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CONNECTIONS ON SOME FUNCTIONAL BUNDLES

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INTRODUCTION

Our starting point was the idea of the Schrödinger connection on a double fibered manifold by Jadczyk and Modugno, [4], [5]. We discuss the “pure case” of two classical fiber bundles $E_1$ and $E_2$ over the same base and define a connection $\Gamma$ on the bundle $\mathcal{F}(E_1, E_2)$ of all smooth maps from a fiber of $E_1$ into the fiber of $E_2$ over the same base point. We study systematically the geometry of the iterated tangent bundle of the infinite dimensional space $\mathcal{F}(E_1, E_2)$ as well as the jet prolongations of $\mathcal{F}(E_1, E_2)$ by means of the ideas introduced by the second author in [9]. Since we deal with functional bundles, our vector fields and connections represent a kind of differential operators. That is why we pay special attention to the case of finite order operators, in which we are able to deduce a very concrete description of the objects and operations in question.

In such a situation we found the simplest way for introducing the curvature of $\Gamma$ in a construction by Ehresmann, [2], which is based on the notion of semiholonomic 2-jets. In the new context we were obliged to rearrange some results, deduced in the finite dimension by direct evaluation, into a more geometrical setting, which could be generalized to our infinite dimensional case. Only then we study the bracket of two vector fields on $\mathcal{F}(E_1, E_2)$. This is a modification of the bracket of two vertical prolongation operators on a classical fibered manifold by Kosmann-Schwarzbach, [11], and the second author, [8]. In Proposition 14 we deduce a satisfactory bracket formula for the curvature of $\Gamma$. We also discuss the absolute differentiation with respect to $\Gamma$ and the special case $E_2$ is a vector bundle.

If we deal with two finite dimensional manifolds and a map between them, we always assume they are of class $C^\infty$, i.e. smooth in the classical sense. On the other

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hand, the idea of smoothness in the infinite dimension is taken from the theory of
smooth structures by Frölicher, [3].

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1. THE TANGENT BUNDLE OF $\mathcal{F}(E_1, E_2)$

Let $p_1: E_1 \to M$ and $p_2: E_2 \to M$ be two classical fiber bundles (i.e. locally trivial
fibered manifolds) over the same base. Consider the set of all fiber maps

$$\mathcal{F}(E_1, E_2) = \bigcup_{x \in M} C^\infty(E_{1x}, E_{2x})$$

and denote by $p: \mathcal{F}(E_1, E_2) \to M$ the canonical projection. We define no topol-
ogy on $\mathcal{F}(E_1, E_2)$, but we introduce the concept of a smooth map from a classical
manifold $Q$ into $\mathcal{F}(E_1, E_2)$.

**Definition 1.** A map $f: Q \to \mathcal{F}(E_1, E_2)$ is called smooth, if
(i) $p \circ f: Q \to M$ is smooth and
(ii) the induced map $\tilde{f}: (p \circ f)^* E_1 \to E_2$,

$$\tilde{f}(q, y) = f(q)(y), \quad (q, y) \in (p \circ f)^* E_1$$

is also smooth.

As usual, $(p \circ f)^* E_1 \to Q$ denotes the bundle induced from $E_1$ by means of $p \circ f$,
i.e.

$$(p \circ f)^* E_1 = \{(q, y) \in Q \times E_1 \mid (p \circ f)(q) = p_1(y)\}.$$

Thus, $\mathcal{F}(E_1, E_2)$ is endowed with a smooth structure in the sense of Frölicher, [3].

For every smooth curve $f: \mathbb{R} \to \mathcal{F}(E_1, E_2)$ we first construct the tangent vector
$X = \frac{\partial}{\partial t}\big|_0 (p \circ f) \in TM$ of its base map at $t = 0$. Write

$$T_x E_1 = (Tp_1)^{-1}(X) \subset TE_1 \quad \text{or} \quad T_x E_2 = (Tp_2)^{-1}(X) \subset TE_2,$$

so that $T_x E_1$ or $T_x E_2$ is an affine bundle over $E_{1x}$ or $E_{2x}$, $x = p(f(0))$, with the
derived vector bundle $T(E_{1x}) := V_x E_1$ or $T(E_{2x}) := V_x E_2$, respectively. Then $f$
defines a map $T_0 f: T_x E_1 \to T_x E_2$ by

$$T_0 f \left( \frac{\partial}{\partial t}\big|_0 h(t) \right) = \frac{\partial}{\partial t}\big|_0 f(t)(h(t))$$

where we may assume that $h: \mathbb{R} \to E_1$ satisfies $p \circ f = p_1 \circ h$.  

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Definition 2. We say that two smooth curves \( f, g : U \to \mathcal{F}(E_1, E_2) \) satisfying \( \frac{\partial}{\partial t} |_0 f = \frac{\partial}{\partial t} |_0 g = X \) determine the same tangent vector at \( f(0) = g(0) = \varphi \), if

\[
T_0f = T_0g : T_X E_1 \to T_X E_2.
\]

The set \( T\mathcal{F}(E_1, E_2) \) of all equivalence classes will be called the tangent bundle of \( \mathcal{F}(E_1, E_2) \).

We write \( \frac{\partial}{\partial t} |_0 f(t) \in T\mathcal{F}(E_1, E_2) \) for the tangent vector determined by \( f \) and \( \pi : T\mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2) \) and \( Tp : T\mathcal{F}(E_1, E_2) \to TM \) for the canonical projections. If \( A \in T\mathcal{F}(E_1, E_2) \), then we denote by \( \tilde{A} : T_{Tp(A)} E_1 \to T_{Tp(A)} E_2 \) the associated map (1).

Remark 1. Let \( \varepsilon \subset \mathcal{F}(E_1, E_2) \) be any subset. Then we define \( T\varepsilon \subset T\mathcal{F}(E_1, E_2) \) by restricting ourselves to the smooth curves with values in \( \varepsilon \).

One sees easily that \( T_0f = T_0g : T_X E_1 \to T_X E_2 \) is an affine bundle morphism over the base map \( \varphi : E_{1x} \to E_{2x} \) with the derived linear morphism \( T\varphi : T(E_{1x}) \to T(E_{2x}) \). Indeed, let \( x^i \) be some local coordinates on \( M \), \( y^p \) or \( z^a \) be some additional coordinates on \( E_1 \) or \( E_2 \) and

\[
x^i = f^i(t), \quad z^a = f^a(y^p, t)
\]

be the coordinate expression of \( f(t) \). Write

\[
Y^p = dy^p, \quad Z^a = dz^a, \quad \varphi^a(y) = f^a(y, 0), \quad \Phi^a(y) = \frac{\partial f^a(y^p, 0)}{\partial t}.
\]

Then the coordinate form of (1) is

\[
Z^a = \frac{\partial \varphi^a(y)}{\partial y^p} Y^p + \Phi^a(y).
\]

Hence the tangent vector to (2) is locally characterized by two systems of numbers and two systems of functions

\[
x^i = f^i(0), \quad X^i = \frac{\partial f^i(0)}{\partial t}, \quad \varphi^a(y^p), \quad \Phi^a(y^p).
\]

The following lemma gives a global assertion of such a type.

