies that \( C(Y), \bar{C}(\bar{Y}) \) also satisfy (42), \( C(Y) \sim \bar{C}(\bar{Y}) \) depends on \( C, \bar{C} \) and \( B \) only and represents the associated map of an element \( C \sim \bar{C} \in T \mathcal{F}(E_1, E_2) \), whose coordinates are

\[
(x^i, \dot{X}^i - \ddot{X}^i, \varphi^a, F^a - \bar{F}^a).
\]

Let \( A, B \) be two differentiable vector fields on \( \mathcal{F}(E_1, E_2) \). Then the maps \( TA \circ B, TB \circ A : \mathcal{F}(E_1, E_2) \to TT \mathcal{F}(E_1, E_2) \) satisfy the condition (42) at every \( \varphi \in \mathcal{F}(E_1, E_2) \).

**Definition 16.** The vector field

\[
[A, B] := TB \circ A - TA \circ B : \mathcal{F}(E_1, E_2) \to T \mathcal{F}(E_1, E_2)
\]

is called the bracket of \( A \) and \( B \).

By (38) we immediately deduce

**Proposition 12.** If \( A \) and \( B \) are projectable over \( A^0 \) and \( B^0 \), then \([A, B]\) is projectable over \([A^0, B^0]\).

Assume \( A \) is of order \( r \) and \( B \) is of order \( s \) with the associated maps \( X^i(x), A^a(x^i, y^p, z^a_\alpha), |\alpha| \leq r \) and \( Y^i(x), B^a(x^i, y^p, z^a_\beta), |\beta| \leq s \), respectively. Analogously to §3, the fourth component of the associated map of \( TA \circ B \) is

\[
\frac{\partial A^a}{\partial x^i} Y^i + \frac{\partial A^a}{\partial z^b} B^b + \frac{\partial A^a}{\partial z^p} D_p B^b + \ldots + \frac{\partial A^a}{\partial z^b} D_b B^b, \quad |\alpha| \leq r
\]

while the fourth component of the associated map of \( TB \circ A \) is

\[
\frac{\partial B^a}{\partial x^i} X^i + \frac{\partial B^a}{\partial z^b} A^b + \frac{\partial B^a}{\partial z^p} D_p A^b + \ldots + \frac{\partial B^a}{\partial z^b} D_b A^b, \quad |\beta| \leq s.
\]

Hence we can summarize by

**Proposition 13.** The bracket \([A, B]\) has the order \( r + s \) and its associated map is \([A^0, B^0]\) and the difference (45)–(44).

We are going to generalize Proposition 9 to connections on \( \mathcal{F}(E_1, E_2) \). First of all we remark that every \( A = j_x^1 \sigma \in J^2 \mathcal{F}(E_1, E_2) \) defines a lifting \( \lambda A : TT_x M \to TT_x \mathcal{F}(E_1, E_2) \) by

\[
\lambda A \left( \frac{\partial}{\partial t} \bigg|_0 f \right) = \frac{\partial}{\partial t} \bigg|_0 \sigma(M f(t))(f(t)).
\]
In coordinates, if $A = (x^i, \varphi^a_i, \varphi_i^a, \varphi_i^{a})$ and $B = \frac{\partial}{\partial t} |_0 f = (x^i, \dot{x}^i, \dot{X}^i)$, then one easily finds the following coordinate form of $\lambda A(B)$:

\begin{equation}
(x^i, \varphi^a_i, \varphi_i^a x^i, \varphi_i^{a} \dot{x}^i, \varphi_i^{a} X^i \dot{x}^j + \varphi_i^{a} \dot{X}^i).
\end{equation}

This directly implies the following generalization of Lemma 3.

**Lemma 5.** Let $C, D \in TT_x M$ satisfy the condition (36) for the strong difference and $A \in J^2 \mathcal{F}(E_1, E_2)$. Then $\lambda A(C), \lambda A(D)$ satisfy (42) and

$$
\Delta A(\pi_T C, T \pi C) = (\lambda A(C) - \lambda A(D)) - \beta_1(A)(C \div D).
$$

Now we need an assumption of technical character (which is fulfilled for every finite order connection).

