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VISIBILITIES AND SETS OF SHORTEST PATHS
IN A CONNECTED GRAPH

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By a graph we mean here an undirected (not necessarily finite) graph without loops and multiple edges. Thus if $G$ is a graph with a vertex set $V(G)$ and an edge set $E(G)$, then $V(G)$ is a nonempty set and $E(G)$ is a subset of the set of all two-element subsets of $V(G)$; $G$ is called finite if $V(G)$ is finite.

The letters $h, i, j, k, m$ and $n$ will be reserved for denoting integers.

Consider a graph $G$. We denote by $W(G)$ the set of all finite sequences of vertices in $G$, including the empty sequence, which will be denoted by $\ast$. Thus $W(G) - \{\ast\}$ is the set of all sequences

(0) 
$v_0, \ldots, v_j,$

where $j \geq 0$ and $v_0, \ldots, v_j \in V(G)$. Similarly to [2], instead of (0) we will write $v_0 \ldots v_j$. Let $u_0, \ldots, u_i, w_0, \ldots, w_k \in V(G)$, where $i, k \geq 0$, and let $\alpha = u_0 \ldots u_i$ and $\beta = w_0 \ldots w_k$. Then we write

$$\alpha \beta = u_0 \ldots u_i w_0 \ldots w_k.$$ 

Moreover, we write $\gamma \ast = \gamma = \ast \gamma$ for every $\gamma \in W(G)$. Let $x_0, \ldots, x_m \in V(G)$, where $m \geq 0$. Put $\delta = x_0 \ldots x_m$. We write

$$||\delta|| = m, \quad F\delta = x_0, \quad L\delta = x_m, \quad \text{and} \quad \delta = x_m \ldots x_0.$$ 

Moreover, we define $\ast = \ast$. Let $y_0, \ldots, y_n \in V(G)$, where $n \geq 0$. We say that $y_0 \ldots y_n$ is a path in $G$ if the vertices $y_0, \ldots, y_n$ are mutually distinct and $\{y_i, y_{i+1}\} \in E(G)$ for every integer $i$ such that $0 \leq i < n$. Let $\mathcal{P}(G)$ denote the set of all paths in $G$. Obviously, $\mathcal{P}(G) \subseteq W(G) - \{\ast\}$. If $\alpha \in \mathcal{P}(G)$, then the number $||\alpha||$ is called the length of $\alpha$. Consider $\mathcal{R} \subseteq \mathcal{P}(G)$ and $u, v \in V(G)$. Define

$$\mathcal{R}_{(u,v)} = \{\alpha \in \mathcal{R}; \ F\alpha = u \text{ and } L\alpha = v\}.$$
We say that $G$ is connected if $\mathcal{P}_{(t,z)} \neq \emptyset$ for every pair of $t, z \in V(G)$, where $\mathcal{P} = \mathcal{P}(G)$.

Consider a connected graph $G$. We define the distance $d_G(x,y)$ of vertices $x$ and $y$ in $G$ as
\[
d_G(x,y) = \min(\|\alpha\|; \alpha \in \mathcal{P}(G), F\alpha = x \text{ and } L\alpha = y).
\]

Let $\xi \in \mathcal{W}(G) - \{\ast\}$; we say that $\xi$ is a shortest path in $G$ if $\xi \in \mathcal{P}(G)$ and $\|\xi\| = d_G(F\xi, L\xi)$. Let $\mathcal{I}(G)$ denote the set of all shortest paths in $G$.

The set $\mathcal{I}(G)$ was characterized by the present author in [2] (under the condition that $G$ is finite); his characterization is “almost non-metric” in the sense that the lengths of paths greater than one are neither considered nor compared in it. In the present paper a more general result will be proved. We will obtain an “almost non-metric” necessary and sufficient condition for a set of paths in a connected graph $G$ to be an element of a certain set of subsets of $\mathcal{I}(G)$. To describe such a set of subsets of $\mathcal{I}(G)$ we introduce the notion of visibility in $G$.