Lemma 1. Let \( F : T_X E_1 \to T_X E_2 \) be an affine bundle morphism over \( \varphi : E_{1x} \to E_{2x} \) with the derived linear morphism \( T\varphi : T(E_{1x}) \to T(E_{2x}) \). Then there exists a smooth curve \( f : \mathbb{R} \to \mathcal{F}(E_1, E_2) \) such that \( F = \tilde{A} \) for the tangent vector \( A = \frac{\partial}{\partial t} |_0 f(t) \).
Proof. Consider some local trivializations $U \times S_1$ and $U \times S_2$ of $E_1$ and $E_2$ over a neighborhood $U \subset M$ of $x$. Then $\mathcal{F}(U \times S_1, U \times S_2) = U \times C^\infty(S_1, S_2)$. The restriction of $F$ to $Y^p = 0$ represents a map $F: S_1 \to TS_2$ along $\varphi$. By Proposition 5 from [16] there exists a smooth curve $\gamma: \mathbb{R} \to C^\infty(S_1, S_2)$ such that $F(y) = \frac{\partial \gamma}{\partial t}(y,0)$, where $\delta: \mathbb{R} \times S_1 \to S_2$ is defined by $\gamma(y, t) = \gamma(t)(y)$. If $\delta: \mathbb{R} \to U$ is any curve with $\frac{\partial \delta}{\partial t}(0) = X$, then the curve $(\delta, \gamma): \mathbb{R} \to U \times C^\infty(S_1, S_2)$ has the required property. \hfill \Box

Now we show that each fiber of $T \mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2)$ is a vector space. Consider $\tilde{A}_1: T_{X_1}E_1 \to T_{X_1}E_2$ and $\tilde{A}_2: T_{X_2}E_1 \to T_{X_2}E_2$ over the same $\varphi$. Given $Y \in (T_{X_1}+x_2E_1)_y$, $y \in E_1$, we take any $W \in (T_{X_2}E_1)_y$, so that $Y - W \in (T_{X_2}E_1)_y$, and we define

$$\tilde{A}_1(Y) + \tilde{A}_2(Y) = \tilde{A}_1(W) + \tilde{A}_2(Y - W).$$

If we select another $\tilde{W} \in (T_{X_1}E_1)_y$, then $W - \tilde{W}$ is a vertical vector. Hence

$$\tilde{A}_1(\tilde{W}) = \tilde{A}_1(W) + T\varphi(\tilde{W} - W), \quad \tilde{A}_2(Y - \tilde{W}) = \tilde{A}_2(Y - W) + T\varphi(W - \tilde{W}),$$

so that our definition is correct. Further, for $0 \neq k \in \mathbb{R}$ we define

$$k\tilde{A}: T_{kX}E_1 \to T_{kX}E_2 \quad \text{by} \quad k\tilde{A}(Y) = k\tilde{A}
\left(\frac{1}{k}Y\right)$$

while for $k = 0$ we prescribe $0\tilde{A}$ to be $T\varphi: T_0E_{1x} \to T_0E_{2x}$. In coordinates, if $A_1 = (x^i, X_1^i, \varphi^a, \Phi_1^a)$ and $A_2 = (x^i, X_2^i, \varphi^a, \Phi_2^a)$, then

$$A_1 + A_2 = (x^i, X_1^i + X_2^i, \varphi^a, \Phi_1^a + \Phi_2^a), \quad kA_1 = (x^i, kX_1^i, \varphi^a, k\Phi_1^a).$$

This proves that each $\pi^{-1}(\varphi)$ is a vector space.

In general, consider another pair $E_3 \to N, E_4 \to N$ of fiber bundles over the same base and subset $\varepsilon \subset \mathcal{F}(E_1, E_2)$.

Definition 3. A map $f: \varepsilon \to \mathcal{F}(E_3, E_4)$ is called smooth, if $f \circ g: Q \to \mathcal{F}(E_3, E_4)$ is smooth for every smooth map $g: Q \to \varepsilon$.

Definition 4. A vector field on $\mathcal{F}(E_1, E_2)$ is a smooth map $A: \mathcal{F}(E_1, E_2) \to T\mathcal{F}(E_1, E_2)$ satisfying $\pi \circ A = \text{id}$. We say that $A$ is projectable, if there exists a classical smooth vector field $A^0: M \to TM$ such that $A^0 \circ \pi = Tp \circ A$.

Write $V \mathcal{F}(E_1, E_2)$ for the kernel of $Tp: T\mathcal{F}(E_1, E_2) \to TM$, which will be called the vertical tangent bundle of $\mathcal{F}(E_1, E_2)$. Then we have an exact sequence

$$0 \to V \mathcal{F}(E_1, E_2) \to T\mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2) \times TM \to 0$$

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Consider a linear splitting $\Gamma: \mathcal{F}(E_1, E_2) \times TM \to TM \mathcal{F}(E_1, E_2)$, i.e. $\pi \circ \Gamma = pr_1, Tp \circ \Gamma = pr_2$ and $\Gamma(\varphi, -): T_x M \to T_{\varphi E_2} \mathcal{F}(E_1, E_2)$ is a linear map for each $\varphi \in \mathcal{F}(E_1, E_2), x = \pi(\varphi)$. Then for every vector field $X: M \to TM$ we have defined its $\Gamma$-lift $\Gamma X: \mathcal{F}(E_1, E_2) \to TM \mathcal{F}(E_1, E_2)$. We say that $\Gamma$ is smooth, if $\Gamma X$ is smooth for every classical smooth vector field $X: M \to TM$.

**Definition 5.** A connection (in tangent form) on $\mathcal{F}(E_1, E_2)$ is a smooth linear splitting $\Gamma: \mathcal{F}(E_1, E_2) \times TM \to TM \mathcal{F}(E_1, E_2)$.

**Remark 2.** If $E_1$ is the trivial fibering $M \to M$, then $\mathcal{F}(E_1, E_2) = E_2$ and we obtain the standard connection on $E_2 \to M$.

### 2. Jet prolongations of $\mathcal{F}(E_1, E_2)$

The simplest way how to define the $r$-th jet prolongation of $\mathcal{F}(E_1, E_2)$ is based on the concept of the fiber $r$-jet, [9], [10]. In general, given a fiber bundle $E \to M$ and a manifold $N$, two maps $f, g: E \to N$ are said to determine the same fiber $r$-jet $j^r_x f = j^r_x g$ at $x \in M$, if $j^r_x f = j^r_x g$ for all $y \in E_x$. Every smooth section $s$ of $\mathcal{F}(E_1, E_2)$ determines the associated base-preserving morphism $\tilde{s}: E_1 \to E_2, \tilde{s}(y) = s(p_1 y)(y)$.

**Definition 6.** Two sections $s_1, s_2: M \to \mathcal{F}(E_1, E_2)$ determine the same $r$-jet $j^r_x s_1 = j^r_x s_2$ at $x \in M$, if $j^r_x \tilde{s_1} = j^r_x \tilde{s_2}$. The set $J^r \mathcal{F}(E_1, E_2)$ of all $r$-jets of the local sections of $\mathcal{F}(E_1, E_2)$ is called the $r$-jet prolongation of $\mathcal{F}(E_1, E_2)$.

However, it will be useful to discuss another approach as well. Since $\tilde{s}: E_1 \to E_2$ is a base-preserving morphism, we can construct its $r$-th jet prolongation $J^r \tilde{s}: J^r E_1 \to J^r E_2$. Write $J^r_{x} \tilde{s} = J^r \tilde{s} | _{J^r_{x} E_1}, x \in M$. By direct evaluation, one easily verifies.