**Definition 17.** A differentiable connection $\Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$ is called strongly differentiable, if $\Gamma X$ is a differentiable vector field on $\mathcal{F}(E_1, E_2)$ for every smooth vector field $X: M \to TM$.

**Proposition 14.** For every strongly differentiable connection $\Gamma$ on $\mathcal{F}(E_1, E_2)$ and for all vector fields $X, Y$ on $M$ we have

$$
C \Gamma(X, Y) = [\Gamma X, \Gamma Y] - \Gamma([X, Y]).
$$

**Proof.** In the same way as in Lemma 4 we deduce $\lambda \tilde{\Gamma}(TX \circ Y) = (T \tilde{\Gamma} X) \circ \Gamma Y$. Then we apply Lemma 5. \hfill \Box

7. **The absolute differentiation**

Let $A, B \in J^1 \mathcal{F}(E_1, E_2)_\varphi$ be two 1-jets with the same target $\varphi$. To deduce that their difference is an element $A - B \in \mathcal{F}(E_1, VE_2 \otimes T^*M)$ over $\varphi$, we consider the associated maps $\tilde{A}, \tilde{B}: J^1_x E_1 \to J^1_x E_2$,

\begin{align*}
A \equiv z^a &= \frac{\partial \varphi^a}{\partial y^p} y^p + \Phi^a_i(y), \\
B \equiv z^a &= \frac{\partial \varphi^a(x, y)}{\partial y^p} y^p + \Psi^a_i(y).
\end{align*}

The element $A(Y) - B(Y)$ is independent of the choice of $Y \in J^1_x E_1$, which defines a map $E_{1x} \to VE_2 \otimes T^*M$ over $\varphi$. (In this sense $J^1 \mathcal{F}(E_1, E_2)$ is an affine bundle with the derived vector bundle $\mathcal{F}(E_1, VE_2 \otimes T^*M)$ analogously to the classical case.)

Let $s: M \to \mathcal{F}(E_1, E_2)$ be a section and $\Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$ a connection.
Definition 18. The absolute differential

\[ \nabla s: M \to \mathcal{F}(E_1, V E_2 \otimes T^*M) \]

is the above difference \( \nabla s(x) = j^1_x s - \Gamma(s(x)) \).

If \( X: M \to TM \) is a vector field, we define the absolute derivative of \( s \) with respect to \( X \) by

(47) \[ \nabla_X s = \langle \nabla s, X \rangle : M \to \mathcal{F}(E_1, V E_2) \]

where \( \langle \, , \, \rangle \) is the extension of the evaluation map \( T \times T^* \to \mathbb{R} \). Having an \( r \)-th order connection with the associated map (22) and a section \( s \) of the form \( z^a = \varphi^a(x, y) \), then the coordinate form of \( \nabla s \) is

(48) \[ \frac{\partial \varphi^a(x, y)}{\partial x^i} - \Phi_i^a(x^i, y^p, \partial \varphi^a(x, y)). \]

To obtain \( \nabla_x s \), we contract (48) with the coordinate functions \( X^i(x) \) of \( X \).

Remark 4. In the case \( E_1 = E_2 := E \) we have a distinguished section \( I: M \to \mathcal{F}(E, E) \), \( I(x) = \text{id}_E \). Analogously to the case of a classical linear connection on \( TM \), the absolute differential \( \nabla I: M \to \mathcal{F}(E, VE \otimes T^*M) \) can be called the torsion of a connection \( \Gamma \) on \( \mathcal{F}(E, E) \). By (48), the coordinate form of the torsion of an \( r \)-th order connection is \( -\Phi_i^p(x^i, y^p, \delta^p_q, 0, \ldots, 0) \).