Let $G$ be a connected graph, and let $Q \subseteq V(G) \times V(G)$. We say that $Q$ is a visibility in $G$ if $Q$ fulfills the following Axioms I–IV (for arbitrary $u, v, x, y \in V(G)$):

I if $(u,v) \in Q$, then $(v,u) \in Q$;
II if $(u,v) \in Q$ and $d_G(u,x) + d_G(x,v) = d_G(u,v)$, then $(u,x) \in Q$;
III if $(u,v) \in Q$, $\{u,x\}, \{v,y\} \in E(G)$ and $d_G(x,v) = d_G(u,v) - 1 = d_G(x,y)$, then $(u,y) \in Q$;
IV if $(u,v) \in Q$, $\{u,x\}, \{v,y\} \in E(G)$ and $d_G(x,v) = d_G(u,v) - 1 \geq 1$, then $(x,y) \in Q$.

We are now prepared to formulate the main result of the present paper.

**Theorem.** Let $G$ be a connected graph, and let $\mathcal{R} \subseteq \mathcal{P}(G)$. Denote $\mathcal{I} = \mathcal{I}(G)$. Then the following statements (1) and (2) are equivalent:

(1) there exists a visibility $Q$ in $G$ such that
\[
\mathcal{R}_{(t,z)} = \mathcal{I}_{(t,z)} \quad \text{if} \quad (t,z) \in Q \quad \text{and}
\]
\[
\mathcal{R}_{(t,z)} = \emptyset \quad \text{if} \quad (t,z) \notin Q;
\]

for every pair of vertices $t$ and $z$ of $G$;

(2) $\mathcal{R}$ fulfills the following Axioms $A_1$–$A_4$ and $B_1$–$B_3$ (for arbitrary $u, v, x, y \in V(G)$ and $\alpha, \beta, \gamma, \delta \in \mathcal{W}(G)$):

$A_1$ if $\alpha \in \mathcal{R}$, then $\overline{\alpha} \in \mathcal{R}$;

$A_2$ if $\alpha u v \in \mathcal{R}$, then $\alpha u \in \mathcal{R}$;

$A_3$ if $uxov \in \mathcal{R}$, $\{v,y\} \in E(G)$, $u \varphi y v \notin \mathcal{R}$ for any $\varphi \in \mathcal{W}(G)$ and $ux\psi y \notin \mathcal{R}$ for any $\psi \in \mathcal{W}(G)$, then $xovy \in \mathcal{R}$;
A_4 \text{ if } u x \alpha, u \beta y v \in \mathcal{R}, \text{ then } \mathcal{R}_{(x,y)} \neq \emptyset; \\
B_1 \text{ if } \alpha u \beta v \gamma, u \delta v \in \mathcal{R}, \text{ then } \alpha u \delta v \gamma \in \mathcal{R}; \\
B_2 \text{ if } u x \alpha, u \beta y v, x u \beta y \in \mathcal{R}, \text{ then } x a v y \in \mathcal{R}; \\
B_3 \text{ if } u x \alpha v \in \mathcal{R}, \text{ then } \{u, v\} \notin E(G).

Proof. Instead of $d_G(t, z)$, where $t, z \in V(G)$, we will write $d(t, z)$.

Part One: $(1) \Rightarrow (2)$. Let $(1)$ hold. We want to prove that $\mathcal{R}$ fulfills Axioms $A_1$–$A_4$ and $B_1$–$B_3$.

Consider arbitrary $u, v, x, y \in V(G)$ and $\alpha, \beta \in \mathcal{W}(G)$.

(Verification of Axiom $A_1$). Suppose $\alpha \in \mathcal{R}$. There exist $t, z \in V(G)$ such that $\alpha \in \mathcal{R}_{(t,z)}$. Hence $\mathcal{R}_{(t,z)} \neq \emptyset$. It follows from $(1)$ that $(t, z) \in Q$ and therefore, $\mathcal{R}_{(t,z)} = \mathcal{I}_{(t,z)}$. We get $\alpha \in \mathcal{I}_{(t,z)}$. Axiom I implies that $(z, t) \in Q$. According to $(1)$, $\mathcal{R}_{(z,t)} = \mathcal{I}_{(z,t)}$. Thus $\alpha \in \mathcal{R}$.