**Proposition 1.** We have $j^r_x s_1 = j^r_x s_2$ iif $J^r_{x} \tilde{s_1} = J^r_{x} \tilde{s_2}$.

Let $z^a = f^a(x^i, y^p)$ be the coordinate expression of $\tilde{s}$. Then the additional coordinate expression of $J^1 \tilde{s}$ is

$$z^a_i = \frac{\partial f^a}{\partial x^i} + \frac{\partial f^a}{\partial y^p} y^p_i$$

where $y^p_i$ or $z^a_i$ are the induced coordinates on $J^1 E_1$ or $J^1 E_2$. For $x = 0$, the functions $\varphi^a(y^p) := f^a(0, y^p)$ are the coordinates of the target $s(0)$ of $j^r_0 s_1$ and $J^r_0 \tilde{s}$ has the form

$$z^a_i = \frac{\partial \varphi^a(y^p)}{\partial y^p} y^p_i + \varphi^a_i(y), \quad \varphi^a_i(y) = \frac{\partial f^a(0, y)}{\partial x^i}$$
It is well-known that $J^1_x E_1$ or $J^1_x E_2$ is an affine bundle over $E_{1x}$ or $E_{2x}$, whose derived vector bundle is $V_x E_1 \otimes T_x^* M$ or $V_x E_2 \otimes T_x^* M$, respectively. Obviously, (8) is an affine bundle morphism over $\varphi$ with the derived linear morphism $T \varphi \otimes \text{id}_{T_x^* M}$. Similarly to §1, we denote by $j^1_x s$ the associated map $J^1_x s : J^1_x E_1 \to J^1_x E_2$. Analogously to Lemma 1, one can prove

**Lemma 2.** Let $S : J^1_x E_1 \to J^1_x E_2$ be an affine bundle morphism over $\varphi : E_{1x} \to E_{2x}$ with the derived linear morphism $T \varphi \otimes \text{id}_{T_x^* M}$. Then there exists a local section $s$ of $\mathcal{F}(E_1, E_2)$ such that $s(x) = \varphi$ and $j^1_x s = S$.

By (8), every $X = j^1_x|_{E_1}$ and every $S = j^1_x s$ define a vector

\[
S(X) = \frac{\partial}{\partial t} \bigg|_{t=0} (s \circ f) \in T_{s(x)} \mathcal{F}(E_1, E_2)
\]

such that $Tp(S(X)) = X$.

**Definition 7.** A connection in the jet form on $\mathcal{F}(E_1, E_2)$ is a smooth section $\Gamma : \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$ of the target jet projection.

**Proposition 2.** The map (9) establishes a bijection between the jet form and the tangent form of connections on $\mathcal{F}(E_1, E_2)$.

**Proof.** Using (8) we find directly that (9) defines a bijection between the linear splittings $T_x M \to T_{s(x)} \mathcal{F}(E_1, E_2)$ of $Tp$ and the elements of $J^1 \mathcal{F}(E_1, E_2)$ of $\varphi$. Assume the jet form of $\Gamma$ is smooth and $f : Q \to \mathcal{F}(E_1, E_2)$ is a smooth map, so that $\Gamma \circ f : Q \to J^1 \mathcal{F}(E_1, E_2)$ is smooth. For every smooth vector field $X : M \to T M$, the map $(\Gamma \circ f)(X \circ p \circ f)$ is also smooth, so that the tangent form of $\Gamma$ is smooth. Conversely, take a local basis $X_1, \ldots, X_m$ of vector fields on $T M$. Then $(\Gamma X_1) \circ f, \ldots, (\Gamma X_m) \circ f$ are smooth maps $Q \to T \mathcal{F}(E_1, E_2)$. By (8) we deduce that $\Gamma \circ f : Q \to J^1 \mathcal{F}(E_1, E_2)$ is smooth.

To define the curvature of a connection of $\mathcal{F}(E_1, E_2)$ in §5, we shall use the second semiholonomic prolongation of $\mathcal{F}(E_1, E_2)$. We recall that $J^1(J^1 E_1 \to M) := J^2 E_1$ is the classical second nonholonomic prolongation of $E_1 \to M$. If $x^i$, $y^p$, $y^p_i$ are the above local coordinates of $J^1 E_1$, then the induced coordinates on $J^2 E_1$ are $y^p_{0i} = \frac{\partial y^p}{\partial x^i}$ and $y^p_{ij} = \frac{\partial y^p_i}{\partial x^j}$. We have the target jet projection $\beta_1 : J^2 E_1 \to J^1 E_1$ and the induced map $J^1 \beta : J^2 E_1 \to J^1 E_1$ of the target jet projection $\beta : J^1 E_1 \to E_1$. An element $Y \in J^2 E_1$ is said to be semiholonomic if $\beta_1(Y) = J^1 \beta(Y)$. In coordinates this is characterized by $y^p_i = y^p_{0i}$. All semiholonomic elements form a subbundle $\bar{J}^2 E_1 \subset J^2 E_1$, and the second holonomic prolongation $J^2 E$ is a subbundle of $\bar{J}^2 E$.

Since we have interpreted $J^1 \mathcal{F}(E_1, E_2)$ as a subset of $\mathcal{F}(J^1 E_1, J^1 E_2)$, we have defined $j^1_x \sigma$ for a local smooth section $\sigma$ of $J^1 \mathcal{F}(E_1, E_2) \to M$ by $j^1_x \sigma$. In this way we
introduce the second nonholonomic prolongation $\tilde{J}^2 \mathcal{F}(E_1, E_2)$ of $\mathcal{F}(E_1, E_2)$. An element $j^1_x\sigma$ is said to be semiholonomic, if $\sigma(x) = j^1_x(\beta \circ \sigma)$, where $\beta: J^1 \mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2)$ is the target jet projection. This defines $\tilde{J}^2 \mathcal{F}(E_1, E_2) \subset J^2 \mathcal{F}(E_1, E_2)$. The inclusion $J^2 \mathcal{F}(E_1, E_2) \subset \tilde{J}^2 \mathcal{F}(E_1, E_2)$ is given by $j^2_x \mapsto j^1_x(j^1_x \sigma)$.

Analogously to the first order case, $j^1_x\sigma$ determines a map $\tilde{j}^1_x\sigma: \tilde{J}^2 x E_1 \to \tilde{J}^2 x E_2$. In coordinates, if $\sigma = (f^a(x, y), f^i(x, y))$, then $\tilde{s}$ is of the form

$$z^a = f^a(x, y), \quad z^i_o = \frac{\partial f^a(x, y)}{\partial y^i} y^i + f^i(x, y).$$

Hence

$$\varphi^a(y) = f^a(0, y), \quad \varphi^i_0 = f^i_0(0, y), \quad \varphi^i_o = \frac{\partial f^a(0, y)}{\partial x^i}, \quad \varphi^a_0 = \frac{\partial f^a(0, y)}{\partial x^i}$$

are the coordinates of $j^1_x\sigma$. From (10) we obtain the coordinate expression of $\tilde{j}^1_x\sigma$ in the form $z^a = \varphi^a(y)$ and

$$z^a_i = \varphi^a_i + \frac{\partial \varphi^a_0}{\partial y^i} y^i + \varphi^a_0, \quad z^a_i = \frac{\partial \varphi^a (y)}{\partial y^i} y^i + \varphi^a_0,$$

Using (12) we deduce directly the following assertion.

**Proposition 3.** $j^1_x\sigma$ is semiholonomic or holonomic iff $\tilde{j}^1_x\sigma$ maps $\tilde{J}^2 x E_1$ into $\tilde{J}^2 x E_2$ or $J^2 x E_1$ into $J^2 x E_2$, respectively.

In coordinates, an element of $\tilde{J}^2 \mathcal{F}(E_1, E_2)$ is characterized by $\varphi^a_i = \varphi^a_0$ and the additional condition for a holonomic element is $\varphi^a_{ij} = \varphi^a_{ji}$.