It might be instructive to discuss a special case in more detail. Let \( E \to M \) be a vector bundle. Consider the subspace \( LE \subset \mathcal{F}(E, E) \) of all linear maps, which is a classical vector bundle over \( M \). A connection \( \Gamma \) on \( LE \) in our sense is a classical general connection on \( LE \). Hence our approach leads to the original idea of the torsion of a general connection \( \Gamma \) on \( LE \). If \( w^p_q \) are the induced fiber coordinates on \( LE \), the usual coordinate expression of \( \Gamma \) is \( dw^p_q = F^{ij}_{qk}(x^j, w^p_k) \, dx^i \). Then \( -F^i_p(x^j, \delta^r_s) \) is the coordinate form of the torsion of \( \Gamma \). Of course, if we take for \( \Gamma \) the tensor product \( \Delta \otimes \Delta^* \) of a linear connection \( \Delta \) on \( E \) and of the dual connection \( \Delta^* \) on \( E^* \), [10], then the torsion of \( \Delta \otimes \Delta^* \) vanishes, for \( I \) is invariant with respect to \( \Delta \otimes \Delta^* \).
Assume \( p : E_2 \to M \) is a vector bundle. Then each fiber of \( \mathcal{F}(E_1, E_2) \) is a vector space, provided the linear operations on \( C^\infty(E_{1x}, E_{2x}) \) are defined by extending the linear operations on \( E_{2x} \). In other words, \( \mathcal{F}(E_1, E_2) \to M \) is a vector bundle over sets, cf. [4]. Such a vector bundle structure is further extended to \( J^1 \mathcal{F}(E_1, E_2) \) by

\[
\tilde{j}^1_x s_1 + \tilde{j}^1_x s_2 = \tilde{j}^1_x (\tilde{s}_1 + \tilde{s}_2), \quad \tilde{j}^1_x (ks) = \tilde{j}^1_x k \tilde{s}, \quad k \in \mathbb{R}
\]

with addition and multiplication by reals in \( E_2 \). Hence \( J^1 \mathcal{F}(E_1, E_2) \to M \) also is a vector bundle over sets.

**Definition 19.** A connection \( \Gamma : \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2) \) is called linear if \( \Gamma \) is a linear morphism over \( M \).

In the case of an \( r \)-th order linear connection, its associated map (22) has the form

\[
(49) \quad \Phi_{ib}^a (x, y) z^b + \Phi_{ib}^q(x, y) z^b + \ldots + \Phi_{ib}^a z^b.
\]

If \( E_2 \) is a vector bundle, then \( VE_2 = E_2 \times E_2 \), which implies

\[
\mathcal{F}(E_1, VE_2 \otimes \Lambda^2 T^* M) = \mathcal{F}(E_1, E_2) \times \mathcal{F}(E_1, E_2 \otimes \Lambda^2 T^* M).
\]

In this case, analogously to the classical situation, the curvature will be interpreted as the second component of the map from Definition 15,

\[
C \Gamma : \mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2 \otimes \Lambda^2 T^* M),
\]

while the first component is the identity.

**Proposition 15.** For every differentiable linear connection \( \Gamma \), the map \( C \Gamma : \mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2 \otimes \Lambda^2 T^* M) \) is a linear morphism over \( M \).

**Proof.** One easily verifies that in the linear case both \( \tilde{\Gamma} \) and \( \Delta \) in Definition 15 are linear morphisms over \( M \). \( \square \)

Quite similarly, if \( E_2 \) is a vector bundle, then the absolute derivative \( \nabla_X s \) of a section \( s \) with respect to a vector field \( X \) on \( M \) is identified with the second component of (47), so that it is section of \( \mathcal{F}(E_1, E_2) \) as well.

We finally remark that several other ideas from the classical theory of connections can be generalized to the case of \( \mathcal{F}(E_1, E_2) \). The most interesting ones could be the vertical prolongation of \( \Gamma \), the connections on \( T \mathcal{F}(E_1, E_2) \subset \mathcal{F}(TE_1 \to TM, TE_2 \to TM) \) or a detailed study of the absolute differentiation in the linear case. Such a research can be based on some general ideas from the theory of classical connections collected in the book [10].
References


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