(Verification of Axiom $A_2$). Suppose $\alpha u v \in \mathcal{R}$. First, let $\alpha = \ast$. According to $(1)$, $u v \in \mathcal{I}$ and $(u, v) \in Q$. Axiom II implies that $(u, u) \in Q$. As follows from $(1)$, $\alpha u = u \in \mathcal{R}$. Let now $\alpha \neq \ast$. There exist $t \in V(G)$ and $\varphi \in \mathcal{W}(G)$ such that $\alpha = t \varphi$. Then $t \varphi u v \in \mathcal{R}_{(t,v)}$. According to $(1)$, $t \varphi u v \in \mathcal{I}$ and $(t, v) \in Q$. Obviously, $t \varphi u \in \mathcal{I}$. We have $d(t, v) = d(t, u) + d(u, v)$. Axiom II implies that $(t, u) \in Q$. According to $(1)$, $\mathcal{R}_{(t,u)} = \mathcal{I}_{(t,u)}$. We get $\alpha u = t \varphi u \in \mathcal{R}$.

(Verification of Axiom $A_3$). Suppose $u x \alpha v \in \mathcal{R}$, $\{v, y\} \in E(G)$, $u \varphi y v \notin \mathcal{R}$ for any $\varphi \in \mathcal{W}(G)$ and $u x \psi y \notin \mathcal{R}$ for any $\psi \in \mathcal{W}(G)$. Clearly, $\{u, x\} \in E(G)$. Since $\mathcal{R}_{(u,v)} \neq \emptyset$, it follows from $(1)$ that $\mathcal{R}_{(u,v)} = \mathcal{I}_{(u,v)}$ and $(u, v) \in Q$. This implies that $u x \alpha v \in \mathcal{I}$ and $u \varphi y v \notin \mathcal{I}$ for any $\varphi \in \mathcal{W}(G)$. Thus $d(x, v) = d(u, v) - 1 \geq 1$ and $d(u, v) \leq d(u, y)$.

Obviously, $d(x, y) \geq d(u, y) - 1$. This means that $d(u, v) - 1 \leq d(x, y) \leq d(u, v)$. Assume that $d(x, y) = d(u, v) - 1$. Axiom III implies that $(u, y) \in Q$. As follows from $(1)$, $\mathcal{R}_{(u,y)} = \mathcal{I}_{(u,y)}$. This means that $u x \psi y \notin \mathcal{I}$ for any $\psi \in \mathcal{W}(G)$. Thus $d(u, y) \leq d(x, y)$. Clearly, $d(u, v) \leq d(u, y) \leq d(x, y) \leq d(u, v) - 1$, which is a contradiction. Hence $d(x, y) = d(u, v)$. We see that $x a v y \in \mathcal{I}$.

Recall that $d(x, v) = d(u, v) - 1 \geq 1$. Axiom IV implies that $(x, y) \in Q$. According to $(1)$, $\mathcal{R}_{(x,y)} = \mathcal{I}_{(x,y)}$. We get $x \alpha v y \in \mathcal{R}$.

(Verification of Axiom $A_4$). Suppose $u x \alpha v, u \beta y v \in \mathcal{R}$. Then $\{u, x\}, \{v, y\} \in E(G)$. According to $(1)$, $u x \alpha v \in \mathcal{I}$ and $(u, v) \in Q$. Since $d(x, v) = d(u, v) - 1 \geq 1$, it follows from Axiom IV that $(x, y) \in Q$. According to $(1)$, $\mathcal{R}_{(x,y)} \neq \emptyset$.