We remark that the higher order nonholonomic and semiholonomic prolongations of $\mathcal{F}(E_1, E_2)$ can be defined in a quite similar way.

### 3. The finite order case

Since both vector fields from §1 and the connections from §2 are defined on a functional bundle, they represent a kind of differential operators. We are going to describe the simplest case of finite order operators.

**Definition 8.** A projectable vector field $A: \mathcal{F}(E_1, E_2) \to T \mathcal{F}(E_1, E_2)$ over $A^0: M \to TM$ is of order $r$, if the condition $j^r_y \varphi = j^r_y \psi, \varphi, \psi \in C^\infty(E_1x, E_2x), y \in E_1x$ implies that the restrictions of $\tilde{A}(\varphi)$ and $A(\psi)$ over $y$ coincide, i.e.

$$\tilde{A}(\varphi)|T_{A^0(x)} E_1_y = A(\psi)|T_{A^0(x)} E_1_y.$$
Let $S(TE_1, TE_2)$ be the set of all affine morphism $(T_x E_1)_y \to (T_x E_2)_z$, $p_1 y = p_2 z = \pi_M X$, where $\pi_M: TM \to M$ is the bundle projection. This is a fibered manifold over $E_1 \times E_2 \times TM$. Write

$$\mathcal{F} J^r(E_1, E_2) = \bigcup_{x \in M} J^r(E_{1x}, E_{2x}).$$

This is a classical manifold as well.

A projectable $r$-th order vector field $A: \mathcal{F}(E_1, E_2) \to T \mathcal{F}(E_1, E_2)$ over $A^0$ defines the associated map $\mathcal{A}: \mathcal{F} J^r(E_1, E_2) \to S(TE_1, TE_2)$ by

$$\mathcal{A}(j^r_y \varphi) = A(\varphi)\big|_{(T_A^0(x) E_1)_y}.$$

**Proposition 4.** The associated map of a projectable $r$-th order vector field on $\mathcal{F}(E_1, E_2)$ is a classical $C^\infty$-map.

**Proof.** This follows from the fact that $A$ is smooth in the sense of Definition 3 quite analogously to [6].

The local coordinates on $\mathcal{F} J^r(E_1, E_2)$ induced by $x^i, y^p$ and $z^a$, $1 \leq |\alpha| \leq r$, where $\alpha$ is a multiindex, the range of which is the fiber dimension of $E_1$. Hence the coordinate form of $\mathcal{A}$ is $X^i(x^j)$ and

$$\Phi^a = \Phi^a(x^i, y^p, z^a), \quad 0 \leq |\alpha| \leq r.$$

The derived linear map of each element of $S(TE_1, TE_2)$ is identified with an element of $\mathcal{F} J^1(E_1, E_2)$. This defines a map $D: S(TE_1, TE_2) \to \mathcal{F} J^1(E_1, E_2)$ and the following diagram commutes:

$$\begin{array}{c}
\mathcal{F} J^1(E_1, E_2) \\
\beta_x \downarrow \\
\mathcal{F} J^r(E_1, E_2) \xrightarrow{\mathcal{A}} S(TE_1, TE_2) \\
\downarrow \\
E_1 \times E_2 \xrightarrow{id \times A^0} E_1 \times E_2 \times TM
\end{array}$$

where $\beta_x$ is the jet projection. Conversely, let $\mathcal{A}: \mathcal{F} J^r(E_1, E_2) \to S(TE_1, TE_2)$ be a smooth map with an underlying vector field $A^0: M \to TM$ such that (16) commutes. Then the rule

$$A(\varphi) = \bigcup_{y \in E_1} \mathcal{A}(j^r_y \varphi).$$
defines a projectable r-th order vector field $A$ on $\mathcal{F}(E_1, E_2)$.

Since $T\mathcal{F}(E_1, E_2)$ is a subset of $\mathcal{F}(TE_1, TE_2)$, we can define the second tangent bundle $T(T^*\mathcal{F}(E_1, E_2))$. This will be described in more detail in §6. Here we restrict ourselves to a general remark, which is related to our study of the order of connections.

**Definition 9.** A vector field $A: \mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2)$ is called differentiable if the formula

$$TA\left(\frac{\partial}{\partial t}\bigg|_0 f\right) = \frac{\partial}{\partial t}\bigg|_0 A \circ f$$

defines a smooth map $TA: T^*\mathcal{F}(E_1, E_2) \to TT\mathcal{F}(E_1, E_2)$.

From (16) we easily deduce (see the coordinate formula in §6) the following assertion.

**Proposition 5.** Every r-th order vector field on $\mathcal{F}(E_1, E_2)$ is differentiable.

**Definition 10.** A connection $\Gamma: \mathcal{F}(E_1, E_2) \to J^1\mathcal{F}(E_1, E_2)$ is of order $r$ if the condition $j^r_y \varphi = j^r_y \psi$, $\varphi, \psi \in C^\infty(E_{1x}, E_{2x})$, $y \in E_{1x}$, implies

$$\overline{\Gamma(\varphi)}|_{J^1_y E_1} = \overline{\Gamma(\psi)}|_{J^1_y E_1}.$$  

Let $S(J^1E_1, J^1E_2)$ be the set of all affine maps $(J^1E_1)_y \to (J^1E_2)_z$ with the derived linear map of the form

$$B \otimes \text{id}_{T^*M} \quad B \in \mathcal{L} \in (V_y E_1, V_z E_2).$$

An r-th order connection $\Gamma: \mathcal{F}(E_1, E_2) \to J^1\mathcal{F}(E_1, E_2)$ defines the associated map $\mathcal{G}: \mathcal{F}J^r(E_1, E_2) \to S(J^1E_1, J^1E_2)$ by

$$\mathcal{G}(j^r_y \varphi) = \overline{\Gamma(\varphi)}|_{J^1_y E_1}.$$  

The coordinate form of $\mathcal{G}$ is

$$\Phi^a_i = \Phi^a_i(x^i, y^p, z^a), \quad 0 \leq |\alpha| \leq r.$$  

Analogously to Proposition 4, one proves

**Proposition 6.** The associated map of an r-th order connection $\mathcal{F}(E_1, E_2) \to J^1\mathcal{F}(E_1, E_2)$ is a classical $C^\infty$-map.
Let $D: S(J^1E_1, J^1E_2) \to \mathcal{F} J^1(E_1, E_2)$ be the map defined by (20). Then the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{F} J^1(E_1, E_2) & \xrightarrow{\beta_*} & \mathcal{F} J^r(E_1, E_2) \\
& \searrow^{D} & \swarrow \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
& S(J^1E_1, J^1E_2) & \to & \mathcal{F} J^1(E_1, E_2)
\end{array}
$$

(23)

Conversely, let $\mathcal{G}: \mathcal{F} J^r(E_1, E_2) \to S(J^1E_1, J^1E_2)$ be a smooth morphism over the identity of $E_1 \times E_2$ such that (23) commutes. Then the rule

$$
\Gamma(\varphi) = \bigcup_{y \in E_1} \mathcal{G}(j_y^* \varphi)
$$

(24)

defines an $r$-th order connection on $\mathcal{F}(E_1, E_2)$.

Analogously to Definition 9, we introduce

**Definition 11.** A connection $\Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$ is called differentiable if the formula

$$
J^1 \Gamma(j^1_s) = j^1_s(\Gamma \circ s)
$$

(25)

defines a smooth map $J^1 \mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2)$.

**Proposition 7.** Every $r$-th order connection is differentiable.

**Proof.** We deduce from (22) the coordinate form of $J^1 \Gamma$ in some coordinates $x^i, \varphi^a, \psi^a_i$ on $J^1 \mathcal{F}(E_1, E_2)$ and $x^i, \varphi^a, \varphi^a_i, \varphi^a_0, \varphi^a_i$ on $\mathcal{F}(E_1, E_2)$. Take a section $\sigma$

$$
z^a = \Psi^a(x^i, y^p)
$$

(26)

so that $\varphi^a = \psi^a(0, y)$ and $\psi^a_i = \frac{\partial \psi^a(0, y)}{\partial x^i}$. Then we obtain for $\Gamma \circ \sigma$

$$
z^a_i = \frac{\partial \psi^a(x, y)}{\partial y^p} y^p_i + \Phi^a_i(x, y, \partial \psi^a(x, y)).
$$

(27)

Now (26) yields

$$
z^a_0 = \frac{\partial \psi^a(0, y)}{\partial y^p} y^p_0 + \frac{\partial \psi^a(0, y)}{\partial x^i}, \text{ i.e. } \varphi^a_0 = \psi^a_i
$$

(28)

and (27) implies

$$
z^a_{ij} = \frac{\partial \psi^a_j}{\partial y^p} y^p_i + \frac{\partial^2 \psi^a}{\partial y^p \partial y^q} y^p_i y^q_j + \frac{\partial \psi^a}{\partial y^p} y^p_{ij} + \frac{\partial \psi^a}{\partial y^p} y^p_{0j} + \frac{\partial \psi^a}{\partial y^p} y^p_{0j} + \frac{\partial \psi^a}{\partial x^i} + \frac{\partial \psi^a}{\partial z^b} \partial_j \psi^b + \ldots + \frac{\partial \psi^a}{\partial z^b} \partial_j \psi^b.
$$

(29)

In particular, (29) shows that $J^1 \Gamma$ is well-defined and smooth.  \qed
Following Virsik, [17], if $\Gamma$ is differentiable and $\Delta$ is another connection $\mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$, we define a section

\[ \Gamma \ast \Delta = J^1 \Gamma \circ \Delta : \mathcal{F}(E_1, E_2) \to J^2 \mathcal{F}(E_1, E_2). \]

The order of such a section can be introduced similarly to Definition 10.

**Proposition 8.** If $\Gamma$ and $\Delta$ are connections of orders $r$ and $s$, respectively, then $\Gamma \ast \Delta$ has the order $r + s$.

**Proof.** We substitute the associated map of $\Delta$ into (28) and (29). \[\square\]

To obtain an explicit formula for the associated map of $\Gamma \ast \Delta$, we introduce the following concept. Having a smooth function $f : \mathcal{F}J^r(E_1, E_2) \to \mathbb{R}$, we define its formal differential $Df$ by

\[ Df : \mathcal{F}J^{r+1}(E_1, E_2) \to V^* E_1, Df(j^{r+1}_y \varphi) = d_y f(j^r \varphi). \]

Then every vertical vector field $\mu$ on $V^* E_1$ determines $(Df, \mu) : \mathcal{F}J^{r+1}(E_1, E_2) \to \mathbb{R}$. For the coordinate vector fields $\frac{\partial}{\partial y^p}$ we obtain the formal derivatives

\[ D_pf = \frac{\partial f}{\partial y^p} + \frac{\partial f}{\partial z^a} z^a_p + \ldots + \frac{\partial f}{\partial z^a} z^a_{p+s}. \]

By iteration, we introduce $D_\beta f : \mathcal{F}J^{r+|\beta|}(E_1, E_2) \to \mathbb{R}$. Let $\Psi^a_i(x^i, y^p, z^a_\beta), 0 \leq |\beta| \leq s$, be associated map of $\Delta$. Then the coordinate form of the main term of (29) is

\[ \varphi_{ij}^a = \frac{\partial \Phi_i^a}{\partial x^j} + \frac{\partial \Phi_i^a}{\partial z^b} \Psi_j^b + \frac{\partial \Phi_i^a}{\partial z^b} D_p \Psi_j^b + \ldots + \frac{\partial \Phi_i^a}{\partial z^a} D_\alpha \Psi_j^a. \]

**Remark 3.** In both cases of connections in the jet form and of projectable vector fields we have a situation somewhat similar to the vertical prolongation operators on classical fibered manifolds studied by Kosmann-Schwarzbach, [11], and the second author, [8]. In [10] Slovak deduced that every vertical prolongation operator is differentiable in the sense of our Definitions 9 and 11. However, his proof is based on quite sophisticated procedures in mathematical analysis, so that we have the feeling that such a problem in our setting is beyond the scope of the present paper.
We describe some properties of connections on a classical fibered manifold \( p: E \to M \) in a way which can be generalized to \( \mathcal{F}(E_1, E_2) \). Given \( A \in J^1_y E \) and \( B \in T_x M, \ x = py \), we denote by \( A(B) \in T_y E \) the \( A \)-lift of \( B \). We show that every \( A \in J^2_y E \) induces similarly a lifting \( \lambda A: TT_x M \to TT_y E \). If \( A = J^1_\sigma \) and \( B = \frac{\partial}{\partial t}\big|_0 f(t) \in TT_x M \), then we construct \( \sigma(\pi(f(t)))(f(t)): \mathbb{R} \to TE \) and set

\[
(34) \quad \lambda A(B) = \frac{\partial}{\partial t}\big|_0 \sigma(\pi(f(t)))(f(t))
\]

where \( \pi: TM \to M \) is the bundle projection. Given some local fiber coordinates \( x^i, y^p \) on \( E \), we have the induced coordinates \( y^p_i, y^p_{0i}, y^p_{ij} \) on \( J^2 E \), the induced coordinates \( X^i, Y^p \) on \( TE \) and the additional coordinates on \( TEE \) denoted by a dot. Then one finds easily the following coordinate form of (34):

\[
(35) \quad Y^p = y^p_i X^i, \quad \dot{y}^p = y^p_{0i} \dot{x}^i, \quad \dot{Y}^p = y^p_{ij} X^i \dot{x}^j + y^p_i \dot{X}^i.
\]

Let \( \kappa \) be the canonical involution of the second tangent bundle. If \( A \in J^2_y E \), then \( \kappa E \circ \lambda A \circ \kappa_M: TT_x M \to TT_y E \) is the lifting of another element \( \kappa A \in J^2_y E \), [15]. In coordinates, \( y^p_{ij}(\kappa A) = y^p_{ji}(A) \). Hence \( A \) is holonomic iff \( \kappa A = A \). Since \( J^2_y E \to J^1 E \) is an affine bundle with the derived vector bundle \( V^* \otimes (\otimes^2 T^* M) \), the points \( \kappa A \) and \( A \) determine a vector \( \Delta(A) := (\kappa A)A \in V_y E \otimes \Lambda^2 T^*_x M \), which is called the deviation (or difference tensor) of \( A \), [7], [12]. The coordinates of \( \Delta(A) \) are \( y^p_{ij} - y^p_{ji} \).

If \( X_1, X_2 \in T_x M \), then we have \( \Delta(A)(X_1, X_2) \in V_y E \).