Thus $\mathcal{R}$ fulfills Axioms $A_1$–$A_4$. Axioms $B_1$–$B_3$ follows from $(1)$ and simple properties of $\mathcal{I}$. Hence $(2)$ holds.

Part Two: $(2) \Rightarrow (1)$. Let $\mathcal{R}$ fulfill Axioms $A_1$–$A_4$ and $B_1$–$B_3$. Combining Axioms $A_1$ and $A_2$, we get

(3) if $u \in V(G)$, $\alpha, \beta \in \mathcal{W}(G)$ and $\alpha u \beta \in \mathcal{R}$, then $\alpha u, u \beta, u \alpha, \beta u \in \mathcal{R}$. 

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Combining Axioms $A_2$ and $A_3$, we get

(4) if $u, v, x, y \in V(G), \alpha \in \mathcal{W}(G), uxav \in \mathcal{R}, \{v, y\} \in E(G)$ and $xavy \notin \mathcal{R}$, then

$\mathcal{R}(u, y) \neq \emptyset$.

This part of the proof will be divided into Sections 1 and 2. In Section 1 we will prove that

(5) if $\mathcal{R}(u, v) \neq \emptyset$, then $\mathcal{R}(u, v) = \mathcal{I}(u, v)$ for every pair of vertices $u$ and $v$ of $G$.

In Section 2 we will prove that

$$\{(u, v); u, v \in V(G) \text{ such that } \mathcal{R}(u, v) \neq \emptyset\}$$

is a visibility in $G$.

Section 1. We denote by $M$ the set of all integers $k$ such that there exist $t, z \in V(G)$ with the property that $d(t, z) = k$. Obviously, either $M$ is the set of all non-negative integers or there exists $h \geq 0$ such that $M = \{0, \ldots, h\}$. For each $m \in M$ we will prove that

(6) if $\mathcal{R}(u, v) \neq \emptyset$, then $\mathcal{I}(u, v) \subseteq \mathcal{R}(u, v)$ for every pair of vertices $u$ and $v$ of $G$ such that $d(u, v) \leq m$,

and

(7) $\mathcal{R}(u, v) \subseteq \mathcal{I}(u, v)$ for every pair of vertices $u$ and $v$ of $G$ such that $d(u, v) \leq m$.

We proceed by induction on $m$. First, let $m = 0$. Since $\mathcal{R} \subseteq \mathcal{P}(G)$, we get $\mathcal{R}(w, w) \subseteq \{w\}$ for each $w \in V(G)$. Hence (60) and (70) follow. Next, let $m = 1$. Consider arbitrary $t, z \in V(G)$ such that $d(t, z) = 1$. Axiom $B_3$ implies that $\mathcal{R}(t, z) \subseteq \{t, z\}$. Hence, (61) and (71) follow.

Now, let $m \geq 2$. Suppose (6$_{m-1}$) and (7$_{m-1}$) hold. This section of the proof will be divided into two subsections. In 1.1, combining (6$_{m-1}$) and (7$_{m-1}$) we will prove that (6$_m$) holds. In 1.2, combining (6$_m$) and (7$_{m-1}$) we will prove that (7$_m$) holds.

1.1. If $\mathcal{R}(t, z) = \emptyset$ for every pair of vertices $t$ and $z$ of $G$ such that $d(t, z) = m$, then (6$_{m-1}$) implies that (6$_m$) holds. Assume that there exist $t, z \in V(G)$ such that $\mathcal{R}(t, z) \neq \emptyset$ and $d(t, z) = m$.

Consider arbitrary $u, v \in V(G)$ such that $\mathcal{R}(u, v) \neq \emptyset$ and $d(u, v) = m$. Consider an arbitrary $\xi \in \mathcal{I}(u, v)$. We want to prove that $\xi \in \mathcal{R}$. Since $\mathcal{R}(u, v) \neq \emptyset$, there exists $\zeta \in \mathcal{R}(u, v)$.

We first assume that $\xi$ and $\zeta$ have a common vertex $w$ such that $u \neq w \neq v$. Then (8) there exist $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathcal{W}(G) - \{\ast\}$ such that $\xi = \varphi_1 w \varphi_2$ and $\zeta = \psi_1 w \psi_2$.

Obviously, $\varphi_1 w \in \mathcal{I}(u, w)$ and $w \varphi_2 \in \mathcal{I}(w, v)$. As follows from (3), $\psi_1 w \in \mathcal{R}(u, w)$ and $w \psi_2 \in \mathcal{R}(w, v)$. It is clear that $d(u, w) < m$ and $d(w, v) < m$. Since $\mathcal{R}(u, w) \neq \emptyset$.
\[\emptyset \not\in R_{(w,v)}, (6_{m-1}) \text{ implies that } \varphi_1 w, w \varphi_2 \in R. \text{ Recall that } \psi_1 w \psi_2 \in R_{(u,v)}. \] Using Axiom \(B_1\) we get \(\psi_1 w \varphi_2 \in R\) and \(\xi = \varphi_1 w \varphi_2 \in R\).

We now assume that \(\xi\) and \(\zeta\) have no common vertex different from \(u\) and \(v\). Put \(n = \|\zeta\|\). Obviously, \(n \geq m \geq 2\). There exist mutually distinct \(x_0, \ldots, x_{m+n-1} \in V(G)\) such that

\[\text{(9) } \xi = x_0 x_{m+n-1} \ldots x_n \text{ and } \zeta = x_0 x_1 \ldots x_n.\]

Obviously, \(x_0 = u\) and \(x_n = v\). Put \(x_{k+m+n} = x_k\) for each \(k \in \{0, \ldots, m+n-1\}\).

Then \(\xi = x_{m+n} x_{m+n-1} \ldots x_n\). We define

\[\text{(10) } \xi_i = x_i x_{i+m+n-1} \ldots x_{i+n} \text{ and } \zeta_i = x_i x_{i+1} \ldots x_{i+n}\]

for each \(i \in \{0, \ldots, m\}\). Obviously, \(\xi_0 = \xi\) and \(\zeta_0 = \zeta\). Recall that we want to prove that \(\xi_0 \in R\). Suppose, to the contrary, that \(\xi_0 \not\in R\). It follows from (3) that \(\zeta_m \not\in R\).

Since \(\xi_0 \not\in R\), \(\zeta_0 \in R\) and \(\zeta_m \not\in R\), there exists \(j \in \{0, \ldots, m-1\}\) such that

(a) \(\xi_j \not\in R\), \(\zeta_j \in R\) and (b) either \(\xi_{j+1} \in R\) or \(\zeta_{j+1} \not\in R\).

Let \(\zeta_{j+1} \in R\). According to (b), \(\xi_{j+1} \in R\). Since \(\zeta_j \in R\), Axiom \(B_2\) implies that \(\xi_j \in R\), which is a contradiction. Thus \(\zeta_{j+1} \not\in R\).

Clearly, \(d(x_j, x_{j+n}) \leq \|\xi_j\| = m\). If \(d(x_j, x_{j+n}) < m\), then—combining \((7_{m-1})\) with the fact that \(\xi_j \in R\)—we get \(\xi_j \in \mathcal{I}\) and therefore \(n = \|\xi_j\| = d(x_j, x_{j+n}) < m\), which is a contradiction. Thus \(d(x_j, x_{j+n}) = m\). This means that \(\xi_j \in \mathcal{I}\). Put

\[\sigma = x_j x_0 x_{m+n-1} \ldots x_{j+n+1}.\]

Then \(\xi_j = \sigma x_{j+n}\). Clearly, \(\sigma \in \mathcal{I}\). Recall that \(\zeta_{j+1} \not\in R\). It follows from (4) that

\[R(x_j, x_{j+n+1}) \neq \emptyset.\]