Let \( \pi_1 = \pi_{TM} = TTM \to TM \) and \( \pi_2 = T\pi_{TM} = TTM \to TM \) be the canonical projections. Consider \( C, D \in TTM \) satisfying

\[
(36) \quad \pi_1(C) = \pi_2(D) \quad \text{and} \quad \pi_1(D) = \pi_2(C).
\]

Since \( \kappa \) exchanges the two projections, \( C \) and \( \kappa D \) are in the same fiber of \( TTM \) with respect to \( \pi_1 \) and satisfy \( \pi_2(C - \kappa D) = 0 \). Hence \( C - \kappa D \) is a tangent vector to a fiber of \( TM \) and such a vector can be identified with an element of \( T_x M \), which will be denoted by \( C \dot{1} D \) and called the strong difference of \( C \) and \( D \). In coordinates, if

\[
(37) \quad C \equiv (a^i, b^i, c^i), D \equiv (b^i, a^i, d^i) \quad \text{then} \quad C \dot{1} D \equiv (c^i - d^i).
\]

In [8] it is deduced the the bracket \([X, Y]\) of two vector fields \( X, Y: M \to TM \) can be expressed by

\[
(38) \quad [X, Y] = TY \circ X \dot{1} TX \circ Y.
\]
Lemma 3. Let $C, D \in TT_xM$ satisfy the condition (36) for the strong difference and $A \in J^2_yE$. Then $\lambda A(C), \lambda A(D)$ also satisfy (36) and

$$\Delta A(\pi_1 C, \pi_2 C) = (\lambda A(C) - \lambda A(D)) - \beta_1(A)(C - D)$$

where $\beta_1 : J^2_yE \to J^1_yE$ is the jet projection.

Proof. By (35) and (37) we have $\lambda A(C) = (y_i^p a^i, y_i^p b^i, y_i^p a^i b^i + y_i^p c^i), \lambda A(D) = (y_i^p b^i, y_i^p a^i, y_i^p b^i a^i + y_i^p d^i)$. This implies our claim. \hfill \Box

According to Remark 2, two connections $\Gamma, \Delta : E \to J^1_yE$ determine $\Gamma \ast \Delta = J^1 \Gamma \circ \Delta : E \to J^2_yE$. For $\Gamma = \Delta$ the values of $\Gamma \ast \Gamma$ lie in $J^2_yE$. In this case we obtain a construction closely related to an idea by Ehresmann, [2].

Definition 12. The map $\tilde{\Gamma} = J^1 \Gamma \circ \Gamma : E \to J^2_yE$ is the Ehresmann prolongation of $\Gamma$. The composition

$$CT := -\Delta \circ \tilde{\Gamma} : E \to VE \otimes \Lambda^2 T^* M$$

is the curvature of $\Gamma$.

To deduce that $CT$ coincides with the standard curvature of $\Gamma$, we need a property of the lifting map

$$\lambda \tilde{\Gamma} : E \times TTM \to TTE.$$

Consider two vector fields $X, Y : M \to TM$, so that $TX \circ Y : M \to TTM$.

Lemma 4. We have

$$\lambda \tilde{\Gamma}(TX \circ Y) = (T \Gamma X) \circ \Gamma Y : E \to TTE.$$

Proof. We have $\tilde{\Gamma}(y) = j_x^1(\Gamma \circ s), j_x^1 s = \Gamma(y)$. If $Y(x) = \frac{\partial}{\partial t}\big|_0 f(t)$, then

$$TX(Y(x)) = \frac{\partial}{\partial t}\big|_0 (X \circ f).$$

By (34),

$$\lambda \tilde{\Gamma}(TX(Y(x))) = \frac{\partial}{\partial t}\big|_0 \Gamma(s(f(t)))(X(f(t))) = (T \Gamma X \circ \Gamma Y)(y).$$

\hfill \Box
**Proposition 9.** For every vector fields $X, Y: M \to TM$, we have

$$\tilde{\Gamma}(X, Y) = [\Gamma X, \Gamma Y] - \Gamma([X, Y]).$$

**Proof.** Consider $TX \circ Y, TY \circ X: M \to TTM$. By Lemma 4 we obtain

$$\lambda \tilde{\Gamma}(TX \circ Y) = T\Gamma X \circ \Gamma Y \quad \text{and} \quad \lambda \tilde{\Gamma}(TY \circ X) = TTY \circ \Gamma X.$$

Then Lemma 3 and (38) imply

$$\Delta \circ \tilde{\Gamma}(X, Y) = (\lambda \tilde{\Gamma}(TX \circ Y) - \lambda \tilde{\Gamma}(TY \circ X)) - \Gamma(TX \circ Y - TY \circ X) =$$

$$= -[\Gamma X, \Gamma Y] + \Gamma([X, Y]).$$

\[ \square \]

5. **The Curvature of a Connection on $\mathcal{F}(E_1, E_2)$**

The deviation of an element $j^1_x \sigma \in \tilde{J}^2 \mathcal{F}(E_1, E_2)$ can be defined by means of the associated map $j^1_x \sigma: \tilde{J}^2_x E_1 \to \tilde{J}^2_x E_2$. In the semiholonomic case we have $\varphi^o_i = \varphi^o_{0i}$. So if we take a holonomic 2-jet $Y \in J^2_x E_1$, then the right-hand side of the second line in (12) is symmetric except the first term. Hence the deviation $\Delta(j^1_x \sigma(Y))$ is independent of $y^p_i$ and $y^p_{ij}$. This defines a map $\Delta(j^1_x \sigma): E_{1x} \to V_x E_2 \otimes \Lambda^2 T^*_x M$ over $\varphi$, i.e. an element of $\mathcal{F}(E_1, V E_2 \otimes \Lambda^2 T^* M)$.

**Definition 13.** $\Delta(j^1_x \sigma)$ is called the deviation of $j^1_x \sigma$. The coordinate form of $\Delta(j^1_x \sigma)$ is $\varphi^o_{ij} - \varphi^o_{ji}$.

**Definition 14.** For a differentiable connection $\Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$, the map $\tilde{\Gamma} := J^1 \Gamma \circ \Gamma: \mathcal{F}(E_1, E_2) \to \tilde{J}^2 \mathcal{F}(E_1, E_2)$ is the Ehresmann prolongation of $\Gamma$.

**Definition 15.** The composition

$$C\Gamma := -\Delta \circ \tilde{\Gamma}: \mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, V E_2 \otimes \Lambda^2 T^* M)$$

is the curvature of a differentiable connection $\Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$.

Clearly, $C\Gamma$ is a section of the canonical projection $\mathcal{F}(E_1, V E_2 \otimes \Lambda^2 T^* M) \to \mathcal{F}(E_1, E_2)$.

Let $\Gamma$ be an $r$-th order connection with the associated map $\Phi^o_i(x^i, y^p, z^o_{ij})$. Then we obtain the associated map of $C\Gamma$ by setting $\Psi^o_{ij} = \Phi^o_{ij}$ in (33) and by antisymmetrizing in $i$ and $j$. This implies