Since \(\sigma \in \mathcal{I}\), it follows from \((6_{m-1})\) that \(\sigma \in R\). Since \(\xi_j \not\in R\), Axiom \(B_1\) implies that

\[\text{(12) } x_j \varphi x_{j+n+1} x_{j+n} \not\in R \text{ for any } \varphi \in \mathcal{W}(G).\]

Combining the fact that \(\zeta_{j+1} \not\in R\) with (12) and Axiom \(A_3\), we see that there exists \(\psi \in \mathcal{W}(G)\) such that

\[x_j x_{j+1} \psi x_{j+n+1} \in R.\]

Put \(\omega = x_{j+1} \psi x_{j+n+1}\). Since \(d(x_j, x_{j+n+1}) = m - 1\), \((7_{m-1})\) implies that \(x_j \omega \in \mathcal{I}\). Since \(\sigma x_{j+n} \in \mathcal{I}\), we get \(x_j \omega x_{j+n} \in \mathcal{I}\). Hence \(\omega x_{j+n} \in \mathcal{I}\) and \(d(x_{j+1}, x_{j+n}) = \|\omega x_{j+n}\| = m - 1\).
Define

$$
\theta = x_{j+1} \ldots x_{j+n}.
$$

Since $\zeta_j \in \mathcal{R}$, (3) implies that $\theta \in \mathcal{R}$. Since $F\theta = x_{j+1}$, $L\theta = x_{j+n}$ and $\omega x_{j+n} \in \mathcal{I}$, it follows from $(6_{m-1})$ that $\omega x_{j+n} \in \mathcal{R}$. Obviously, $x_j \theta \in \mathcal{R}$. According to Axiom $B_1$, $x_j \omega x_{j+n} \in \mathcal{R}$. Since $L\omega = x_{j+n+1}$, we get a contradiction with (12).

We have proved that $\xi \in \mathcal{R}$. This means that $(6_m)$ holds.

1.2. Consider arbitrary $u, v \in V(G)$ such that $d(u, v) = m$. If $\mathcal{R}_{(u,v)} = \emptyset$, then $\mathcal{R}_{(u,v)} \subseteq \mathcal{I}_{(u,v)}$. Let $\mathcal{R}_{(u,v)} \neq \emptyset$. Consider an arbitrary $\zeta \in \mathcal{R}_{(u,v)}$. We want to prove that $\zeta \in \mathcal{I}$. Obviously, there exists $\xi \in \mathcal{I}_{(u,v)}$.

We first assume that $\xi$ and $\zeta$ have a common vertex $w$ such that $u \neq w \neq v$. Then (8) holds. Clearly, $d(u, w) < m$ and $d(w, v) < m$. As follows from $(7_{m-1})$, $\psi_1 w \in \mathcal{I}_{(u,w)}$ and $w \psi_2 \in \mathcal{I}_{(w,v)}$. This implies that $\zeta \in \mathcal{I}$.

We now assume that $\xi$ and $\zeta$ have no common vertex different from $u$ and $v$. Put $n = ||\zeta||$. Obviously, $n \geq m = d(u, v)$. Recall that we want to prove that $\zeta \in \mathcal{I}$. Suppose, to the contrary, that $\zeta \notin \mathcal{I}$. Then $n > m$. There exist mutually distinct $x_0, \ldots, x_{m+n-1} \in V(G)$ such that (9) holds. We adopt the convention (10) and define $\xi_i$ and $\zeta_i$ as in (11) for each $i \in \{0, \ldots, m\}$. Recall that

$$
\zeta_0 = \zeta = x_0 \ldots x_m \ldots x_n, \quad \zeta_m = x_m \ldots x_n \ldots x_{m+n} \quad \text{and} \quad x_{m+n} = x_0.
$$

If $\zeta_m \in \mathcal{R}$, then Axioms $A_1$ and $B_1$ imply that

$$
x_m \ldots x_n \ldots x_m \ldots x_0 \in \mathcal{R},
$$

which contradicts the fact that $\mathcal{R} \subseteq \mathcal{P}(G)$. Hence $\zeta_m \notin \mathcal{R}$.

Since $\zeta_0 \in \mathcal{I}, \zeta_0 \in \mathcal{R}$ and $\zeta_m \notin \mathcal{R}$, there exists $j \in \{0, \ldots, m-1\}$ such that

(a) $\xi_j \in \mathcal{I}, \xi_j \in \mathcal{R}$ and (b) either $\xi_{j+1} \notin \mathcal{I}$ or $\zeta_{j+1} \notin \mathcal{R}$.

Since $\xi_j \in \mathcal{I}$, it follows from $(6_m)$ that $\xi_j \in \mathcal{R}$. Axiom $A_4$ implies that

$$
\mathcal{R}_{(x_{j+1}, x_{j+n+1})} \neq \emptyset.
$$

Let $\xi_{j+1} \in \mathcal{I}$. According to $(6_m)$, $\xi_{j+1} \in \mathcal{R}$. Recall that $\xi_j, \xi_j \in \mathcal{R}$. Axiom $B_2$ implies that $\zeta_{j+1} \in \mathcal{R}$, which contradicts (b).

Thus $\xi_{j+1} \notin \mathcal{I}$. This means that $d(x_{j+1}, x_{j+n+1}) \leq m - 1$. Hence $d(x_{j+1}, x_{j+n}) \leq m$. Define $\rho$ as in (13). Assume that $d(x_{j+1}, x_{j+n}) \leq m - 1$; then $(7_{m-1})$ implies that
\(\varphi \in \mathcal{S}\); therefore \(n - 1 \leq m - 1\), which is a contradiction. Thus \(d(x_{j+1}, x_{j+n}) = m\). This means that \(d(x_{j+1}, x_{j+n+1}) = m - 1\). There exists \(\psi \in \mathcal{W}(G)\) such that

\[
x_{j+1}\psi x_{j+n+1} x_{j+n} \in \mathcal{S}.
\]

Similarly to 1.1, put \(\omega = x_{j+1}\psi x_{j+n+1}\). Then \(\|\omega\| = m - 1\). It follows from (6m) that \(\omega x_{j+n} \in \mathcal{R}\). Since \(\zeta \in \mathcal{R}\), Axiom B1 implies that \(x_j \omega x_{j+n} \in \mathcal{R}\). According to (3), \(x_j \omega \in \mathcal{R}\). Since \(d(x_j, x_{j+n+1}) = m - 1\), \((7m-1)\) implies that \(x_j \omega \in \mathcal{S}\). But \(\|x_j \omega\| = m > d(x_j, x_{j+n+1})\), which is a contradiction.

We have proved that \(\zeta \in \mathcal{S}\). This means that \((7m)\) holds.

Summarizing the results of 1.1 and 1.2, we see that \((5)\) holds.

Section 2. Denote

\[
Q = \{(t, z); t, z \in V(G) \text{ such that } \mathcal{R}_{(t, z)} \neq \emptyset\}.
\]

We want to prove that \(Q\) fulfills Axioms I–IV.

Consider arbitrary \(u, v, x, y \in V(G)\). Suppose \((u, v) \in Q\). Then \(\mathcal{R}_{(u, v)} \neq \emptyset\). According to \((5)\), \(\mathcal{R}_{(u, v)} = \mathcal{S}_{(u, v)}\).

(Verification of Axiom I) It follows from Axiom A1 that \(\mathcal{R}_{(v, u)} \neq \emptyset\). We get \((v, u) \in Q\).

(Verification of Axiom II) Suppose \(d(u, v) = d(u, x) + d(x, v)\). If \(x = v\), then it is obvious that \((u, x) \in Q\). Let \(x \neq v\). Then there exist \(\alpha, \beta \in \mathcal{W}(G)\) such that \(\alpha \beta \in \mathcal{S}_{(u, v)}\). Hence \(\alpha \beta \in \mathcal{R}_{(u, v)}\). It follows from (3) that \(\alpha \beta \in \mathcal{R}_{(u, x)}\). Therefore, \(\mathcal{R}_{(u, x)} \neq \emptyset\). We get \((u, x) \in Q\).