**Proposition 10.** The curvature of an $r$-th order connection has the order $2r$.  

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As remarked in §3, the inclusion $T \mathcal{F}(E_1, E_2) \subset \mathcal{F}(TE_1 \to TM, TE_2 \to TM)$ defines the second tangent bundle $T(T \mathcal{F}(E_1, E_2)) = T T \mathcal{F}(E_1, E_2)$. We have a projection $TTp: TT \mathcal{F}(E_1, E_2) \to TTM$ and two projections $\pi_T, T\pi: TT \mathcal{F}(E_1, E_2) \to T \mathcal{F}(E_1, E_2)$. In the above coordinates, consider an element $F \in TT \mathcal{F}(E_1, E_2)$ tangent to a curve $x^j(t), X^i(t), f^a(y, t)$ and

$$Z^a = \frac{\partial f^a(y, t)}{\partial y^p} Y^p + \Phi^a(y, t).$$

Then its associated map $\tilde{F}: TT_x E_1 \to TT_x E_2, X = T T p(F)$, is of the form

$$Z^a = \frac{\partial \varphi^a}{\partial y^p} Y^p + \Phi^a(y), \dot{z}^a = \frac{\partial \varphi^a}{\partial y^p} \dot{y}^p + f^a(y)$$

$$\dot{Z}^a = F^a(y) + \frac{\partial \Phi^a}{\partial y^p} \dot{y}^p + \frac{\partial f^a}{\partial y^p} Y^p + \frac{\partial^2 \varphi^a}{\partial y^p \partial y^q} Y^p \dot{y}^q + \frac{\partial \varphi^a}{\partial y^p} \dot{Y}^p.$$ 

So $\varphi^a, \Phi^a, f^a, F^a$ are the functional coordinates of $F$, which are completed by the coordinates $x^i, X^i, \dot{x}^i, \dot{X}^i$ of $X \in TTM$. The coordinate form of $\pi_T$ or $T\pi$ is

$$\pi_T(x^i, X^i, \dot{x}^i, \dot{X}^i, \varphi^a, \Phi^a, f^a, F^a) = (x^i, X^i, \varphi^a, \Phi^a),$$

$$T\pi(x^i, X^i, \dot{x}^i, \dot{X}^i, \varphi^a, \Phi^a, f^a, F^a) = (x^i, \dot{x}^i, \varphi^a, f^a).$$

Consider the canonical involution $\kappa_{E_1}$ or $\kappa_{E_2}$ of the second tangent bundle.

**Proposition 11.** For every $F \in TT \mathcal{F}(E_1, E_2)$ over $X \in TTM$ there exists a unique element $\kappa F \in TT \mathcal{F}(E_1, E_2)$ such that its associated map $\kappa \tilde{F}: TT_{\kappa M \times E_1} E_1 \to TT_{\kappa M \times E_2}$ is $\kappa \tilde{F} = \kappa_{E_2} \circ \tilde{F} \circ \kappa_{E_1}$.

**Proof.** This follows from (40). \qed

Obviously, the coordinate form of $\kappa$ is

$$\kappa(x, X, \dot{x}, \dot{X}, \varphi, \Phi, f, F) = (x, X, \dot{x}, \dot{X}, \varphi, f, \Phi, F).$$

Consider $C, \tilde{C} \in TT \mathcal{F}(E_1, E_2)$ over $X, \tilde{X} \in TTM$ satisfying

$$\pi_T(C) = T\pi(\tilde{C}) \quad \text{and} \quad \pi_T(\tilde{C}) = T\pi(C).$$

Then we define the strong difference $C \dot{\!}/\! \tilde{C} \in T \mathcal{F}(E_1, E_2), Tp(C \dot{\!}/\! \tilde{C}) = X \dot{\!}/\! \tilde{X}$, as follows. For every $B \in (T_{X \dot{\!}/\! \tilde{X}} E_1)_y$ we take any $Y, \bar{Y} \in (TT E_1)_y$ over $X, \tilde{X}$.
such that \( Y \sim \tilde{Y} = B \). Then one easily verifies that \( C(Y), \tilde{C}(\tilde{Y}) \) also satisfy (42), \( C(Y) \sim \tilde{C}(\tilde{Y}) \) depends on \( C, \tilde{C} \) and \( B \) only and represents the associated map of an element \( C \sim \tilde{C} \in T \mathcal{F}(E_1, E_2) \), whose coordinates are

\[
(43) \quad (x^i, \dot{x}^i - \ddot{x}^i, \varphi^a, F^a - \tilde{F}^a).
\]

Let \( A, B \) be two differentiable vector fields on \( \mathcal{F}(E_1, E_2) \). Then the maps \( TA \circ B, TB \circ A : \mathcal{F}(E_1, E_2) \rightarrow TT \mathcal{F}(E_1, E_2) \) satisfy the condition (42) at every \( \varphi \in \mathcal{F}(E_1, E_2) \).

**Definition 16.** The vector field

\[
[A, B] := TB \circ A - TA \circ B : \mathcal{F}(E_1, E_2) \rightarrow T \mathcal{F}(E_1, E_2)
\]

is called the bracket of \( A \) and \( B \).

By (38) we immediately deduce

**Proposition 12.** If \( A \) and \( B \) are projectable over \( A^0 \) and \( B^0 \), then \( [A, B] \) is projectable over \( [A^0, B^0] \).

Assume \( A \) is of order \( r \) and \( B \) is of order \( s \) with the associated maps \( X^i(x), A^a(x^i, y^p, z^a_\alpha), |\alpha| \leq r \) and \( Y^i(x), B^a(x^i, y^p, z^a_\beta), |\beta| \leq s \), respectively. Analogously to §3, the fourth component of the associated map of \( TA \circ B \) is

\[
(44) \quad \frac{\partial A^a}{\partial x^i} Y^i + \frac{\partial A^a}{\partial z^b} B^b + \frac{\partial A^a}{\partial z^b} D_p B^b + \ldots + \frac{\partial A^a}{\partial z^b_\alpha} D_\alpha B^b, \quad |\alpha| \leq r
\]

while the fourth component of the associated map of \( TB \circ A \) is

\[
(45) \quad \frac{\partial B^a}{\partial x^i} X^i + \frac{\partial B^a}{\partial z^b} A^b + \frac{\partial B^a}{\partial z^b} D_p A^b + \ldots + \frac{\partial B^a}{\partial z^b_\beta} D_\beta A^b, \quad |\beta| \leq s.
\]

Hence we can summarize by

**Proposition 13.** The bracket \([A, B]\) has the order \( r + s \) and its associated map is \([A^0, B^0]\) and the difference \((45)-(44)\).

We are going to generalize Proposition 9 to connections on \( \mathcal{F}(E_1, E_2) \). First of all we remark that every \( A = j_x^1 \sigma \in \tilde{j}_x^2 \mathcal{F}(E_1, E_2) \) defines a lifting \( \lambda A : TT_x M \rightarrow TT_\varphi \mathcal{F}(E_1, E_2) \) by

\[
\lambda A \left( \frac{\partial}{\partial t} \bigg|_0 f \right) = \frac{\partial}{\partial t} \bigg|_0 \sigma(\pi_M(f(t))(f(t)).
\]
In coordinates, if $A = (x^a, \varphi^a, \varphi^a_0, \dot{\varphi}^a_{ij})$ and $B = \frac{\partial}{\partial t}\big|_0 f = (x^i, \dot{x}^i, \ddot{X}^i)$, then one easily finds the following coordinate form of $\lambda A(B)$:

$$
(x^i, \varphi^a, \varphi^a_i X^i, \varphi^a_i \dot{x}^i, \varphi^a_{ij} X^i \dot{x}^j + \varphi^a_i \ddot{X}^i).
$$

This directly implies the following generalization of Lemma 3.

**Lemma 5.** Let $C, D \in TT_{x}M$ satisfy the condition (36) for the strong difference and $A \in J^2 \mathcal{F}(E_1, E_2)$. Then $\lambda A(C), \lambda A(D)$ satisfy (42) and

$$
\Delta A(\pi_T C, T\pi C) = (\lambda A(C) - \lambda A(D)) - \beta_1(A)(C - D).
$$

Now we need an assumption of technical character (which is fulfilled for every finite order connection).