(Verification of Axiom III) Suppose \(\{u, x\}, \{v, y\} \in E(G)\) and \(d(x, y) = d(u, v) - 1 = d(x, y)\). Clearly, \(x \neq v\). There exists \(\alpha \in \mathcal{W}(G)\) such that \(uxav \in \mathcal{S}\). Since \(d(u, v) = d(x, y)\), we have \(xav \notin \mathcal{S}\). Since \(uxav \in \mathcal{S}\), we have \(uxav \in \mathcal{R}\). Since \(xav \notin \mathcal{S}\), (5) implies that \(xav \notin \mathcal{R}\). It follows from (4) that \(\mathcal{R}_{(u, y)} \neq \emptyset\). We get \((u, y) \in Q\).

(Verification of Axiom IV) Suppose \(\{u, x\}, \{v, y\} \in E(G)\) and \(d(x, v) = d(u, v) - 1 \geq 1\). There exists \(\alpha \in \mathcal{W}(G)\) such that \(uxav \in \mathcal{S}\). Hence \(uxav \in \mathcal{R}\). If \(xav \in \mathcal{R}\), then \(\mathcal{R}_{(x, y)} \neq \emptyset\). Let \(xav \notin \mathcal{R}\). If there exists \(\beta \in \mathcal{W}(G)\) such that \(\beta x \gamma \notin \mathcal{S}\), then (3) implies that \(x \beta y \in \mathcal{R}\), and thus \(\mathcal{R}_{(x, y)} \neq \emptyset\). Let \(uxav \in \mathcal{R}\) for any \(v \in \mathcal{W}(G)\). Axiom A3 implies that there exists \(\gamma \in \mathcal{W}(G)\) such that \(w \gamma y \in \mathcal{R}\). Since \(uxav \in \mathcal{R}\), Axiom A4 implies that \(\mathcal{R}_{(x, y)} \neq \emptyset\). We get \((x, y) \in Q\).

We have proved that \(Q\) is a visibility in \(G\).

The proof of the theorem is complete. \(\square\)
The following corollary is similar to the result which was (under the condition that $G$ is finite) originally proved in [2]:

**Corollary.** Let $G$ be a connected graph, and let $\mathcal{R} \subseteq \mathcal{P}(G)$. Then $\mathcal{R} = \mathcal{I}(G)$ if and only if $\mathcal{R}$ fulfills Axioms $A_1$–$A_3$, $B_1$–$B_3$ and the following Axiom $A_0$ (for arbitrary $u, v, x, y \in V(G)$ and $\alpha, \beta, \gamma, \delta \in \mathcal{W}(G)$):

$$A_0 \quad \mathcal{R}_{(u,v)} \neq \emptyset.$$

**Proof.** Let $\mathcal{R} = \mathcal{I}(G)$. Then $\mathcal{R}$ fulfills Axiom $A_0$. Our theorem implies that $\mathcal{R}$ fulfills Axioms $A_1$–$A_3$ and $B_1$–$B_3$.

Conversely, let $\mathcal{R}$ fulfill Axioms $A_0$–$A_3$ and $B_1$–$B_3$. Axiom $A_0$ implies that $\mathcal{R}$ fulfills Axiom $A_4$. According to our theorem, there exists a visibility $Q$ in $G$ such that (1) holds. Axiom $A_0$ states that $\mathcal{R}_{(u,v)} \neq \emptyset$ for every pair of vertices $u, v$ of $G$. Combining this fact with (1), we get $\mathcal{R} = \mathcal{I}(G)$, which completes the proof.  

**Remark.** Let $G$ be a finite connected graph. The set $\mathcal{I}(G)$ is closely related to the interval function of $G$ in the sense of H.M. Mulder [1]. An “almost non-metric” characterization of the interval function of $G$ was given in [3].

**References**


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