**Definition 17.** A differentiable connection $\Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$ is called strongly differentiable, if $\Gamma X$ is a differentiable vector field on $\mathcal{F}(E_1, E_2)$ for every smooth vector field $X: M \to TM$.

**Proposition 14.** For every strongly differentiable connection $\Gamma$ on $\mathcal{F}(E_1, E_2)$ and for all vector fields $X, Y$ on $M$ we have

$$
C\Gamma(X, Y) = [\Gamma X, \Gamma Y] - \Gamma([X, Y]).
$$

**Proof.** In the same way as in Lemma 4 we deduce $\lambda \Gamma(TX \circ Y) = (T\Gamma X) \circ \Gamma Y$. Then we apply Lemma 5. \qed

### 7. The Absolute Differentiation

Let $A, B \in J^1 \mathcal{F}(E_1, E_2)$ be two 1-jets with the same target $\varphi$. To deduce that their difference is an element $A - B \in \mathcal{F}(E_1, VE_2 \otimes T^*M)$ over $\varphi$, we consider the associated maps $\hat{A}, \hat{B}: J^1_x E_1 \to J^1_x E_2$,

$$
A \equiv z^a_i = \frac{\partial \varphi^a}{\partial y^p} y^p_i + \Phi_i^p(y), \quad B \equiv z^a_i = \frac{\partial \varphi^a(x, y)}{\partial y^p} y^p_i + \Psi_i^p(y).
$$

The element $A(Y) - B(Y)$ is independent of the choice of $Y \in J^1_x E_1$, which defines a map $E_{1x} \to VE_2 \otimes T^*M$ over $\varphi$. (In this sense $J^1 \mathcal{F}(E_1, E_2)$ is an affine bundle with the derived vector bundle $\mathcal{F}(E_1, VE_2 \otimes T^*M)$ analogously to the classical case.)

Let $s: M \to \mathcal{F}(E_1, E_2)$ be a section and $\Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2)$ a connection.
Definition 18. The absolute differential

$$\nabla s : M \to \mathcal{F}(E_1, VE_2 \otimes T^*M)$$

is the above difference $\nabla s(x) = j^1_s - \Gamma(s(x))$.

If $X : M \to TM$ is a vector field, we define the absolute derivative of $s$ with respect to $X$ by

$$(47) \quad \nabla_X s = (\nabla s, X) : M \to \mathcal{F}(E_1, VE_2)$$

where $(\ , \ )$ is the extension of the evaluation map $T \times T^* \to \mathbb{R}$. Having an $r$-th order connection with the associated map (22) and a section $s$ of the form $z^a = \varphi^a(x, y)$, then the coordinate form of $\nabla s$ is

$$(48) \quad \frac{\partial \varphi^a(x, y)}{\partial x^i} - \Phi^a_i(x^i, y^p, \partial_{\alpha} \varphi^a(x, y)).$$

To obtain $\nabla_x s$, we contract (48) with the coordinate functions $X^i(x)$ of $X$.

Remark 4. In the case $E_1 = E_2 := E$ we have a distinguished section $I : M \to \mathcal{F}(E, E)$, $I(x) = \text{id}_E$. Analogously to the case of a classical linear connection on $TM$, the absolute differential $\nabla I : M \to \mathcal{F}(E, VE \otimes T^*M)$ can be called the torsion of a connection $\Gamma$ on $\mathcal{F}(E, E)$. By (48), the coordinate form of the torsion of an $r$-th order connection is $-\Phi^p_i(x^i, y^p, \delta_p^r, 0, \ldots, 0)$.

It might be instructive to discuss a special case in more detail. Let $E \to M$ be a vector bundle. Consider the subspace $LE \subset \mathcal{F}(E, E)$ of all linear maps, which is a classical vector bundle over $M$. A connection $\Gamma$ on $LE$ in our sense is a classical general connection on $LE$. Hence our approach leads to the original idea of the torsion of a general connection $\Gamma$ on $LE$. If $w^p_q$ are the induced fiber coordinates on $LE$, the usual coordinate expression of $\Gamma$ is $\text{dw}^q_p = F^p_{q_i}(x^i, w^r_s) \text{dx}^i$. Then $-F^p_{q_i}(x^i, \delta^r_s)$ is the coordinate form of the torsion of $\Gamma$. Of course, if we take for $\Gamma$ the tensor product $\Delta \otimes \Delta^*$ of a linear connection $\Delta$ on $E$ and of the dual connection $\Delta^*$ on $E^*$, [10], then the torsion of $\Delta \otimes \Delta^*$ vanishes, for $I$ is invariant with respect to $\Delta \otimes \Delta^*$. 

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8. THE VECTOR BUNDLE CASE

Assume \( p: E_2 \to M \) is a vector bundle. Then each fiber of \( \mathcal{F}(E_1, E_2) \) is a vector space, provided the linear operations on \( C^\infty(E_1x, E_2x) \) are defined by extending the linear operations on \( E_2x \). In other words, \( \mathcal{F}(E_1, E_2) \to M \) is a vector bundle over sets, cf. [4]. Such a vector bundle structure is further extended to \( J^1 \mathcal{F}(E_1, E_2) \) by

\[
j^1_x s_1 + j^1_x s_2 = j^1_x (s_1 + s_2), \quad j^1_x (ks) = j^1_x k \bar{s}, \quad k \in \mathbb{R}
\]

with addition and multiplication by reals in \( E_2 \). Hence \( J^1 \mathcal{F}(E_1, E_2) \to M \) also is a vector bundle over sets.

**Definition 19.** A connection \( \Gamma: \mathcal{F}(E_1, E_2) \to J^1 \mathcal{F}(E_1, E_2) \) is called linear if \( \Gamma \) is a linear morphism over \( M \).

In the case of an \( r \)-th order linear connection, its associated map (22) has the form

\[
\Phi_{ib}^a(x, y)z^b + \Phi_{ib}^{aq}(x, y)z^b_q + \ldots + \Phi_{ib}^{aq} z^b_{\alpha}.
\]

If \( E_2 \) is a vector bundle, then \( VE_2 = E_2 \times E_2 \), which implies

\[
\mathcal{F}(E_1, VE_2 \otimes \Lambda^2 T^* M) = \mathcal{F}(E_1, E_2) \times \mathcal{F}(E_1, E_2 \otimes \Lambda^2 T^* M).
\]

In this case, analogously to the classical situation, the curvature will be interpreted as the second component of the map from Definition 15,

\[
CG: \mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2 \otimes \Lambda^2 T^* M),
\]

while the first component is the identity.

**Proposition 15.** For every differentiable linear connection \( \Gamma \), the map \( CG: \mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2 \otimes \Lambda^2 T^* M) \) is a linear morphism over \( M \).

**Proof.** One easily verifies that in the linear case both \( \hat{\Gamma} \) and \( \Delta \) in Definition 15 are linear morphisms over \( M \). \( \square \)

Quite similarly, if \( E_2 \) is a vector bundle, then the absolute derivative \( \nabla_X s \) of a section \( s \) with respect to a vector field \( X \) on \( M \) is identified with the second component of (47), so that it is section of \( \mathcal{F}(E_1, E_2) \) as well.

We finally remark that several other ideas from the classical theory of connections can be generalized to the case of \( \mathcal{F}(E_1, E_2) \). The most interesting ones could be the vertical prolongation of \( \Gamma \), the connections on \( T \mathcal{F}(E_1, E_2) \subset \mathcal{F}(TE_1 \to TM, TE_2 \to TM) \) or a detailed study of the absolute differentiation in the linear case. Such a research can be based on some general ideas from the theory of classical connections collected in the book [10].
References


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